Lecture Note Sketches

Applied Nonlinear Dynamics 322

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Contents

References

1 Introduction

Dynamical systems: predict temporal evolution

Characterize the state of the system by a set of quantities: $\mathbf{x}(t)$

Evolution will in general depend on the history of the system at earlier times

$$\{\mathbf{x}(t)|t \leq t_0\} \qquad \Rightarrow \qquad \mathbf{x}(t_p)$$

It is sufficient to know the rate of change of the system at any given time:

- $\frac{d\mathbf{x}}{dt}$ could depend on the states over a whole earlier interval
  e.g.
  $$\frac{dx(t)}{dt} = \int_{-\infty}^{t} \mathbf{f}(\mathbf{x}(t'))dt'$$
  with f a possibly nonlinear vector function of $\mathbf{x}(t)$. In that case one would get an integro-differential equation.

- most often it is sufficient to know the state at the current time
  e.g. $\mathbf{x}(t)$ is described by differential equation
  $$\frac{d\mathbf{x}(t)}{dt} = \mathbf{f}(\mathbf{x}(t))$$

We focus on differential equations: What is the challenge?

The differential equation

$$\frac{d^2x}{dt^2} + x = 0$$

has the two independent solutions

$$x_1(t) = \cos t \qquad x_2(t) = \sin t$$

Linear superposition

$$x_g(t) = Ax_1(t) + Bx_2(t)$$

provides the solution for any *initial* condition

$$x(0) = x_0 \qquad \left.\frac{dx}{dt}\right|_{t=0} = v_0$$

Superposition possible only for linear equations.

Here we focus on **nonlinear dynamics**

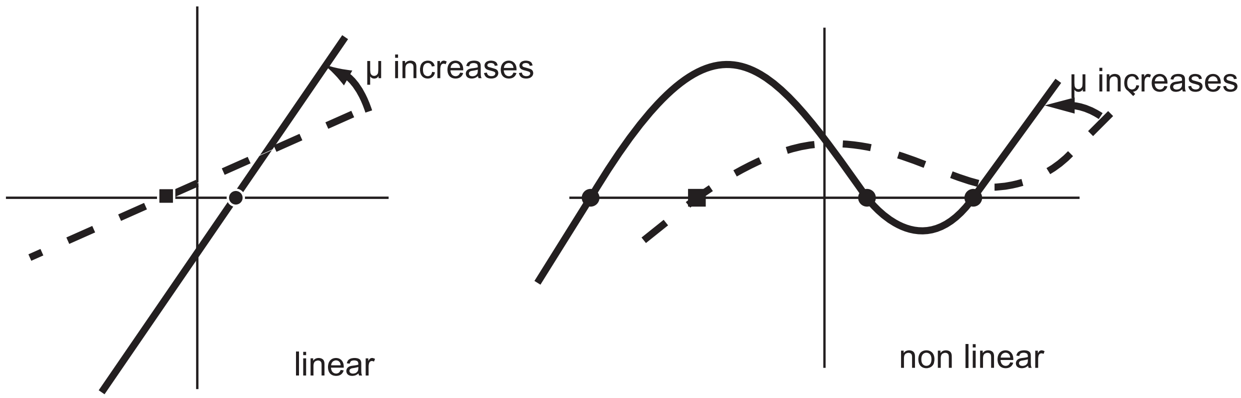
- differential equations much more difficult to solve

- systems exhibit vast richness of phenomena

Simple illustration: **linear vs. nonlinear**

Consider zeros x_0 of a function f that depends on a parameter μ

$$f(x, \mu) = 0$$



Linear Systems:

- change in parameter leads to quantitative change of solution
- unique solution for a given value of the parameter

Nonlinear Systems:

- not only quantitative changes but also **qualitative changes** in the behavior at transition points:
  solutions can depend on parameters in a **non-smooth** fashion (e.g. Taylor vortex flow)
- multiplicity of solutions for the same parameter values: hysteresis
  (e.g. rolls vs. spiral-defect chaos convection)
- chaotic dynamics
  many frequencies, coexisting (unstable) periodic solutions (e.g. Belousov-Zhabotinsky reaction)

Examples:

- Fluid flow between rotating cylinders
- Flow of a fluid in a layer heated that is heated from below
- Oscillations in chemical reactions

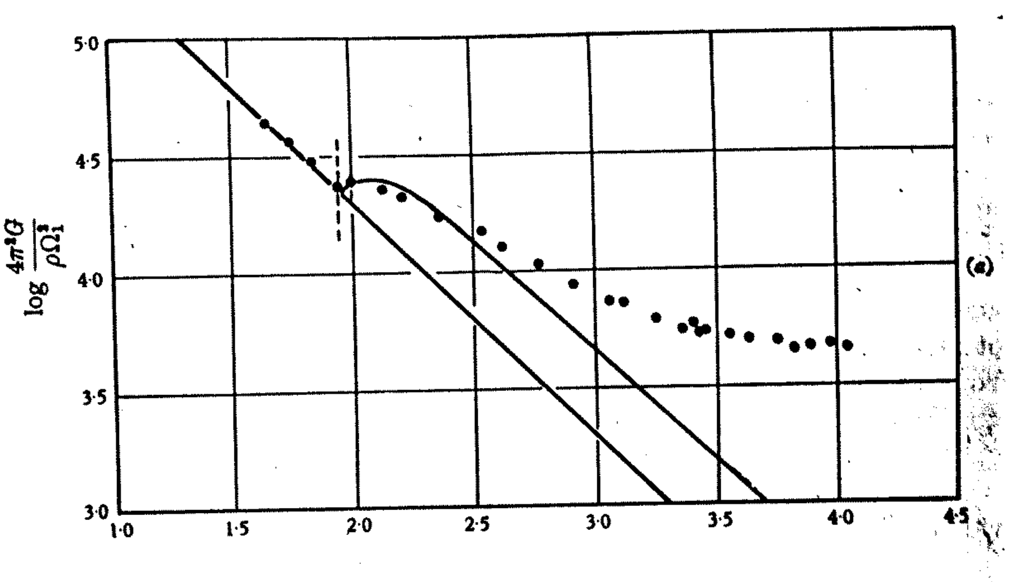


Figure 1: Fluid flow between two rotating cylinders. The torque needed to rotate the inner cylinder increases suddenly (non-smoothly) when the flow changes qualitatively to form Taylor vortics (Donnelly and Simon, J. Fluid Mech. 7 (1960) 401). (Note, in the base flow the torque G increases linearly with the rotation rate Ω_1 and $G/\Omega_1^2 \propto 1/\Omega_1$)

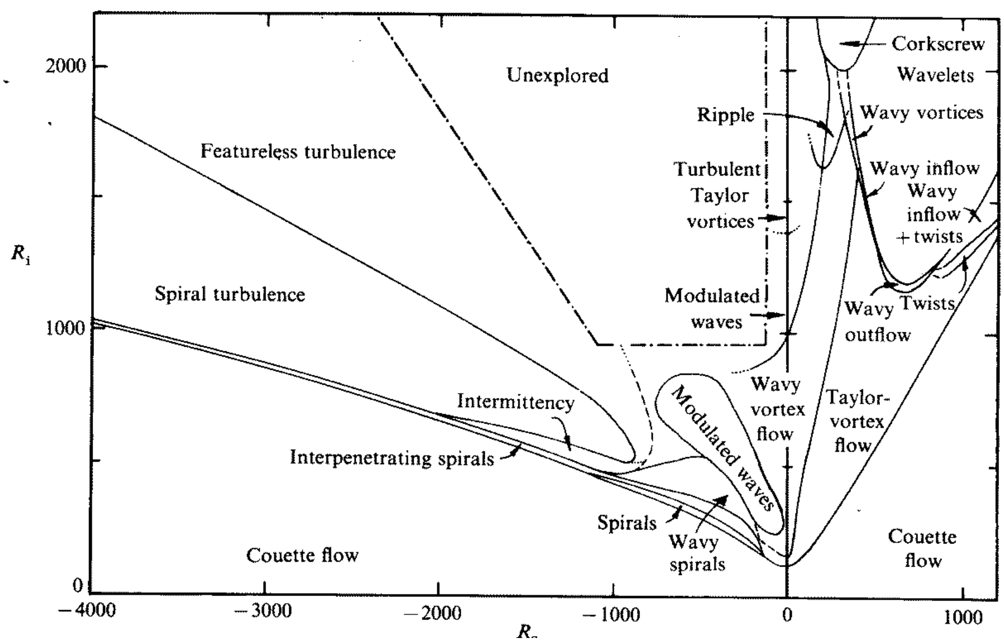


Figure 2: Taylor vortex flow exhibits a bewildering multitude of qualitatively different behaviors when the rotation rates are changed (Andereck, Liu and Swinney, J. Fluid Mech. 164 (1986) 155).

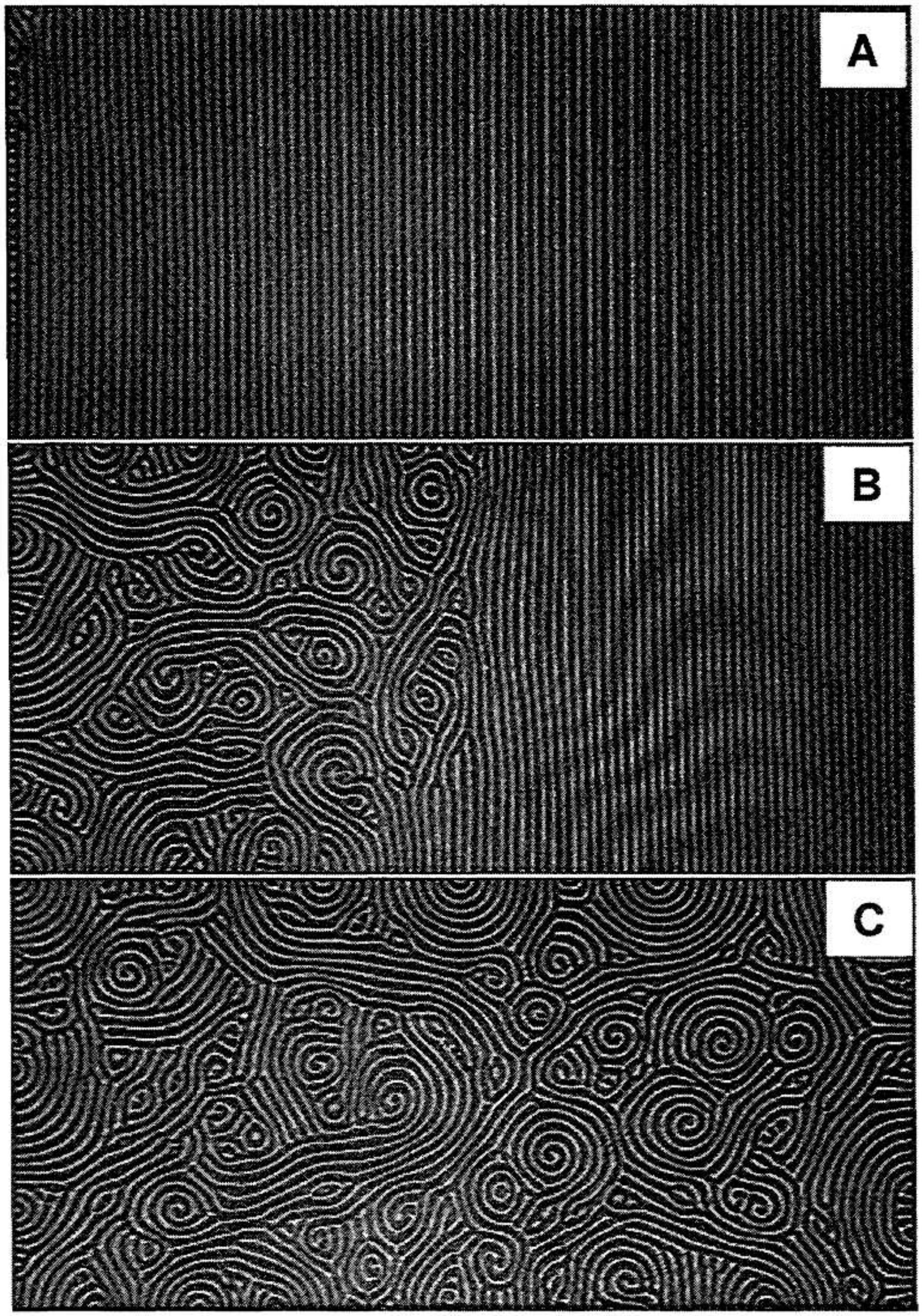


Figure 3: Convection of a fluid heated from below. Steady roll patterns and spiral defect chaos are stable for the same parameter values (Melnikov, Egolf, Jeanjean, Plapp, Bodenschatz AIP Conf. Prof. 501 (2000) 36)

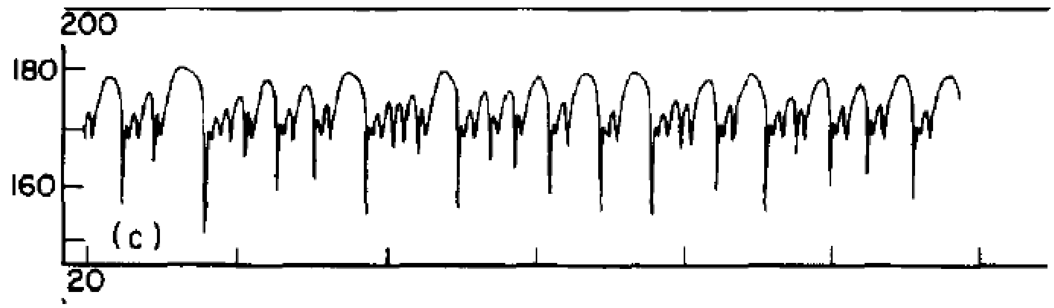


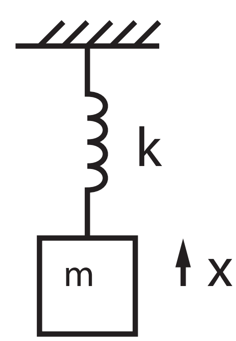
Figure 4: Chaotic oscillations in the Belousov-Zhabotinsky reaction (Schmitz, Graziani, and Hudson, J. Chem. Phys. 67 (1977) 3040).

Possible approaches

- exact analytical solutions for nonlinear systems are available only in rare cases even if analytical solution is available it is often so complicated that it is diffcult to extract from it insight about the mechanisms underlying the behavior of the system
- numerical solution
  - confirms the model/basic equations: of great interest if mathematical model has not been established, e.g., chemical oscillations, heart muscle
  - gives quantitative details for **specific** values of system parameters: these details may not be accessible in experiments: 3d fluid flow, turbulent, chemical concentrations of each species
  - insight into the mechanisms?
- qualitative aspects: transitions between different states
  - approximate solutions near transition points using perturbative methods
- visualization: **geometry of dynamics**, phase portraits
  - overview of **all** possible behaviors

Simple Example: Mass-spring-system

$$\frac{d^2x}{dt^2} = -\beta\frac{dx}{dt} - \frac{k}{m}x$$



with friction β.

We will write all differential equations as first-order systems:

$$\begin{aligned}\dot{x} &= v \\ \dot{v} &= -\beta v - \frac{k}{m}x\end{aligned}$$

Note:

- notation: we often write $\dot{x}$ for $\frac{dx}{dt}$

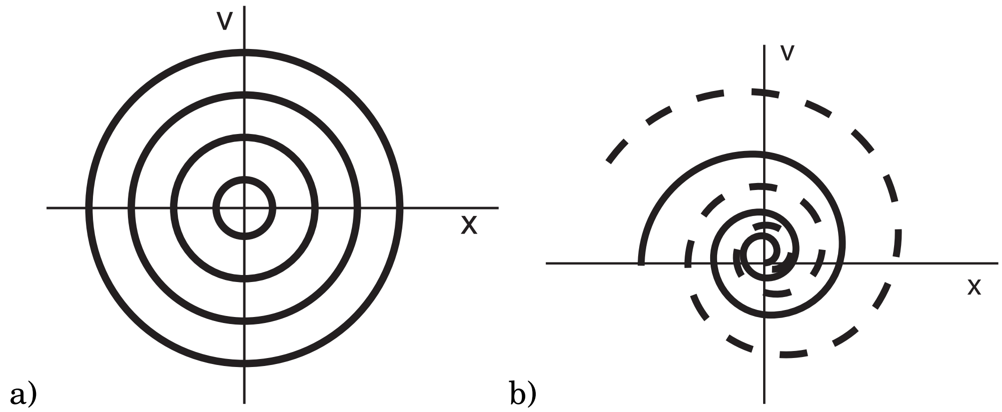


Figure 5: Phase portraits for a harmonic oscillator. a) without friction ($\beta = 0$) any initial condition leads to harmonic oscillations. b) with damping ($\beta > 0$) any initial condition leads to oscillatory convergence to a fixed point.

The phase portrait provides a **complete overview** of all possible solutions
(here linear $\rightarrow$ not much going on.)

Conservative Systems (no friction/dissipation)

- almost all different initial condition lead to different states

Dissipative Systems

- a range of initial conditions leads to the same state: **attractors**
- **transitions**: **qualitative** change in the number and type of attractors

We will mostly focus on dissipative systems.

2 1-d Flow

2.1 Flow on the Line[1]

Any first-order differential equation with constant coefficients,

$$\dot{x} = f(x),$$

can be solved exactly for any $f(x)$ (by separation of variables).

$$\int_{x_0}^{x} \frac{dx}{f(x)} = t - t_0$$

[1]cf. Strogatz Ch.2

Example:

$$\dot{x} = \sin x$$

$$t = \int \frac{dx}{\sin x} = \int \csc x dx$$

$$t = -\ln|\csc x + \cot x| + C$$

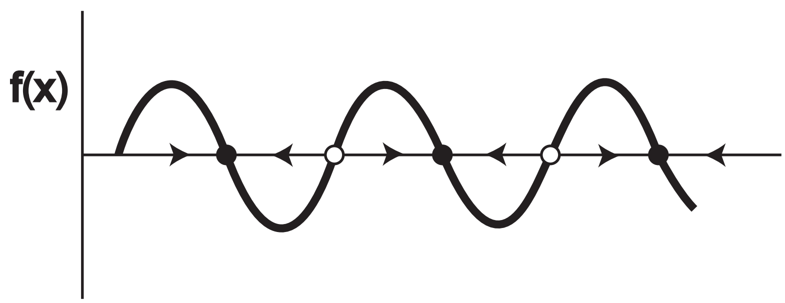
Now what? What have we learned?

Even if we could solve for x, would we have an overview of the behavior of system for arbitrary initial conditions?

Geometrical picture: **phase space** (or phase line in 1 dimension)

$\dot{x} = f(x)$ defines a *flow* in phase space or a *vector field*

For 1d: plot in addition $f(x)$



The phase portrait gives us a complete picture of the qualitative behavior of the system.

Conclude: any i.c. ends up in one of the **fixed points** at $x_n = (2n+1)\pi$.
Fixed points are **stagnation points** of the flow

Stability:

- flow *into* $x_n = (2n+1)\pi$: *stable*
- flow *out of* $x_n = 2n\pi$: *unstable*

Of course: for **quantitative results** ('numbers') we need the detailed solution

Example: Population Growth with Limited Resources

Write N for the number of animals (size of population)

$$\dot{N} = g(N)N$$

with $g(N) =$ net birth/death rate

Limited food/space:
births decrease, deaths increase with increasing N

Minimal model

$$g(N) = \alpha - \beta N$$

$\Rightarrow$ Logistic growth model

$$\dot{N} = \alpha N - \beta N^2$$

Make dimensionless:

$$[\alpha] = \frac{1}{s} \qquad [\beta] = \frac{1}{s}\frac{1}{\#}$$

$\frac{1}{\alpha}$ characteristic growth time $\qquad \dfrac{\alpha}{\beta}$ characteristic population size

Introduce

$$\tau = \alpha t \qquad n = \frac{\beta}{\alpha} N$$

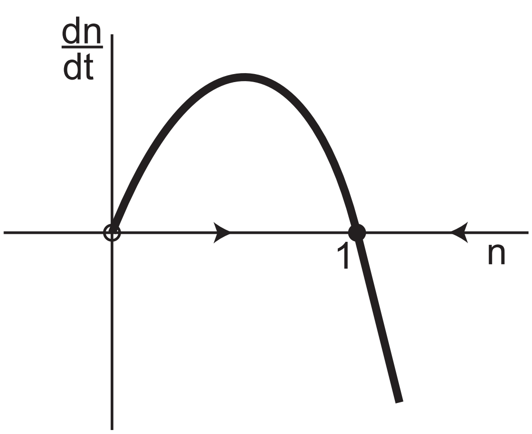
Question: If population goes to some equilibrium, what size would you expect?

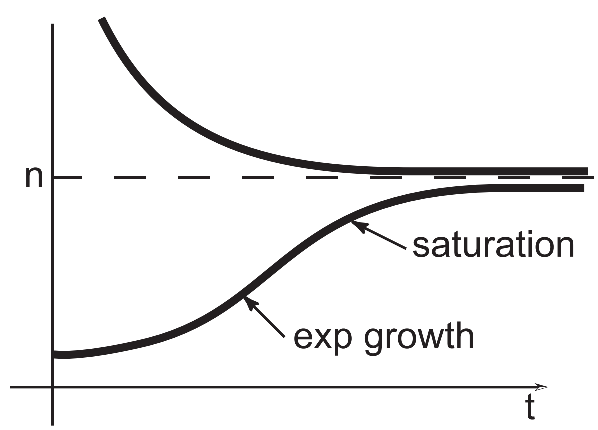
$\frac{\alpha}{\beta}$ is the only characteristic size after initial condition is forgotten $\Rightarrow$ expect N $\rightarrow \frac{\alpha}{\beta}$

$$\frac{dn}{d\tau} = n - n^2$$

Could solve the differential equation by partial fraction

Instead, consider phase space:





fixed points: $n = 0$, $n = 1$

flow indicates: $n = 0$ unstable, $n = 1$ stable indeed: **all i.c.** go to $N = \frac{\alpha}{\beta}$.

In the one-dimensional plot the stability of the fixed points can be read off easily. In general (higher dimensions) much more difficult $\Rightarrow$ determine the *linear* stability of the fixed points, i.e. consider small perturbations around them.

Linear Stability:

Study effect of small perturbation away from fixed point

Linearize around fixed points:

Insert the expansion

$$n = n_0 + \epsilon n_1(\tau) \qquad \epsilon \ll 1$$

into the equation for n

$$\frac{d(n_0 + \epsilon n_1(\tau))}{d\tau} = n_0 + \epsilon n_1(\tau) - (n_0 + \epsilon n_1(\tau))^2$$

Expanding the square we get

$$\epsilon \frac{dn_1(\tau)}{d\tau} = n_0 - n_0^2 + \epsilon \left(n_1(\tau) - 2n_0 n_1(\tau) \right) - \epsilon^2 n_1(\tau)^2$$

Equating like powers in ϵ on both sides we get

$$\begin{aligned} \mathcal{O}(\epsilon^0): \qquad 0 &= n_0 - n_0^2 \\ &\Rightarrow n_0 = 1 \qquad \textbf{or} \qquad n_0 = 0 \end{aligned}$$

$$\begin{aligned} \mathcal{O}(\epsilon^1): \qquad \frac{dn_1}{d\tau} &= n_1 - 2n_0 n_1 = (1 - 2n_0)\, n_1 \\ &\Rightarrow n_1 \propto e^{(1-2n_0)\tau} \end{aligned}$$

Thus,

$$\begin{array}{lll} n_0 = 1 \Rightarrow & 1 - 2n_0 < 0 & \text{that fixed point is stable} \\ n_0 = 0 \Rightarrow & 1 - 2n_0 > 0 & \text{that fixed point is unstable} \end{array}$$

More generally

$$\dot{x} = f(x)$$

To determine the linear stability of the fixed point x_0 we make the ansatz:

$$x = x_0 + \epsilon x_1(t)$$

and expand

$$\dot{x}_1 = f\left(x_0 + \epsilon x_1\right) = f(x_0) + f'(x_0)\, \epsilon x_1 + \frac{1}{2} f''(x_0)\, \left(\epsilon x_1\right)^2 + \ldots$$

Again equation like powers

$$\begin{aligned} \mathcal{O}(\epsilon^0): \qquad 0 &= f(x_0) \\ \mathcal{O}(\epsilon^1): \qquad \dot{x}_1 &= f'(x_0) x_1 \end{aligned}$$

Thus,

$$x_1(t) \propto e^{f'(x_0)t}$$

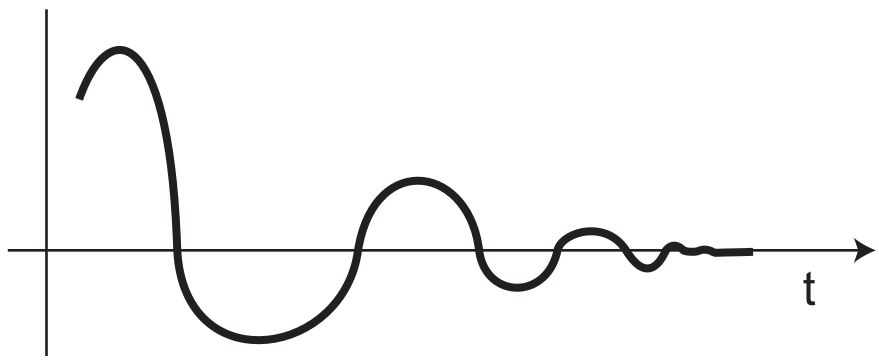
and

the fixed point is stable: $f'(x_0) < 0$
the fixed point is unstable: $f'(x_0) > 0$

Note: for coupled systems $f'(x)$ is replaced by Jacobian matrix:
eigenvalues determine stability (see below).

2.1.1 Impossibility of Oscillations:

Can the solution approach fixed point via **oscillations**?



No! To get oscillations $x(t)$ would have to come back to an earlier value at a later time, i.e. $x(t_2) = x(t_1)$ with $t_2 > t_1$. But the sign of $\dot{x}(t)$ would have to be opposite at the two times, in particular $f(x(t_1)) \neq f(x(t_2))$, which is not possible since $x(t_2) = x(t_1)$.

$\rightarrow$ system evolves monotonically between fixed points

More general concept: potential

For one-dimensional systems with time-independent coefficients one can always write

$$\dot{x} = f(x) = -\frac{dV}{dx} \qquad \text{with } V = -\int f(x)dx$$

Consider:

$$\frac{dV}{dt} = \frac{dV}{dx}\dot{x} = -\dot{x}^2 \leq 0$$

any change in x reduces $V \Rightarrow V$ cannot return to previous, higher value $\Rightarrow$ no periodic motion possible

$$\frac{dV}{dt} = 0 \qquad \Rightarrow \qquad \dot{x} = 0 \qquad \text{fixed point}$$

x either goes to a fixed point or it diverges to $-\infty$ (if V is not bounded from below).

Notes:

- V is called a Lyapunov function for the differential equation

Compare: mechanical system in overdamped limit

$$m\ddot{x} = -\beta\dot{x} + F(x)$$

for very small mass (no inertia)

$$\dot{x} = \frac{1}{\beta}F(x)$$

Overshoot requires inertia, i.e. 2nd derivative.

Note: The concept of the potential is not limited to one-dimensional systems. However, only few higher-dimensional systems can be derived from a potential.

2.2 Existence and Uniqueness

So far we assumed that starting from a given initial condition we always get a **unique** solution for **all times**:

- at any time 'we know where to go'
- we can continue this forever

Solutions to

$$\dot{x} = f(x)$$

1. do not have to exist for all times:
   for a given initial condition the solution may cease to exist beyond some time
2. do not have to be unique:
   the same initial condition can lead to different states later.

1. Existence

solution can disappear by becoming infinite

if this happens in *finite* time then there is no solution beyond that time

Example:

$$\dot{x} = +x^{\alpha} \qquad \text{with} \qquad x(0) = x_0 > 0$$

$$\begin{aligned} \int x^{-\alpha}\,dx = \int \frac{dx}{x^{\alpha}} &= t + C \\ \frac{1}{1-\alpha}x^{1-\alpha} &= t + C \end{aligned}$$

initial conditions:

$$\begin{aligned} C &= \frac{1}{1-\alpha}x_0^{1-\alpha} \\ x(t) &= \left((1-\alpha)t + x_0^{1-\alpha}\right)^{\frac{1}{1-\alpha}} \end{aligned}$$

Solution diverges at

$$t^* = \frac{x_0^{1-\alpha}}{\alpha - 1} > 0 \qquad \text{if} \qquad \alpha > 1$$

i.e. for $\alpha > 1$ the divergence occurs in **finite time.**

Note:

- a divergence in *infinite* time represents no problem: $x(t) = e^t$

2. Uniqueness

Consider the previous example for $0 < \alpha < 1$

$$\Rightarrow x = 0 \qquad \text{for} \qquad t^* = \frac{x_0^{1-\alpha}}{\alpha - 1} < 0$$

Solution can start at t^* with $x(t^*) = 0$ and grow from there.

But: $\tilde{x}(t) \equiv 0$ is also a solution for all times.

Thus:

- two *different* solutions $\tilde{x}(t)$ and $x(t)$ satisfy the *same* initial condition, $x(t^*) = 0$.

Moreover, we can construct another solution by starting with $\tilde{x}(t) \equiv 0$ for $t < t^*$ and 'switch' to $x(t) > 0$ beyond t^*,

$$\hat{x}(t) = \begin{cases} 0 & t \le t^* \\ \left((1-\alpha)t + x_0^{1-\alpha}\right)^{\frac{1}{1-\alpha}} & t > t^* \end{cases}$$

The combined solution $\hat{x}(t)$ is continuous and satisfies the differential equation.

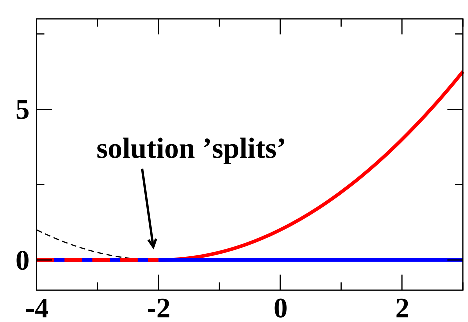


Figure 6: Non-uniqueness of an initial value problem: both, the blue and the red function satisfy the differential equation $\dot{x} = x^{\frac{1}{2}}$ with the initial condition $x(t_0) = 0$ for any $t_0 \le t^* = -2$.

Worse: t^* varies with x_0

$\Rightarrow$ can pick any t^* and patch the solutions at that t^*

$\Rightarrow$ infinitely many solutions with identical i.c. $x = 0$.

Note:

- in the linear case, $\alpha = 1$, one would have $x(t) \propto e^t$ and $x = 0$ would be reached only in the infinite past ($t \to -\infty$), which would not allow a connection to the solution $x \equiv 0$.

Theorem[2]**:**

If for $\dot{x} = f(x,t)$

- $f(x,t)$ is continuous in the vicinity of the initial condition $x(t_0) = x_0$, i.e. for $|t-t_0| < \Delta t$ and $|x - x_0| \leq \Delta x$, and
- $f(x,t)$ satisfies Lipschitz condition within that vicinity,

$$|f(x_1,t) - f(x_2,t)| \leq K\,|x_1 - x_2| \qquad \text{for all } |x_{1,2} - x_0| \leq \Delta x \text{ and for all } \; |t - t_0 \leq \Delta t|$$

  with some constant K (K can be thought of as a maximal slope in that vicinity)

then the solution exists for a finite time interval around t_0, $|t - t_0| \leq \Delta T$, and is unique there. The interval is given by

$$\Delta T = \min\left(\Delta t, \frac{\Delta x}{M}\right)$$

where M is the maximum of $|f(x,t)|$ within $|t - t_0| < \Delta t$ and $|x - x_0| \leq \Delta x$.

Notes:

- $f(x) = |x|^\alpha$ does not satisfy Lipschitz condition at $x = 0$ for $0 < \alpha < 1$:
  we would need

$$|x|^\alpha \leq Kx \quad \text{for all } x \; \text{ near } \; x = 0$$

  i.e. $K \geq |x|^{\alpha-1}$. But for $|x|^{\alpha-1} \to \infty$ for $x \to 0$ and $\alpha < 1$.
  $\Rightarrow$ the theorem does not guarantee the uniqueness of solution; and, in fact, it is not unique.

- If $f'(x)$ is continuous near x_0 then the slope of $f(x)$ has a maximum and $f(x)$ satisfies the Lipschitz condition ,

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h} \qquad \to \qquad f(x_0 + h) - f(x_0) = \mathcal{O}(h),$$

  and the solution is unique near x_0

2.3 Bifurcations in 1 Dimension[3]

We had: in 1d final state always fixed point (if dynamics are bounded)

How many fixed points? How can the number of fixed points change?

[2] see, e.g., Lin & Segel, *Mathematics applied to deterministic problems in the natural sciences*, p.57
[3] cf. Strogatz Ch.3

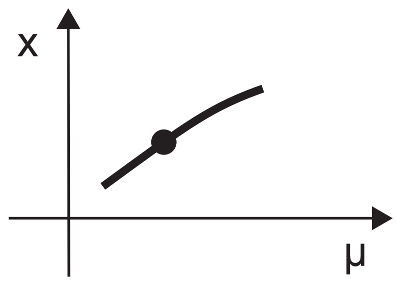
⇒ Introduce parameter μ

$$f(x,\mu)=0$$

To study the creation or elimination of fixed point we need to study only small changes in μ

⇒ analysis in the *vicinity* of some specific value of μ

Question: Does the solution persist when the parameter is changed? Is it unique?



2.3.1 Implicit Function Theorem

Local analysis in the vicinity of a fixed point for small changes in μ:

Taylor expansion

$$f(x,\mu)=\underbrace{f(x_0,\mu_0)}_{=0}+\frac{\partial f}{\partial x}(x-x_0)+\frac{\partial f}{\partial \mu}(\mu-\mu_0)+\frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x-x_0)^2+\cdots$$

(All derivatives evaluated at x_0,μ_0)

fixed point: $f(x_0,\mu_0)=0$

If $\frac{\partial f}{\partial x}|_{x_0,\mu_0}\neq 0 \Rightarrow$ solve uniquely for x

$$x-x_0=-(\mu-\mu_0)\frac{\frac{\partial f}{\partial u}}{\frac{\partial f}{\partial x}}+O\left((\mu-\mu_0)^2\right)+\underbrace{\mathcal{O}\left((x-x_0)^2\right)}_{\mathcal{O}\left((\mu-\mu_0)^2\right)}$$

With $x-x_0=\mathcal{O}(\mu-\mu_0)$ the higher-order terms can be consistently neglected and the fixed point exists for all values of μ in the vicinity of μ_0:
Thus, in this case there is a *branch* of solutions.

More generally for higher dimensions:

Implicit function theorem:

Consider the solutions of

$$\mathbf{f}(\mathbf{x},\mu)=0 \quad \mathbf{x}\in\mathcal{R}^n \qquad \mathbf{f} \text{ smooth in } \mathbf{x} \text{ and } \mu$$

If

$$\mathbf{f}(\mathbf{x}=\mathbf{x}_0,\mu=\mu_0)=0$$

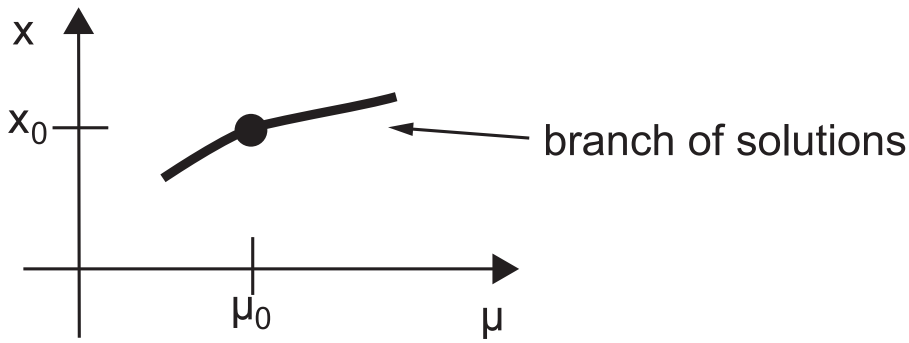
and

$$\det\left(\frac{\partial f_i}{\partial x_j}\right) \neq 0 \qquad \text{at } \mu = \mu_0 \text{ and } \mathbf{x} = \mathbf{x}_0,$$

then there is a **unique** differentiable $\mathbf{X}(\mu)$ that satisfies in the vicinity of $\mu = \mu_0$

$$\mathbf{f}(\mathbf{X}(\mu), \mu) = 0 \quad \text{and} \quad \mathbf{X}(\mu = \mu_0) = \mathbf{x}_0.$$

Thus: if $\det\left(\frac{\partial f_i}{\partial xj}\right) \neq 0$ there is a **branch** of solutions going through $\mathbf{x} = \mathbf{x}_0$ as μ is varied.



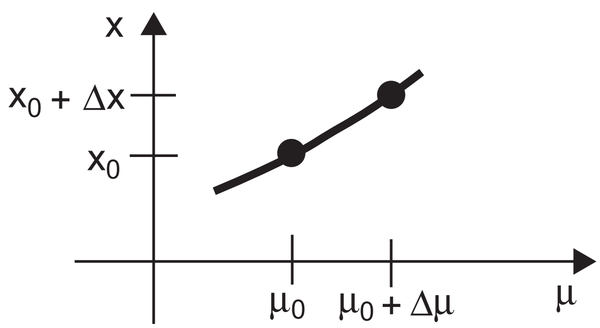
Notes:

- In 1d: $\det \frac{\partial f_i}{\partial xj} \to \frac{df}{dx} = f'(x)$
  $\Rightarrow$ as seen in explicit calculation: if $f'(x) \neq 0$ branch persists uniquely
- change in x is smooth in μ if $\frac{\partial f}{\partial x} \neq 0$

$$\Delta x \sim \Delta\mu \tag{1}$$

- **generic properties** are those properties that do not require **any tuning** of the parameters

  When picking parameters at random one expects $\frac{\partial f}{\partial x}|_{x_0,\mu_0} \neq 0$,
  i.e. we would need to *tune* μ to get $\frac{\partial f}{\partial x}|_{x_0,\mu_0} = 0$
  $\Rightarrow$ **generically** there is a smooth branch



Since the stability changes if $f'(x) = 0$

A **change in the number of fixed points** requires a **change in the (linear) stability** of the fixed point.

2.3.2 Saddle-Node Bifurcation

What happens when $\frac{\partial f}{\partial x}=0$?

Need to go to higher order in Taylor expansion (for simplicity choose $x_0=0, \mu_0=0$)

$$0=f(x,\mu)=\underbrace{f(0,0)}_{=0}+\underbrace{\frac{\partial f}{\partial x}}_{=0}\,x+\frac{\partial f}{\partial \mu}\,\mu+\frac{1}{2}\frac{\partial^2 f}{\partial x^2}\,x^2+\frac{\partial^2 f}{\partial x\partial \mu}\,x\mu+\frac{1}{2}\frac{\partial^2 f}{\partial \mu^2}\,\mu^2+\ldots$$

Solve again:

$$x^2=-\frac{2}{\frac{\partial^2 f}{\partial x^2}}\left\{\frac{\partial f}{\partial \mu}\mu+\frac{\partial^2 f}{\partial x\partial \mu}\,x\mu+\frac{1}{2}\frac{\partial^2 f}{\partial \mu^2}\mu^2+\cdots\right\}$$

In the smooth case we had (cf. (1))

$$x\sim\mu, \qquad \text{e.g.} \qquad x=\alpha\mu,$$

With this scaling we would get

$$\underbrace{x^2}_{\alpha^2\mu^2=O(\mu^2)}=-\frac{2}{\frac{\partial^2 f}{\partial x^2}}\left\{\frac{\partial f}{\partial \mu}\underbrace{\mu}_{\mathcal{O}(\mu)}+\frac{\partial^2 f}{\partial x\partial \mu}\underbrace{x\mu}_{\alpha+\mu^2=\mathcal{O}(\mu^2)}+\frac{1}{2}\frac{\partial^2 f}{\partial \mu^2}\underbrace{\mu^2}_{\mathcal{O}(\mu^2)}+\cdots\right\}$$

To solve this equation we collect all terms of a given order in μ, i.e. μ^0, μ^1, μ^2, $\ldots$and solve the equations order by order in μ:
There is no term of $\mathcal{O}(\mu)$ on the left-hand side, the first term on the right-hand side is the only term of $\mathcal{O}(\mu)$. At $\mathcal{O}(\mu)$ we therefore obtain the equation

$$\frac{\partial f(0,0)}{\partial \mu}=0$$

This is in general not the case $\Rightarrow$ assuming the scaling $x\sim\mu$ leads therefore to a contradiction.

Try a different balance of x and μ

$$x\sim\mu^{1/2} \qquad \text{e.g.} \qquad x=\alpha\mu^{1/2}$$

Then

$$\underbrace{x^2}_{\alpha^2\mu=\mathcal{O}(\mu)}=-\frac{2}{\frac{\partial^2 f}{\partial x^2}}\left\{\frac{\partial f}{\partial \mu}\mu+\frac{\partial^2 f}{\partial x\partial \mu}\underbrace{x\mu}_{\alpha\mu=\mathcal{O}(\mu^{3/2})}+\frac{1}{2}\frac{\partial^2 f}{\partial \mu^2}\mu^2+\cdots\right\}$$

and we can balance the first term on the right-hand side with the left-hand side and obtain

$$x_{1,2}=\pm\sqrt{-2\frac{\frac{\partial f}{\partial \mu}}{\frac{\partial^2 f}{\partial x^2}}\;\mu}$$

Notes:

- If the implicit function theorem fails one gets a nonlinear, higher-order equation with multiple solutions (depending on the parameters)
- the change in x is **not smooth** in μ

Dynamics:

$$\dot{x} = f(x, \mu) = a\mu + bx^2 + \text{h.o.t.} \tag{2}$$

with

$$a = \left.\frac{\partial f}{\partial \mu}\right|_{x=x_0,\mu=\mu_0} \equiv \partial_\mu f \qquad b = \left.\frac{\partial^2 f}{\partial x^2}\right|_{x=x_0,\mu=\mu_0} \equiv \partial_x^2 f$$

and $h.o.t.$ denotes higher-order terms.

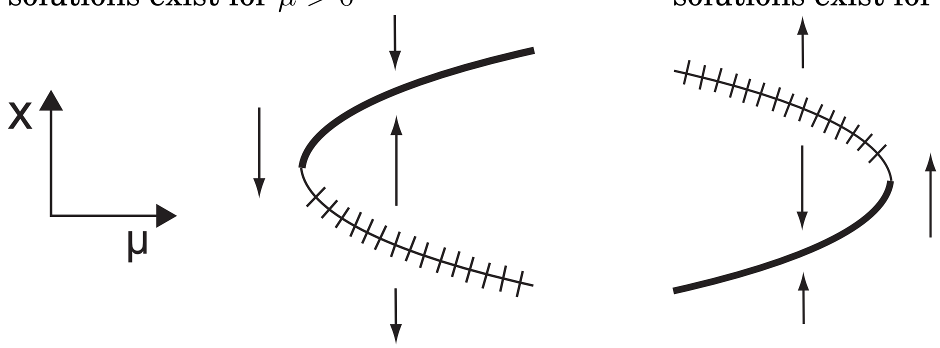
Bifurcation diagrams: plot all solution branches as a function of μ

Relevant parameters:

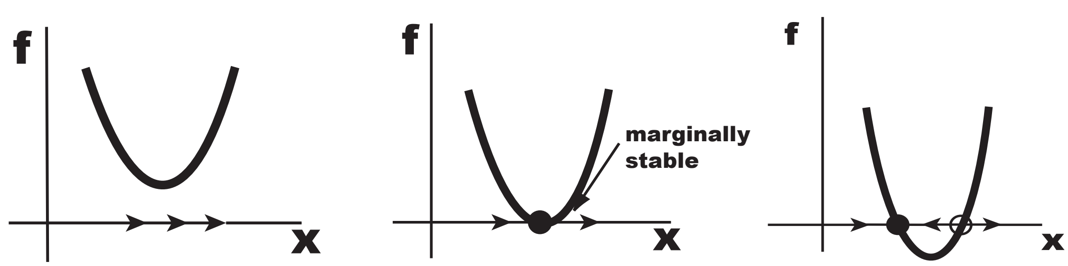
$$\frac{a}{b} = \frac{\partial_\mu f}{\partial_x^2 f} \equiv \frac{\frac{\partial f}{\partial \mu}}{\frac{\partial^2 f}{\partial x^2}}$$

$\frac{a}{b} < 0$ and $a > 0$
solutions exist for $\mu > 0$

$\frac{a}{b} > 0$ and $a > 0$
solutions exist for $\mu < 0$



The y-direction corresponds to the phase line for a given value of μ. The arrows indicate the flow on the phase line for that value of μ.



Notes:

- 2 fixed points are created/destroyed. Single solutions cannot simply pop up or disappear: merging and annihilation of 2 solutions

- The coinciding fixed points at $\mu = 0$ are linearly marginally stable:
  $\partial_x f$ changes sign going along the solution branch: **change in stability**

- The only condition for a saddle-node bifurcation to occur is $\partial_x f = 0$, which is the condition for **any** bifurcation to occur. No additional tuning of parameters is required.

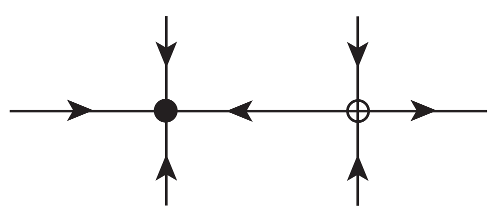
  – Therefore, if a bifurcation occurs, one should expect a saddle-node bifurcation, unless the system has some special properties.

  – (2) is the universal form of the equation describing the dynamics near a saddle-node bifurcation

- The flow changes direction **only locally**:
  only when μ goes through 0 and only near the bifurcation point $x = 0$ does the flow change direction.
  Away from the bifurcation point the flow is qualitatively unchanged when μ changes (arrows far away remain the same).

- The saddle-node bifurcation is sometimes also called "blue-sky bifurcation", because two solutions seem to appear from nowhere as the parameter μ is changed.

- in higher dimensions: saddle-node bifurcation



stable $\sim$ node unstable $\sim$ saddle

Example:

- In Taylor vortex flow in a short cylinder the transition to vortices arises through a saddle-node bifurcation and exhibits hysteresis.
  Here the saddle-node bifurcation is part of a larger bifurcation scenario

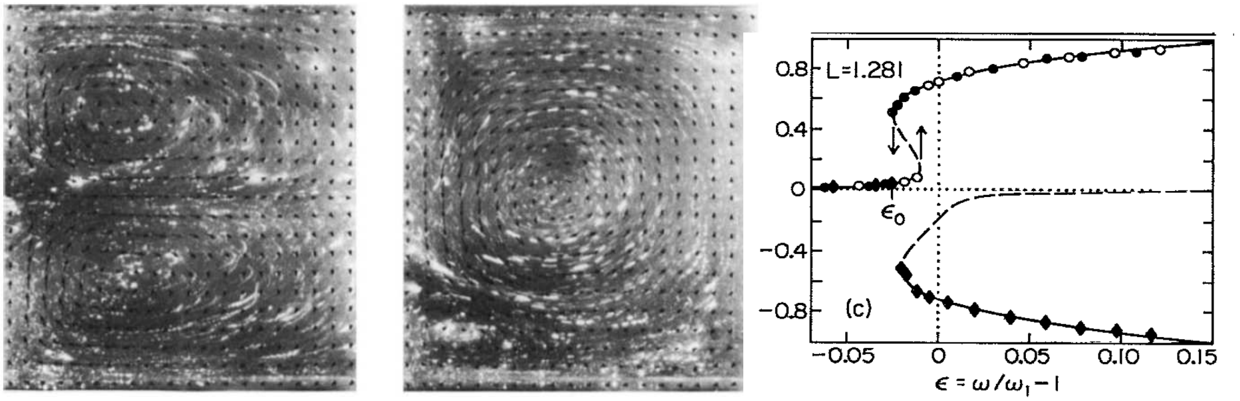


Figure 7: Two saddle-node bifurcations in Taylor vortex flow in a short cylinder. a) symmetric vortices below the bifurcation and jump to asymmetric vortices above the transition [**?**]. b) bifurcation diagram [**?**].

2.3.3 Transcritical Bifurcation

Consider a system that satisfies an additional condition beyond that of the occurrence of a bifurcation:
Assume one fixed-point solution exists for all μ. For simplicity assume that solution is $x = 0$:

$$f(0,\mu) = 0 \qquad \text{for all} \quad \mu$$

Taylor expansion around $x = 0$ at the bifurcation point $\mu = 0$

$$f(x,\mu) = \underbrace{f(0,0)}_{=0} + \underbrace{\partial_x f(0,0)}_{=0}\, x + \underbrace{\partial_\mu f(0,0)}_{=0}\,\mu + \frac{1}{2}\partial_x^2 f(0,0)\, x^2 + \partial_{x\mu}^2 f(0,0)\, x\mu + \underbrace{\frac{1}{2}\partial_\mu^2 f(0,0)}_{=0}\,\mu^2 + \ldots$$

Vanishing terms

- $x = 0$ is a fixed point for $\mu = 0$: $f(0,0) = 0$
- a bifurcation occurs: $\partial_x f(0,0) = 0$
- $x = 0$ is a solution for all values of μ: $\partial_\mu f(0,0) = 0$, $\partial_\mu^2 f(0,0) = 0$

Universal evolution equation

$$\dot{x} = x\,(a\,\mu + b\,x) + \cdots$$

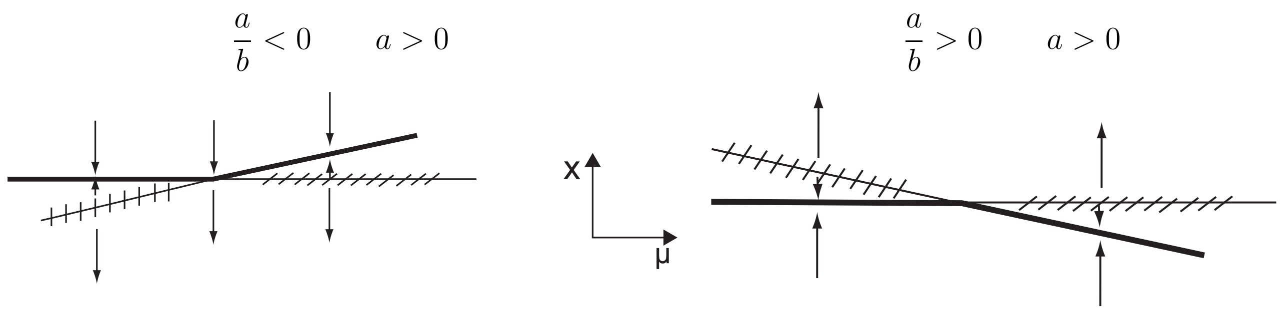
with

$$a = \partial_{x\mu}^2 f\big|_{x=0,\mu=0} \qquad b = \frac{1}{2}\,\partial_x^2 f\big|_{x=0,\mu=0}$$

Two fixed points:

$$x_1 = 0 \qquad x_2 = -\frac{a}{b}\mu \equiv -\frac{\partial_{x\mu}^2 f}{\frac{1}{2}\partial_x^2 f}\,\mu$$

There are four cases:



Two more cases for $a < 0$: the fixed point $x_1 = 0$ is then stable for $\mu > 0$ and unstable for μ <0 with the corresponding other branch depending on the sign of a/b.

Notes:

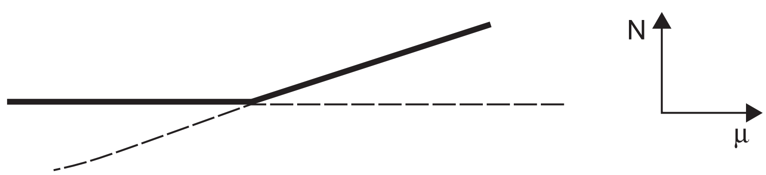
- Both fixed points exist below and above the bifurcation ($\mu < 0$ and $\mu > 0$)
- 'Exchange of stability' between the two branches of solutions
- Sufficiently large perturbation can lead away from the (linearly) stable fixed points.

Examples:

1. Logistic equation for population dynamics

$$\dot{N} = \mu N - N^2$$

for $\mu < 0$ the lower branch is unphysical since $N > 0$ is required



2. Rayleigh-Benard convection in a fluid layer heated from below:

   - The state without fluid flow (modeled with $x = 0$) exists for all temperature differences
   - Hexagonal flow patterns arise in a transcritical bifurcation connected with a saddle-node bifurcation
   - Large perturbations can kick the solution without fluid flow above the unstable branch of the transcritical bifurcation and trigger the formation of hexagonal convection patterns.

- For $\mu > 0$ the lower branch is unstable in a different way (instability not contained in the single equation)

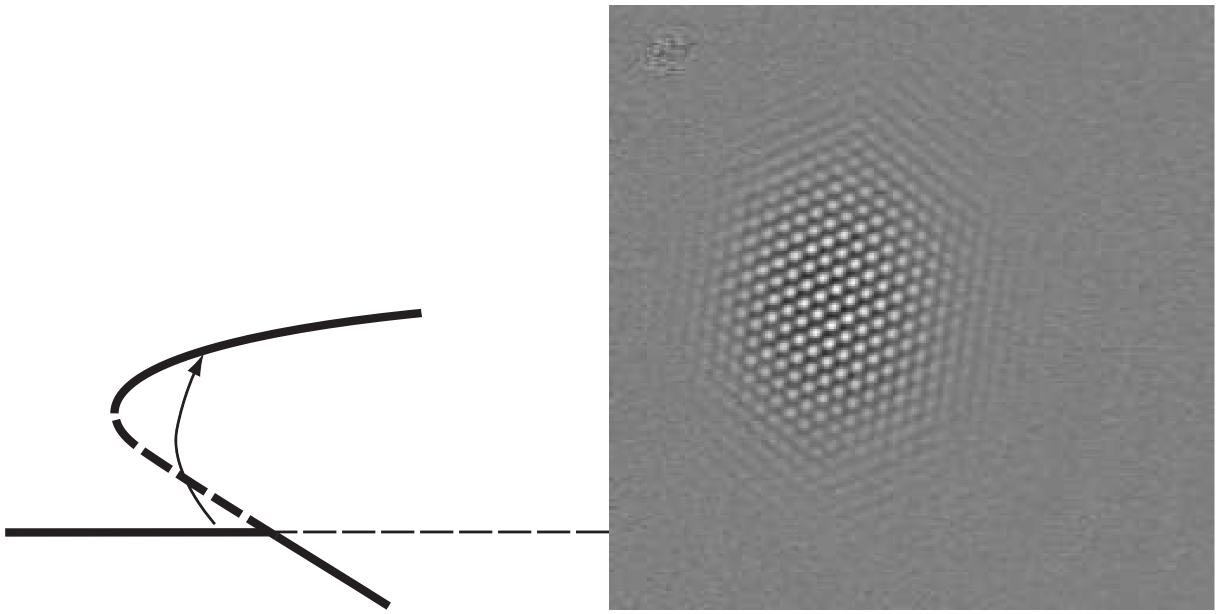
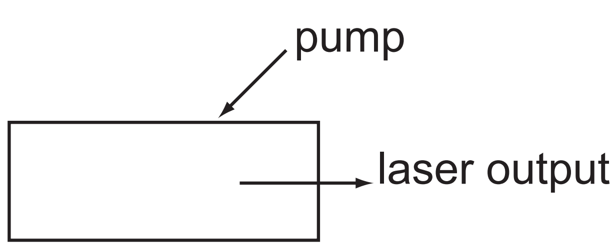


Figure 8: Convection in very thin fluid layers sets in via a transcritical bifurcation to hexagonal convection patterns. The hexagons and the convection-less state are simultaneously linearly stable in a (very small) range of parameters. If the heating is increased the pattern expands into the whole system [**?**].

3. Simple Model for Laser



Optical cavity with excitable atoms

Dynamics of atoms:

- atoms are excited by pump P [4]
- atoms emit photons and go to ground state
  - spontaneously: spontaneous emission
  - due to other photon: **stimulated emission**
    a photon triggers the emission of a photon from excited atom

[4] Atoms are also excited by photons already present; effect much smaller than pump (n is small near onset of lasing)

$$\dot{N} = P - fN - gnN$$

N: number of excited atoms, P: pump, f: decay through spontaneous emission, g: 'collision' with photon takes atom to ground state (stimulated emission), n: number of photons in the cavity

Dynamics of photons:

- photons generated by **stimulated emission**
- photons leave through end mirrors

$$\dot{n} = gNn - \kappa n$$

g: gain, κ: output/loss

Note: n counts only photons with correct phase (only those generated by stimulated emission)

Now: we have 2 equations: *too difficult* for now

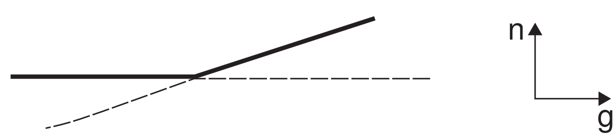
model the N-equation ad hoc: steady state # of excited atoms will be reduced by photons

$$N = N_0 - \alpha n$$

then

$$\dot{n} = g(N_0 - \alpha n)\, n - \kappa n = (gN_0 - \kappa)\, n - \alpha g n^2$$

again same equation as for logistic growth



Note:

- we will learn under what conditions the model for N is justified:
  reduction from many ode's to few/single ode by *center-manifold reduction*.

2.3.4 Pitchfork Bifurcation

Consider systems with reflection symmetry $x \to -x$, i.e. systems for which, if $x(t)$ is a solution, $-x(t)$ is also a solution.
Thus, assume $x(t)$ a solution,

$$\dot{x} = f(x(t), \mu),$$

and require that the reflected function $-x(t)$ is also a solution,

$$-\dot{x} = f(-x(t), \mu).$$

Thus we have

$$\dot{x} = -f(-x(t), \mu).$$

Since also $\dot{x} = f(x(t), \mu)$, we have that $f(x, \mu)$ must be *odd* in x

$$f(-x, \mu) = -f(x, \mu) \qquad \text{for all } \mu.$$

Therefore, in a Taylor expansion around $x = 0$ no even powers in x can appear

$$f(x,\mu) = \underbrace{\partial_x f(0,0)}_{=0}\, x + \underbrace{\partial^2_{x\mu} f(0,0)}_{a}\, x\mu + \frac{1}{2}\underbrace{\partial^2_x f(0,0)}_{=0}\, x^2 + \underbrace{\frac{1}{6}\partial^3_x f(0,0)}_{b}\, x^3 + \dots$$

This yields

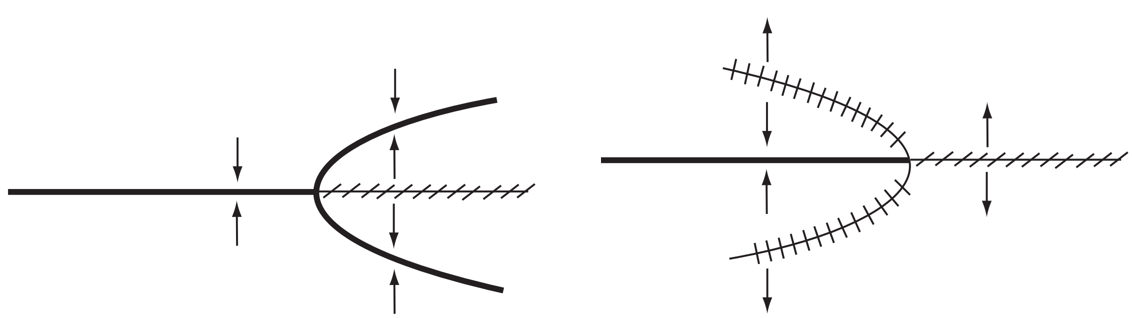
$$\dot{x} = a\mu x + bx^3$$

Depending on the value of μ there are one or three fixed points:

$$\begin{aligned} x_0 &= 0 \\ x_{2,3} &= \pm\sqrt{-\frac{a}{b}\mu} \end{aligned}$$

Consider the two cases:

$$\frac{a}{b} < 0 \qquad a > 0 \qquad\qquad \frac{a}{b} > 0 \qquad a > 0$$



supercritial subcritical

Notes:

- Supercritical: new branches arise on the side where the state $x = 0$ is linearly unstable $\Rightarrow$ nonlinear saturation of the instability
- Subcritical $\Rightarrow$ no saturation to cubic order $\Rightarrow$ need higher-order terms
- The system has the reflection symmetry $x \to -x$

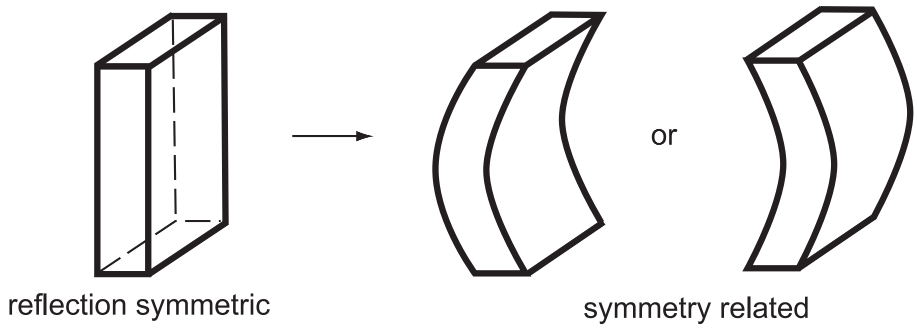
  The solution $x_0 = 0$ has that symmetry as well.

  The solutions $x_{2,3} = \pm\sqrt{-\frac{a}{b}\mu}$ break the reflection symmetry:
  The symmetry of the system is manifested in the fact that the two solutions are symmetrically related.

  $\Rightarrow$ The pitchform bifurcation is a *symmetry-breaking* bifurcation.

Examples:

a) buckling of a beam



b) Electro-convection in nematic liquid crystals:
AC-voltage can induce a pitch-fork bifurcation to stripe-like convection patterns in nematic liquid crystals.

The reflection symmetry arises effectively from a translation symmetry: shifting by half a wavelength corresponds to a flipping of the velocity vector[5]:

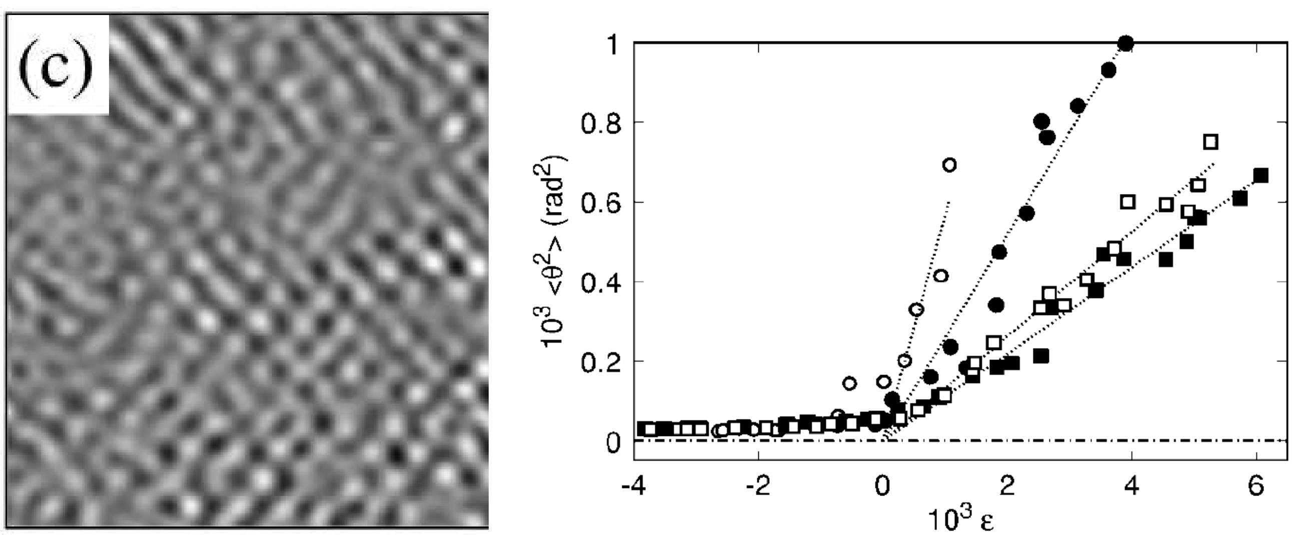


Figure 9: a) Disordered convection pattern very close to the bifurcation point in electroconvection of a nematic liquid crystal. b) The square of the pattern amplitude grows linearly at the bifurcation point reflecting the square-root law for the amplitude. As the electrical conductivity of the liquid crystal is changed (different symbols) a tricritical point is approached: the pitch-fork bifurcation eventually becomes subcritical. [**?**].

[5]intermediate positions are also possible $\Rightarrow$ system has larger symmetry

c) Ferromagnets

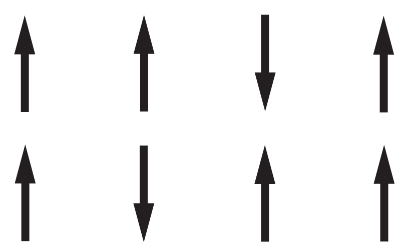
Bifurcations are closely related to phase transitions, they both describe qualitative changes in the behavior of the system, like from a liquid to a gas:

- phase transitions can often described in terms of a single scalar quantity that describes the state of the system:
  as such an *order parameter* serves the density of a gas/liquid or the magnetization of a ferromagnet
- the order parameter typically satisfies a nonlinear equation, which exhibits bifurcations as the temperature, say, is changed.

Consider ferromagnets:

- as temperature T increased beyond the critical temperature T_c: ferromagnetic $\Rightarrow$ paramagnetic ('non-magnetic')

Ising model: each atom carries a magnetic moment (spin): $s_i = \pm 1$



Overall magnetization arises if the spins align on average in one direction or the other: *spontaneous symmetry breaking*

Interactions:

- energy of spins in external magnetic field:

$$E_H = -Hs_i \qquad \text{want to be parallel to field}$$

- energy of spin - spin interaction:

$$E_S = -\sum_{i,j} J_{ij} s_i s_j \qquad J_{ij} > 0, \qquad \text{want to be parallel to each other}$$

$\sum_{i,j}$ is a sum over neighbors

Note:

- Macroscopic magnets want to align anti-parallel: north is attracted by south.
  The parallel alignment in ferromagnets is a quantum-mechanical effect.

- Total energy:

$$E(s_1 \ldots, s_N) = -\sum_i H s_i - \sum_{i,j} J_{ij}\, s_i s_j$$
$$= -\sum_i \underbrace{\left(H + \sum_j J_{ij} s_j\right)}_{H_i^{eff}(s_j)} s_i$$

each spin s_i feels a field that depends on its neighbors

$$H_i^{eff}(s_j) = H + \sum_j J_{ij} s_j$$

For finite temperature T the probability of spin i to have value s_i depends on its energy E_i

$$P(s_i) \propto e^{-E_i/kT} = e^{H_i^{eff}(s_j)\, s_i/kT} \qquad \text{Boltzmann factor}$$

k Boltzmann constant

The average value of s_i is then given by

$$\bar{s}_i = \sum_{s_i=\pm 1} s_i P(s_i) = P(1) - P(-1)$$

However:

$H_i^{eff}(s_j)$ still contains the unknown orientation of all the other interacting spins, which at any given moment will depend on $s_i \Rightarrow P(s_i)$ very difficult to calculate.

Note:

- In one spatial dimension the model was solved by Ising in his thesis in 1925. In two dimensions it was solved by Onsager in 1944. In three dimensions no analytical solution is known.

Mean field approximation: replace the local spin value s_i by the average $\bar{s}_i$, which is the same for all spins since the system is spatially homogeneous, $\bar{s}_i = \bar{s}$.

$$H_i^{eff}(s_j) \to \bar{H} = H + \sum_j J_{ij}\bar{s}$$
$$= H + \bar{s} \underbrace{\sum_j J_{ij}}_{\bar{J}}$$

Then

$$P(s_i) = \frac{1}{\mathcal{N}} e^{\bar{H}\, s_i/kT} \qquad \text{with} \qquad \bar{H} = H + \bar{s}\bar{J}$$

Normalization of probability:

$$1 = P(+1) + P(-1) \qquad \Rightarrow \qquad \mathcal{N} = e^{\bar{H}/kT} + e^{-\bar{H}/kT}$$

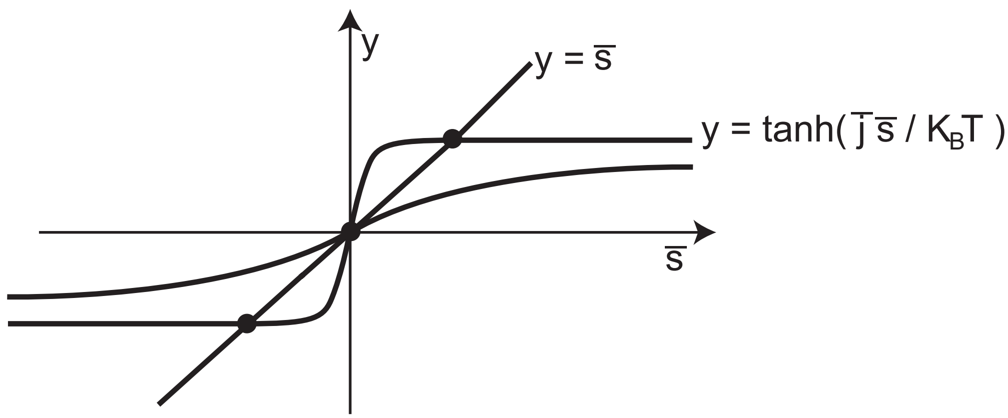
Thus the average magnetization satisfies:

$$\bar{s} = \frac{(+1)\, e^{\bar{H}/kT} + (-1)\, e^{-\bar{H}/kT}}{e^{\bar{H}/kT} + e^{-\bar{H}/kT}} = \tanh\left\{\frac{(H + \bar{s}\bar{J})}{kT}\right\}$$

Consider $H = 0$:

$$\bar{s} = \tanh(\frac{\bar{s}\bar{J}}{kT})$$

Graphically



Using

$$\tanh\theta = \theta - \frac{1}{3}\theta^3 + \mathcal{O}(\theta^5)$$

we get

$$\bar{s} = \frac{\bar{s}\bar{J}}{kT} - \frac{1}{3}\left(\frac{\bar{s}\bar{J}}{kT}\right)^3 + \mathcal{O}\left(\bar{s}^5\right)$$

$$0 = \left(\frac{\bar{J}}{kT} - 1\right)\bar{s} - \frac{1}{3}\left(\frac{\bar{J}}{kT}\right)^3 \bar{s}^3 + \mathcal{O}\left(\bar{s}^5\right)$$

i.e. to leading order

$$0 = \mu\bar{s} + b\bar{s}^3$$

with

$$\mu = \frac{\bar{J}}{kT} - 1 \qquad b = -\frac{1}{3}\left(\frac{\bar{J}}{kT}\right)^3$$

Since the bifurcation (phase transition) occurs at $\mu = 0$ the critical temperature T_c is given by

$$T_c = \frac{\bar{J}}{k}$$

Note:

- since $|\mu| \ll 1$we can set $T = T_c$ in b
$$b = -\frac{1}{3}$$

Notes:

- pitchfork bifurcation since symmetry $\bar{s} \to -\bar{s}$
buckling, thermal convection, ferromagnets very different physical systems: transition described by the same equation because they have the same symmetries
- supercritical pitchfork bifurcation $\Leftrightarrow$ phase transition of 2^{nd} order.
- $H \neq 0$ breaks reflection symmetry $\Rightarrow$ the pitchfork bifurcation is perturbed $\Rightarrow$ see Section 2.3.6.

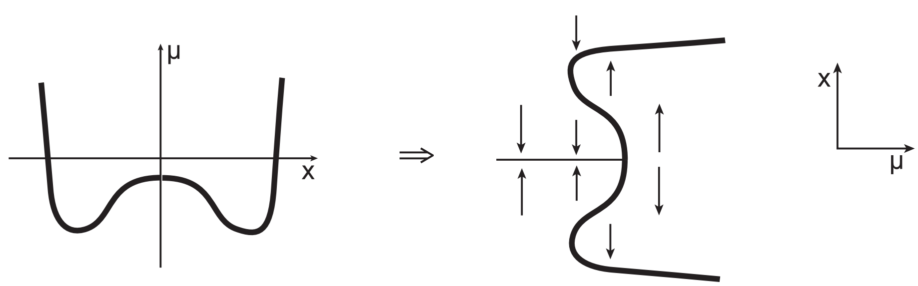
2.3.5 Subcritical Pitchfork Bifurcation:

If the cubic term does not lead to a saturation of the amplitude we need to include a quintic term:

$$\dot{x} = \mu x + \underbrace{bx^3}_{\text{destabilizing for } b>0} - \underbrace{cx^5}_{\text{stabilizing for } c>0}$$

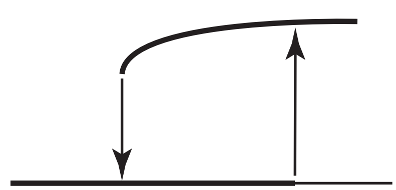
Assume $c > 0$. In general need not saturate at quintic order

To get the bifurcation diagram: plot $\mu = \mu(x)$



Note:

- 2 saddle-node bifurcations
- hysteresis loop & bistability



- Taylor expansion in x:
analysis strictly valid only if x is sufficiently small on the upper branch.
Since
$$x_{1,2}^2 = \frac{b \pm \sqrt{b^2 - 4\mu c}}{c}$$

validity requires that b is small, i.e. we need to expand around the 'tricritical' point $b = 0$
the bifurcation has to be *weakly subcritical*

2.3.6 Imperfect Bifurcations

The saddle-node bifurcation requires only the tuning of a single parameter.
For the transcritical and for the pitch-fork bifurcation to occur we needed 2 conditions

- bifurcation occurs: $\partial_x f|_{x_0,\mu_0} = 0$
- additional coefficients 'happen to vanish', e.g., because of some symmetry

Question:
What happens when the additional conditions are only satisfied approximately, e.g. the symmetries are **weakly broken**?

Example: In the ferromagnet the external field H breaks the up-down symmetry $\bar{s} \to -\bar{s}$.

We had:

$$\bar{s} = \tanh(\beta(H + \bar{J}\bar{s})\}$$

Using again

$$\tanh\theta = \theta - \frac{1}{3}\theta^3 + \mathcal{O}(\theta^5)$$

the equation for the magnetization $\bar{s}$ becomes

$$\bar{s} = \beta(H + \bar{J}\bar{s}) - \frac{1}{3}\left(\beta(H + \bar{J}\bar{s})\right)^3 + \mathcal{O}\left(\left(\beta(H + \bar{J}\bar{s})\right)^5\right).$$

For small $\bar{s}$ and H the quintic term can be neglected.

With $\mu \equiv \beta\bar{J} - 1$ one has near the bifurcation

$$|\mu| \ll 1$$

and one obtains to leading order in μ, $\bar{s}$, and H

$$0 = \mu\bar{s} - \frac{1}{3}\beta^3\bar{J}^3\bar{s}^3 + \beta H$$

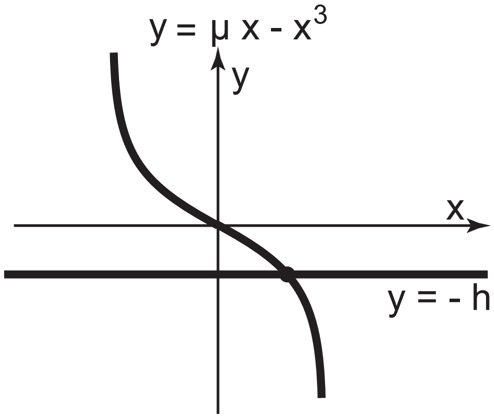
Thus, consider the perturbed pitchfork bifurcation

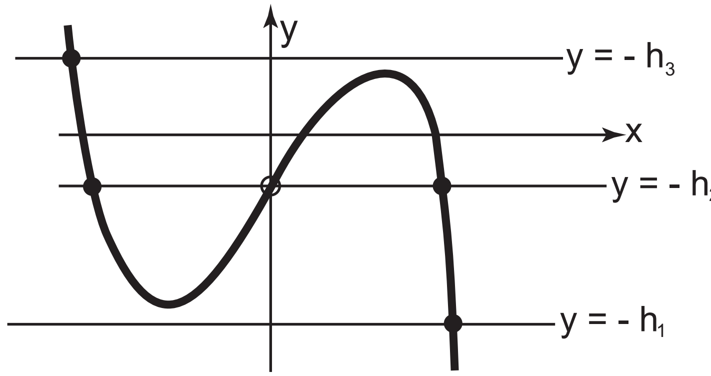
$$\dot{x} = \mu x - x^3 + h$$

Solving this cubic equation directly for fixed points is cumbersome (although possible).

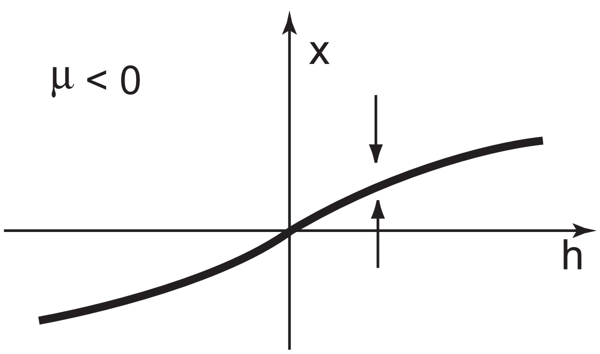
Graphical solution:

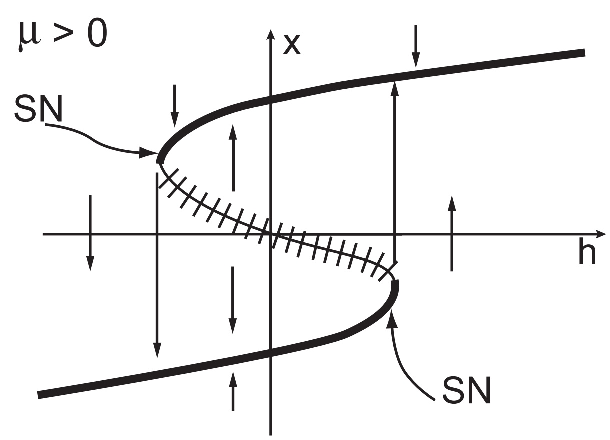
$\mu < 0$ $\mu > 0$





Vary h for fixed μ:





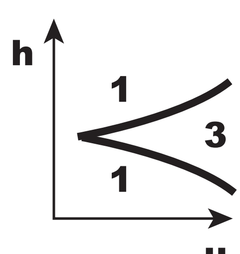
Note:

- varying h up and down beyond the saddle-node bifurcations induces a *hysteresis loop*: in the magnetic system the magnetization jumps and switches sign.

The saddle-node bifurcations occur at the extrema of $\mu x - x^3$, thus for given μ

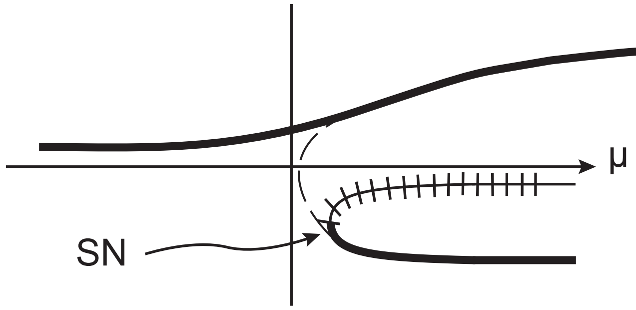
$$x_{SN} = \pm\sqrt{\frac{1}{3}\mu} \qquad h_{SN} = \mp\sqrt{\frac{1}{3}\mu}\,\frac{2}{3}\mu$$

$h_{SN}(\mu)$ for $\mu > 0$ gives lines of saddle-node bifurcations in the (μ, h)-parameter plane: for $|h| < h_{SN}(\mu)$ there are 3 solutions, otherwise only 1 solution.



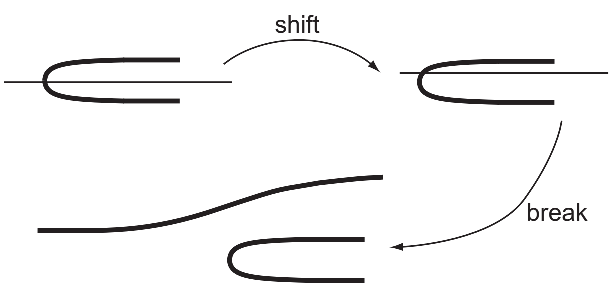
The parameter plane can be traversed in different directions.

Vary μ for $h > 0$ fixed:



Notes:

- A bifurcation is called *degenerate* if additional conditions "happen" to be satisfied, i.e. if additional coefficients in the Taylor vanish due to the tuning of some parameter
  - in a system that does not have a reflection symmetry a pitch-fork bifurcation would be degenerate.
- *Unfolding* of a degenerate bifurcation: introduce sufficiently many parameters so that no degeneracy is left.



Unfolding the pitchfork bifurcation:

- break the symmetry $x \to -x$, but keep the solution $x = 0$ for all $\mu \Rightarrow$ transcritical bifurcation
- break the transcritical bifurcation by dropping the condition that $x = 0$ is always a solution$\Rightarrow$ only saddle-node bifurcation remains

Note:

- the number of parameters that have to be tuned to get a certain bifurcation is called the *codimension* of that bifurcation: if the system has N parameters and the bifurcation has codimension d then the bifurcation occurs on a $N - d$-dimensional surface ('manifold') in the N-dimensional parameter space.
- to get an unperturbed pitch-fork bifurcation in a system without reflection symmetry we have to tune 2 parameters

$$\mu = 0 \qquad \& \qquad h = 0$$

  codimension-2 bifurcation

- in systems with reflection symmetry the pitch-fork bifurcation has codimension 1

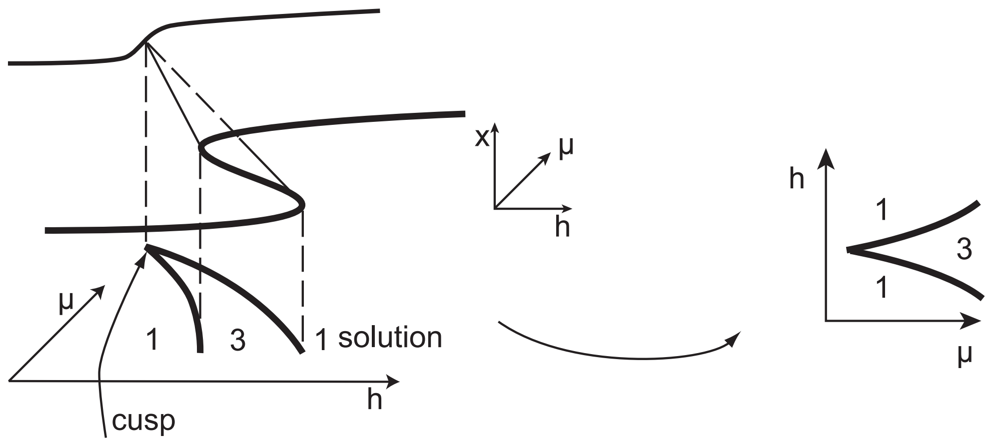
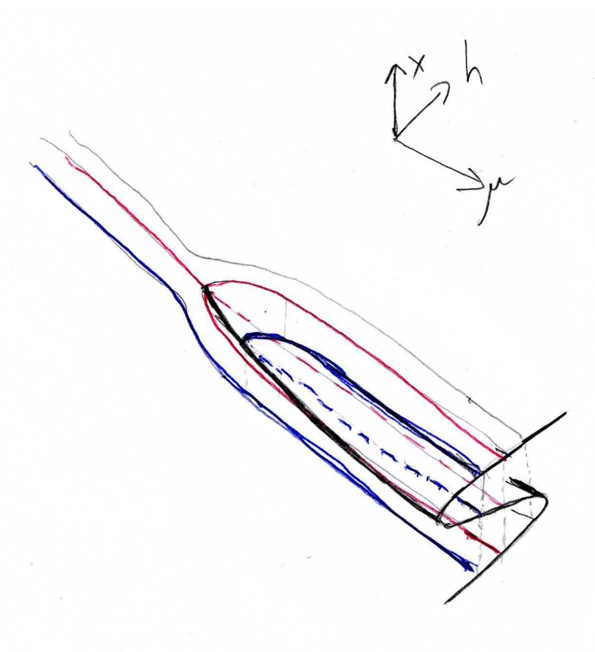


Figure 10: Solution surface. The pitch-fork bifurcation occurs at the codimension-2 point $(\mu = 0, h = 0)$. Saddle-node bifurcations occur along lines, which have codimension 1.

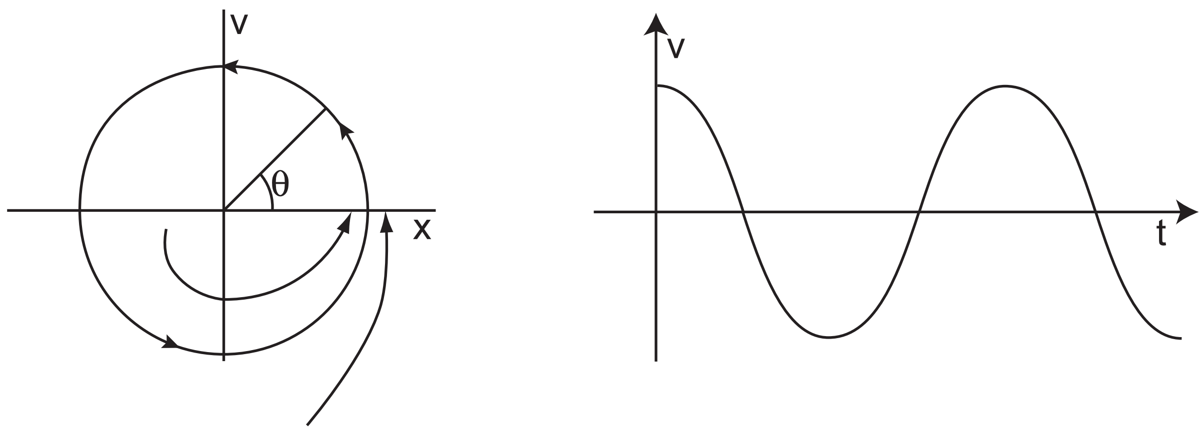


This is the surface of a cusp catastrophe:

- *catastrophes* occur as saddle-node bifurcations are crossed:
  solution **jumps** to other branch
  minute changes lead to **drastic** results.

2.4 Flow on a Circle[6]

For oscillations to be possible the system needs to allow a return: two dimensions needed



Consider the dynamics on the periodic orbit:

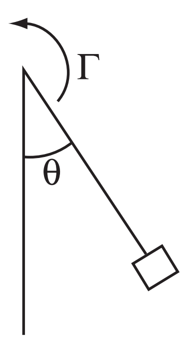
Flow on a circle

$$\dot{\theta} = f(\theta) \qquad \theta \in [0, 2\pi]$$

Notes:

- $f(\theta)$ cannot be arbitrary: has to be single-valued, i.e. 2π-periodic
- $f(\theta)$ gives the instantaneous frequency

Example: Overdamped Pendulum with Torque



$$m\ell^2\ddot{\theta} + \beta\dot{\theta} = -mg\ell\sin\theta + \tilde{\Gamma}$$

consider large damping

$$\dot{\theta} = \Gamma - a\sin\theta$$

with $a = mg\ell/\beta$ and $\Gamma = \tilde{\Gamma}/\beta$

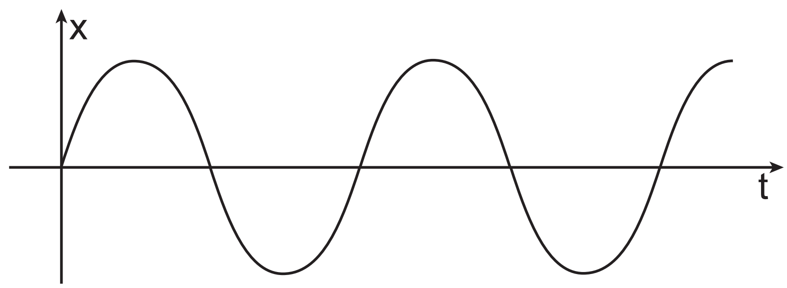
[6] cf. Strogatz Ch.4

i) $a = 0$ (no gravity)

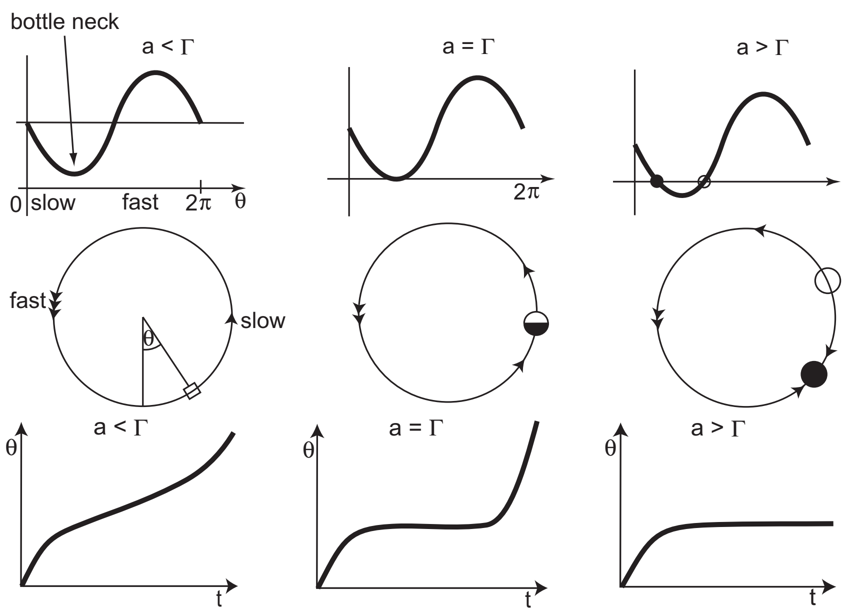
$$\theta = \theta_0 + \Gamma t \qquad \text{whirling motion}$$

oscillation in horizontal coordinate:

$$x = \ell \sin\theta = \ell \sin(\theta_0 + \Gamma t)$$



ii) $\alpha > 0$ (with gravity)



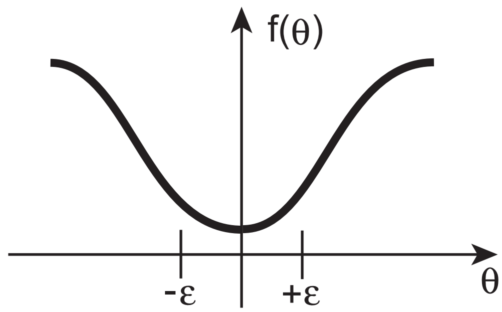
'Ghost' of the saddle-node bifurcation:
for a just below the saddle-node bifurcation, $a \lesssim \Gamma$, the evolution becomes extremely slow near the location on the orbit where the two saddle and the node are 'borne' at $a = \Gamma$.

Note:

- quite generally: near a steady bifurcation the dynamics become slow: growth/decay rates go to 0 ('critical slowing down').

Estimate the period near the bifurcation point:

$$T = \int dt = \int_0^{2\pi} \frac{d\theta}{\dot{\theta}} = \int_0^{2\pi} \frac{d\theta}{\Gamma - a\sin\theta}$$



Consider the general case near a saddle-node bifurcation

$$\dot{\theta} = f(\theta)$$

assume that minimum of $f(\theta)$ is at $\theta = 0$

$$f(0) = \mu, \qquad f'(0) = 0$$

$$\Rightarrow f(\theta) = \mu + \underbrace{\frac{1}{2}f''(0)}_{a}\,\theta^2 + \mathcal{O}(\theta^3)$$

$$T = \int_0^{2\pi} \frac{d\theta}{f(\theta)} \underbrace{=}_{f(\theta)\text{ periodic}} \int_{-\epsilon}^{2\pi-\epsilon} \frac{d\theta}{f(\theta)} + \int_{-\epsilon}^{+\epsilon} \underbrace{\frac{d\theta}{\mu + a\theta^2 + \mathcal{O}(\theta^3)}}_{\text{diverges as }\mu\to 0} + \underbrace{\int_{\epsilon}^{2\pi-\epsilon} \frac{d\theta}{f(\theta)}}_{\text{finite as }\mu\to 0}$$

$$= \int_{-\epsilon}^{+\epsilon} \frac{d\theta}{\mu + a\theta^2} + T_0$$

extract the divergence for $\mu \to 0$ (at fixed ϵ) using $\psi = \frac{\theta}{\sqrt{\mu}}$

$$T = \frac{1}{\mu} \int_{-\frac{\epsilon}{\mu^{1/2}}}^{\frac{\epsilon}{\mu^{1/2}}} \frac{\mu^{1/2}\,d\psi}{1 + a\psi^2} + T_0 \;\to\; \frac{1}{\mu^{1/2}} \int_{-\infty}^{\infty} \frac{d\psi}{1 + a\psi^2} + T_0 \;\propto\; \mu^{-1/2}$$

Notes:

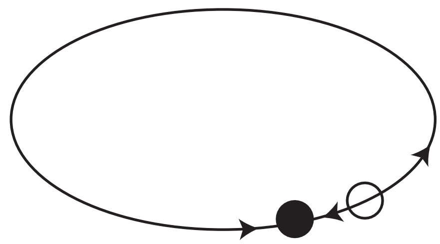
- A saddle-node bifurcation on an invariant circle is **one** way to generate oscillations. This bifurcation is often called a SNIC bifurcation. *Generically* one has for this bifurcation

$$T \propto \mu^{-1/2}$$

- other types of bifurcations to oscillatory behavior lead to different $T(\mu)$, e.g. at a Hopf bifurcation one has

$$T(\mu = 0) = T_0 \text{ finite}.$$

- the fact that the saddle-node bifurcation leads to oscillations is a **global** feature of the system:
  one needs a **global connection** between from saddle to the node (in addition to the direct connection)



Examples for SNIC Bifurcations:

i) Synchronization of Oscillators and Fireflies

Videos:

- synchronized fireflies `https://www.youtube.com/watch?v=a-Vy7NZTGos`
- metronomes falling into lock-step `https://www.youtube.com/watch?v=kqFc4wriBvE`

Such synchronization is not just curious but technologically and scientifically relevant:

- couple lasers to achieve high power
- coherent brain activity (EEG)
  - synchronized firing by multiple neurons has more impact on neurons reading this output.
  - synchronization has been associated with attention to stimuli
  - too much synchrony bad: epileptic seizure
- synchronous activity of heart muscle cells is essential for functioning of the heart: asynchrony (fibrillation) can be deadly.

Minimal model of synchronization of oscillators, e.g. fireflies

- light up periodically
- respond to neighboring fireflies

Consider single firefly with periodic light source representing another firefly

Light source:

$$\dot{\psi} = \Omega$$

Firefly:

$$\dot{\phi} = \omega + a\sin(\psi - \phi)$$

The coupling must be 2π-periodic.

For $a > 0$ firefly speeds up if it lags behind $\psi > \theta \Rightarrow \dot{\phi}$ is increased.

rewrite: $\theta = \phi - \psi$

$$\dot{\theta} = \underbrace{\omega - \Omega}_{\Gamma} - a\sin(\theta)$$

Γ: frequency mismatch = detuning

- *Fixed point:* firefly flashes are entrained by the light source if their detuning is not too large

  $$\underbrace{|\omega - \Omega|}_{\text{range of entrainment}} < a \qquad \text{and} \qquad \theta_0 = \arcsin\frac{\omega - \Omega}{a} \neq 0$$

  Firefly flashes lag behind/pull ahead, but phase difference fixed: **phase-locked** state

- *"Whirling" motion:* $|\omega - \Omega| > a$

  flashes are not synchronized with the light source.

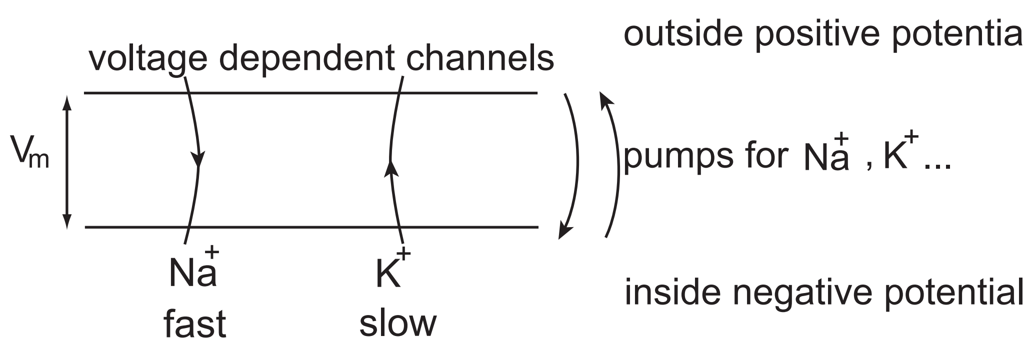
Notes:

- Entrainment is a common feature of coupled oscillators
- In general the coupling is bidirectional.

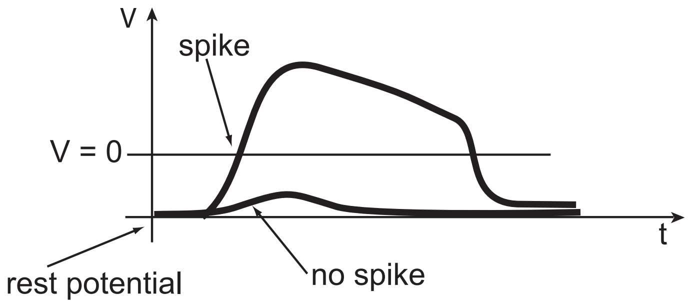
ii) Excitability in Neurons

Main features of the neurons that are to be modeled

- The voltage across the membrane of the nerve cell plays a central role: the communication between neurons is mostly achieved via voltage pulses ('action potentials')



- Neurons are excitable:
  - Brief small stimulations evoke only small responses.
  - Brief super-threshold stimulations evoke large voltage excursions: action potential



- Steady stimulation can lead to periodic firing of action potentials

Biophysical model:

Voltage-dependent membrane currents 'charge' the 'capacitance'. In addition, there are in general currents representing synaptic inputs from other neurons.

The voltage-dependent membrane currents follow Ohm's law with a bias due to different ion concentrations inside and outside of the cell

$$
\begin{aligned}
C\frac{dV}{dt} &= -\underbrace{g_{Na}m^3h\,(V-V_{Na})}_{\text{sodium current}} - \underbrace{g_K n^4\,(V-V_K)}_{\text{potassium current}} - \underbrace{g_L\,(V-V_L)}_{\text{leak current}} + I_{input} \\
\tau_n(V)\,\frac{dn}{dt} &= n_\infty(V) - n \\
\tau_h(V)\,\frac{dh}{dt} &= h_\infty(V) - h \\
\tau_m(V)\,\frac{dm}{dt} &= m_\infty(V) - m
\end{aligned}
$$

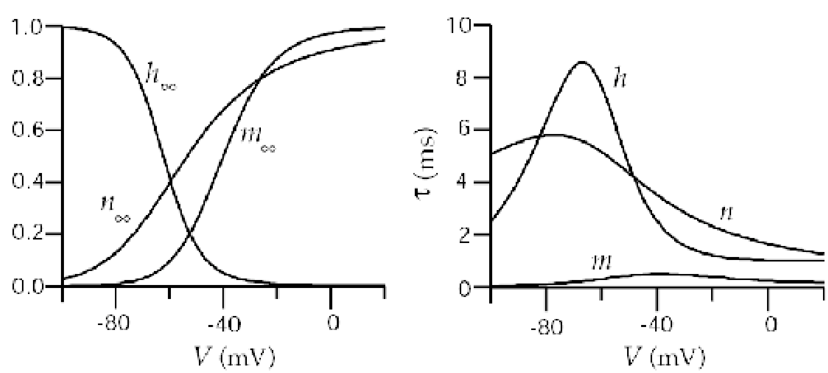


Figure 11: Steady-state values of the gating variables m, n, h (left) and the corresponding time constants (right) for the potassium and the sodium channels, respectively.

Generation of the action potential

- Sufficiently large depolarization (V less negative) $\Rightarrow$ Na^+ channels open fast (m increases) $\Rightarrow$ the cell becomes rapidly yet more depolarized
- Depolarization of the neuron $\Rightarrow$ the slower K^+ channels open (n increases), V becomes negative: cell becomes hyperpolarized again

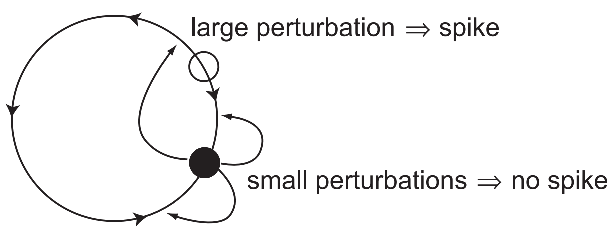
Periodic spiking can arise through a SNIC bifurcation. In that case the normal form for the description of the system dynamics for parameter values near the bifurcation is the θ-model [**?**] (see also article in Scholarpedia `http://www.scholarpedia.org/article/Ermentrout-Kopell_canonical_model`)

$$\begin{aligned}\dot{\theta} &= 1 - \cos\theta + (1 + \cos\theta)\,(-r + I_{inj}) \\ &= 1 - r + I_{inj} - (1 + r - I_{inj})\cos\theta\end{aligned}$$

where θ characterizes the phase on the cycle.

SNIC bifurcation occurs at $f(\theta) = 0 = f'(\theta)$, i.e.

$$\theta = 0 \qquad I_{inj} = r$$



Notes:

- Near the SNIC the conductance model shows the same power-law scaling of the period with the distance to the bifurcation point as the θ-model and the generic SNIC bifurcation.

3 Two-dimensional Systems

New aspects:

- 'true' oscillations without periodic "boundary" conditions
- reduction of dynamics to lower dimension

3.1 Classification of Linear Systems[7]

We would like to obtain a complete overview of the dynamics of a two-dimensional system. For linear systems this is possible and we will see that this will provide essential information about the local neighborhood of fixed points.

Therefore consider first a general linear system

$$\dot{\underline{x}} = \underline{\underline{L}}\,\underline{x} \qquad \text{with} \qquad \underline{x}(0) = \underline{x}_0$$

One can give the formal solution in terms of a matrix exponential

$$\underline{x}(t) = e^{\underline{\underline{L}}\,t}\underline{x}_0$$

which is defined via the expansion

$$e^{\underline{\underline{L}}\,t} = 1 + \underline{\underline{L}}\,t + \frac{1}{2}\underline{\underline{L}}^{\,2}t^2 + \ldots$$

We can simplify $\underline{\underline{L}}$ by similarity transformation:

If the eigenvalues of $\underline{\underline{S}}$ are different from each other $\underline{\underline{L}}$ can be diagonalized

$$\underline{\underline{S}}^{\,-1}\underline{\underline{L}}\,\underline{\underline{S}} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

The eigenvectors of $\underline{\underline{L}}$ are given by the columns of $\underline{\underline{S}}$

$$\underline{\underline{L}}\,\underline{v}_{1,2} = \lambda_{1,2}\,\underline{v}_{1,2}$$

$$\begin{aligned} \underline{\underline{S}}^{\,-1}\underline{\underline{L}}\,\underline{\underline{S}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} &= \lambda_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \Rightarrow \underline{\underline{L}}\,\underbrace{\underline{\underline{S}}\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\underline{v}_1} &= \lambda_1 \underbrace{\underline{\underline{S}}\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\underline{v}_1} \end{aligned}$$

Note:

- In general the eigenvectors need not be orthogonal to each other

[7] cf. Strogatz Ch.6

The dynamics in the eigendirections are simple

$$\begin{aligned} e^{\underline{\underline{L}}\,t}\,\underline{v}_i &= \{1+\underline{\underline{L}}\,t+\frac{1}{2}(\underline{\underline{L}}\,t)^2+\cdots\}\,\underline{v}_i = \\ &= \{1+\lambda_i t+\frac{1}{2}\lambda_i^2 t^2+\cdots\}\,\underline{v}_i = \\ &= e^{\lambda_i t}\,\underline{v}_i \end{aligned}$$

i.e., along the eigendirections we have simple exponential time dependence.

The general solution can be written in terms of the eigenvectors

$$\underline{x}(t) = e^{\lambda_1 t}\underline{v}_1 A_1 + e^{\lambda_2 t}\underline{v}_2 A_2$$

with the amplitudes A_i determined by the initial conditions

$$\underline{x}_0 = A_1\underline{v}_1 + A_2\underline{v}_2$$

Notes:

- The eigenvalues can be complex

$$\begin{aligned} \lambda_{1,2} &= \sigma_{1,2}+i\omega_{1,2} \\ \underline{x}(t) &= A_1 e^{\sigma_1 t}e^{i\omega_1 t}\underline{v}_1 + A_2 e^{\sigma_2 t}e^{i\omega_2 t}\underline{v}_2 \end{aligned}$$

  If $\underline{\underline{L}}$ is a real matrix the eigenvalues and eigenvectors are complex conjugates of each other, $\sigma_1=\sigma_2$, $\omega_1=-\omega_2$, and the solution is real

$$\underline{x}(t) = e^{\sigma t}(A_1 e^{i\omega t}\underline{v}_1 + A_1^* e^{-i\omega t}\underline{v}_1^*)$$

- If $\underline{\underline{L}}$ has repeated eigenvalues it cannot always be diagonalized. But it always can be reduced to the Jordan normal form

$$\underline{\underline{S}}^{-1}\underline{\underline{L}}\,\underline{\underline{S}} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

We are interested in the trajectories (orbits) $y(x)$ in the phase plane, which are parametrized by the time t. Consider for simplicity a diagonal $\underline{\underline{L}}$,

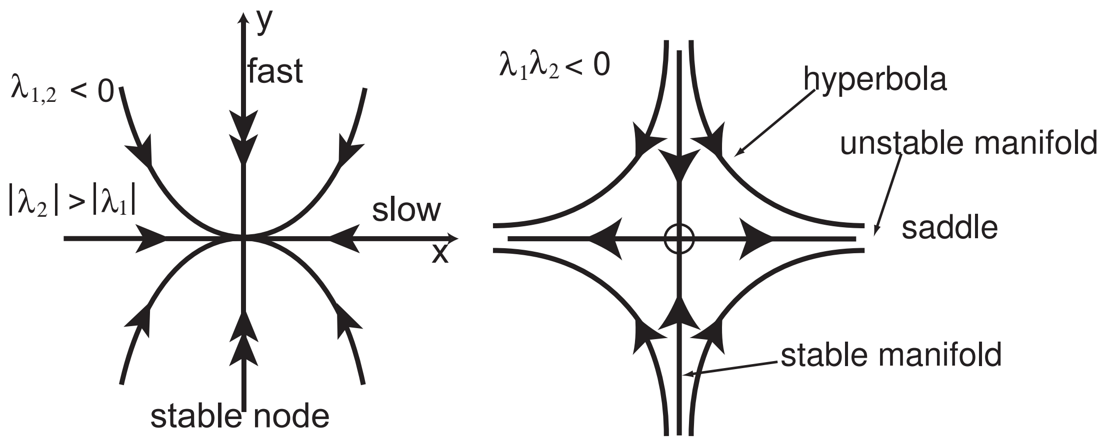
$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{array}{c} x = e^{\lambda_1 t}x_0 \\ y = e^{\lambda_2 t}y_0 \end{array}$$

$$\Rightarrow \qquad e^t = (\frac{x}{x_0})^{1/\lambda_1}$$

$$y(t) = \left(\left(\frac{x}{x_0}\right)^{1/\lambda_1}\right)^{\lambda_2} y_0 = y_0\left(\frac{x}{x_0}\right)^{\frac{\lambda_2}{\lambda_1}}$$

Thus

$$y(t) = C\,x(t)^{\frac{\lambda_2}{\lambda_1}}$$



Definitions:

- Stable manifold of a fixed point x_0:

$$\{\underline{x} \,|\, \underline{x}(0) = \underline{x} \Rightarrow \underline{x}(t) \rightarrow \underline{x}_0 \text{ for } t \rightarrow +\infty\}$$

- Unstable manifold of a fixed point x_0:

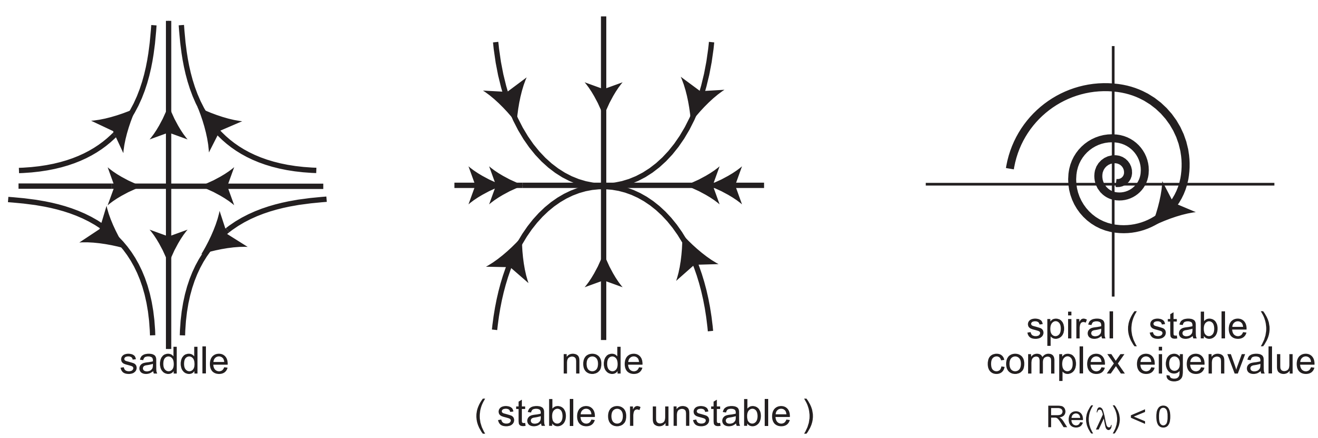
$$\{\underline{x} \,|\, \underline{x}(0) = \underline{x} \Rightarrow \underline{x}(t) \rightarrow \underline{x}_0 \text{ for } t \rightarrow -\infty\}$$

Note:

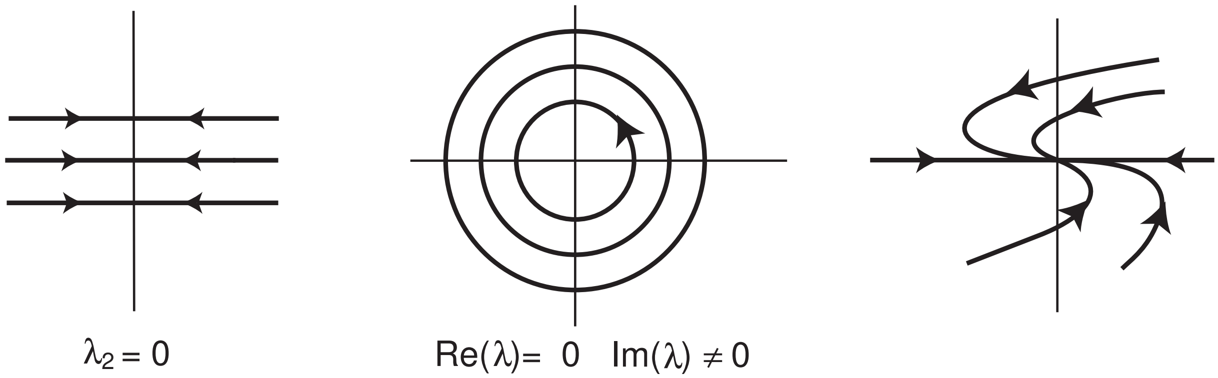
- The stable and unstable manifolds of a fixed point are quite informative for the flow in the vicinity of the fixed point.

Possible Phase Portraits:

i) Generic cases, i.e. the phase portraits do not change qualitatively when a parameter is changed slightly:



ii) Special cases, i.e. a parameter has to be tuned to a special value to obtain these diagrams. Small changes in a parameter can change the diagrams qualitatively:



center degenerate node

At a degenerate node the linearization has a double eigenvalue with only a single eigenvector

$$\underline{\underline{\mathbf{L}}} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \qquad \lambda < 0$$

the system is *almost* oscillating.

In two dimensions the eigenvalues can be written simply in terms of $\det \underline{\underline{L}}$ and $tr\underline{\underline{L}}$

$$\det L = \det(\underline{\underline{S}}^{-1}\underline{\underline{L}}\,\underline{\underline{S}}) = \lambda_1\lambda_2 \qquad tr\underline{\underline{S}}^{-1}\underline{\underline{L}}\,\underline{\underline{S}} = \lambda_1 + \lambda_2$$

$$\lambda_{1,2} = \frac{+tr\underline{\underline{L}} \pm \sqrt{(tr\underline{\underline{L}})^2 - 4\det\underline{\underline{L}}}}{2}$$

Change in stability: Re $(\lambda_i) = 0$

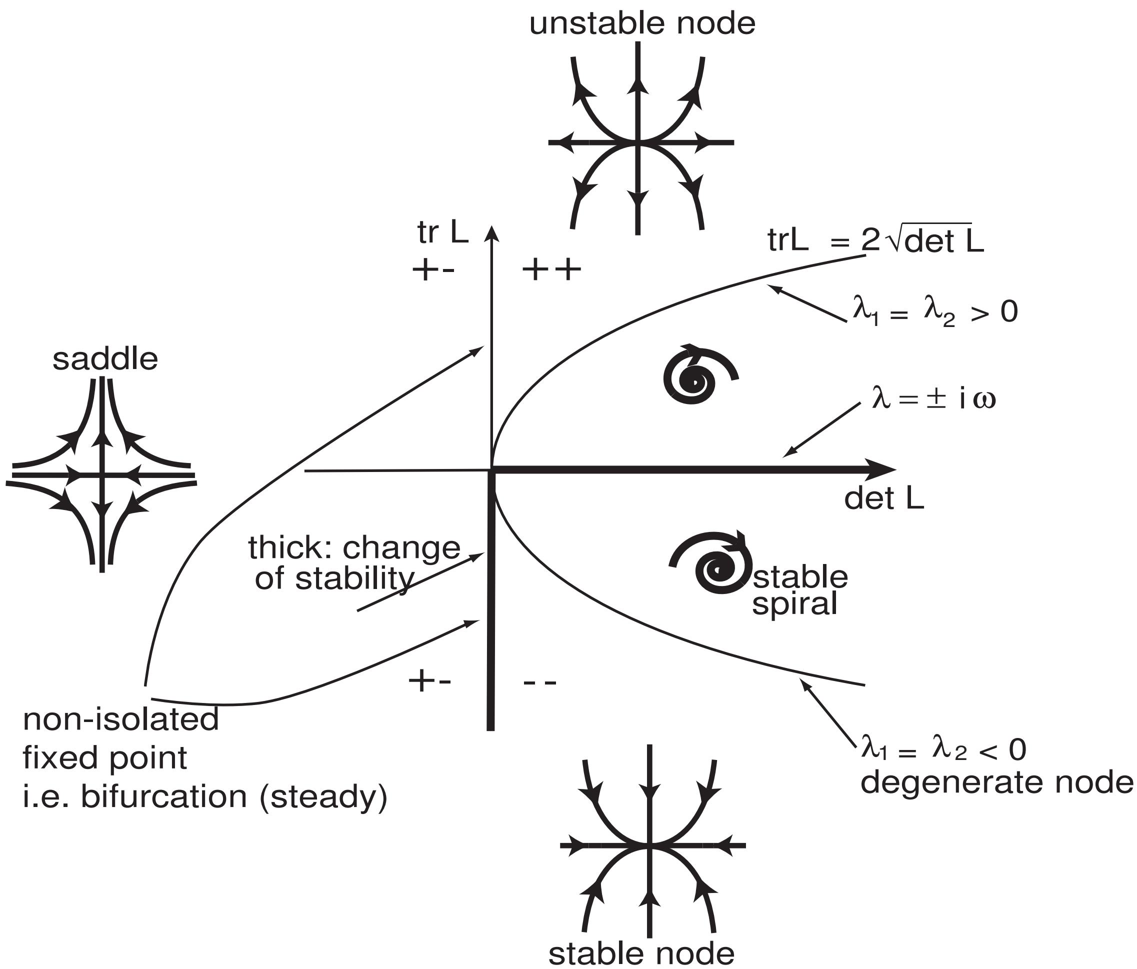
i) $tr\underline{\underline{L}} = 0$ and $\det\underline{\underline{L}} > 0$ $\Rightarrow$ $\lambda = \pm i\omega$ complex pair crossing imaginary axis

ii) $tr\underline{\underline{L}} < 0$ and $\det\underline{\underline{L}} = 0$ $\Rightarrow$ λ_1=0 , $\lambda_2 < 0$ single zero eigenvalue

Change in character:

Transition between real $\leftrightarrow$ complex

$$\left(tr\underline{\underline{L}}\right)^2 = 4\det\underline{\underline{L}}$$



Notes:

- Degenerate node ⇒ border between nodes and spirals, does not quite oscillate
- Non-isolated fixed points: steady bifurcation, one or more fixed points are created/annihilated (details depend on the nonlinearities).

3.2 Stability

So far we have considered only linear stability. There are also other notions of stability.

Linear stability in a nonlinear system

- all *infinitesimal* perturbations decay eventually
- the dynamics of infinitesimal perturbations can be determined by a *linearization* of the equations around the points in question

Example:
Damped-driven pendulum

$$m\ell^2\,\ddot{\theta} + \beta\,\dot{\theta} = -mg\ell\,\sin\theta + \Gamma$$

rewrite as first-order system using $x = \theta$ and $y = \dot{\theta}$:

$$\begin{aligned}\dot{x} &= y \equiv F_x(x,y)\\ \dot{y} &= -\frac{\beta}{m\ell^2}y - \frac{mg\ell}{m\ell^2}\sin x + \Gamma \equiv F_y(x,y)\end{aligned}$$

Fixed points:

$$y_0 = 0 \qquad \& \qquad mg\ell \sin x_0 = \Gamma$$

Expand around the fixed points

$$\begin{aligned}x &= x_0 + \epsilon x_1(t) \qquad \epsilon \ll 1\\ y &= y_0 + \epsilon y_1(t)\end{aligned}$$

Insert the expansion

$$\begin{aligned}\epsilon \dot{x}_1 &= F_x\left(x_0 + \epsilon x_1(t), y_0 + \epsilon y_1(t)\right) =\\ &= \underbrace{F_x(x_0, y_0)}_{0} + \epsilon x_1\, \partial_x F_x|_{(x_0,y_0)} + \epsilon y_1 \partial_y F_x|_{(x_0,y_0)} + \mathcal{O}(\epsilon^2)\end{aligned}$$

Analogously for $\dot{y}_1$.

In matrix form:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{y}_1 \end{pmatrix} = \underbrace{\begin{pmatrix} \partial_x F_x & \partial_y F_x \\ \partial_x F_y & \partial_y F_y \end{pmatrix}}_{\text{Jacobian } \underline{\underline{L}}} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

$\Rightarrow$ the linear stability is determined by the eigenvalues of the Jacobian

For the pendulum we have

$$\underline{\underline{L}} = \begin{pmatrix} 0 & 1 \\ -\frac{g}{\ell}\cos x_0 & -\frac{\beta}{m\ell^2} \end{pmatrix}$$

Thus, the eigenvalues are given by

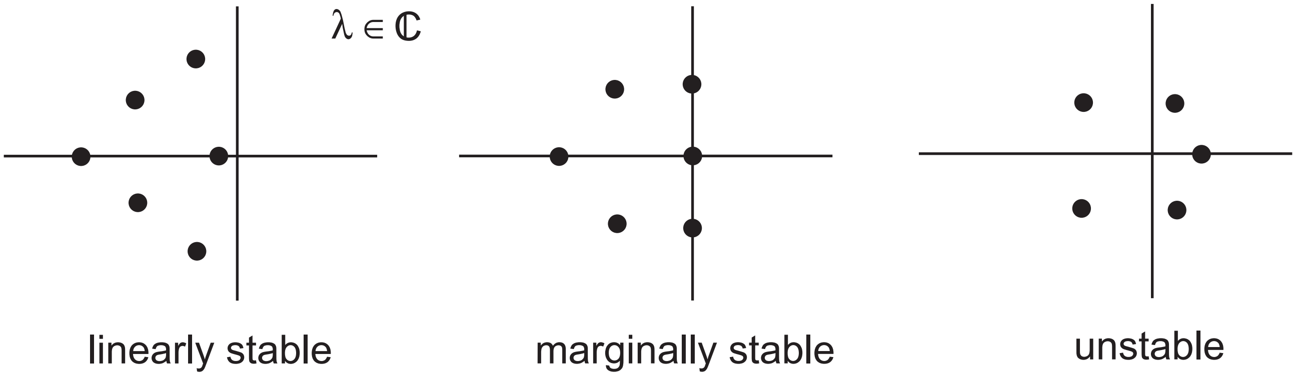
$$\mathbf{det}\left(\underline{\underline{L}} - \lambda \underline{\underline{I}}\right) = 0$$

yielding in this case

$$(-\lambda)(-\lambda - \frac{\beta}{m\ell^2}) + \frac{g}{\ell}\cos x_0 = 0$$

$$\begin{aligned}\lambda^2 &+ \lambda\frac{\beta}{m\ell^2} + \frac{g}{\ell}\cos x_0 = 0\\ \lambda_{1,2} &= -\frac{\beta}{2m\ell^2} \pm \frac{1}{2}\sqrt{\left(\frac{\beta}{m\ell^2}\right)^2 - 4\frac{g}{\ell}\cos x_0}\end{aligned}$$

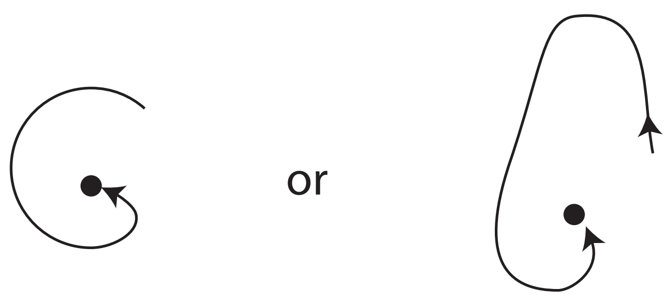
In general, for n first-order equations the Jacobian of the linearization is an $n \times n$ matrix and has n eigenvalues in the complex plane:



Other stability notions:

Attractor:
A set of points (e.g. a fixed point) is attracting if all trajectories that start close to it converge to it, i.e.

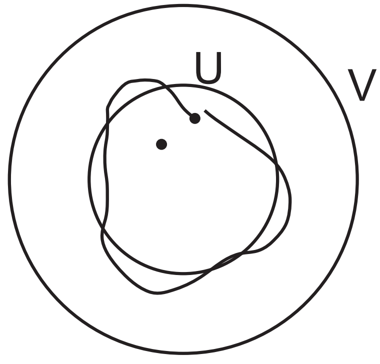
$$\text{for all } \mathbf{x}(0) \text{ near } \mathbf{x}_{FP}: \qquad \mathbf{x}(t) \to \mathbf{x}_{FP} \quad \text{for} \quad t \to \infty$$



Note:

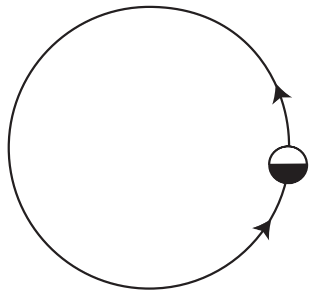
- The trajectory need not approach the attractor right away.
- The set of points that eventually reach the attractor form the *basin of attraction* of the attractor.

Lyapunov Stability:
A set is (Lyapunov) stable if all orbits that start close to it remain close to it for all times. Technically, for any neighborhood V of $\mathbf{x}_{FP}$ one can find a $U \subseteq V$ such that if $\mathbf{x}(0) \in U$ then $\mathbf{x}(t) \in V$ for all times.



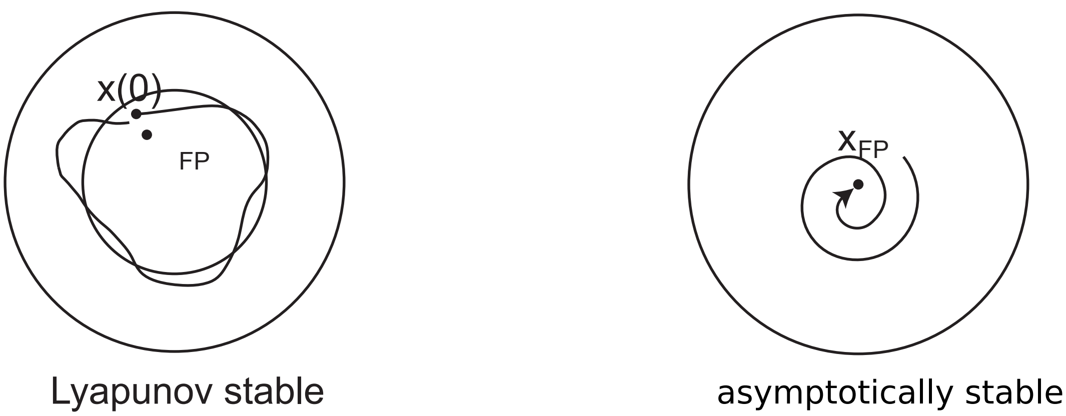
Notes:

- Lyapunov stability of a set does not imply that the set is an attractor:
- An attractor does not have to be Lyapunov stable



This fixed point is not Lyapunov stable (one cannot find neighborhood to which excursion are confined), but due to the global connection it is attracting.

Asymptotic Stability:
A set is asymptotically stable if it is attracting and Lyapunov stable, i.e. if all orbits that start sufficiently close to a fixed point converge to it as $t \to \infty$ without leaving its neighborhood.



Notes:

- A fixed point is asymptotically stable $\Rightarrow$ the fixed point is attracting, it is an attractor.
- Linear stability $\Rightarrow$ asymptotic stability $\Rightarrow$ Lyapunov stability
- Linear instability $\Rightarrow$ instability
- **But:** asymptotic or Lyapunov stability **do not imply** linear stability

Examples: see homework

3.3 General Properties of the Phase Plane

3.3.1 Hartman-Grobman theorem

Linear systems: can be completely understood

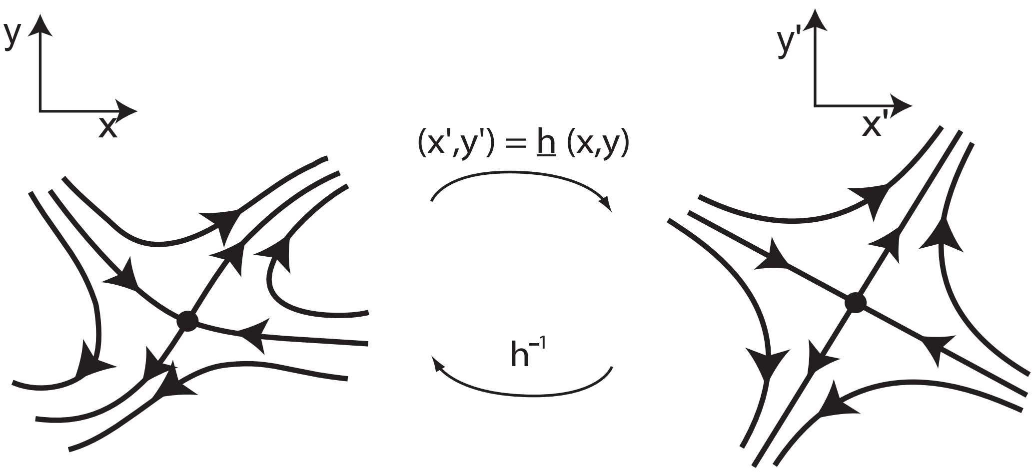
How much of that can be transferred to nonlinear systems?

Definition: A fixed point $\underline{x}_0$ of $\dot{\underline{x}} = \underline{f}(\underline{x})$ is called hyperbolic if all eigenvalues of $\frac{\partial f_i}{\partial x_j}$ have non-zero real parts.

Thus: in all directions a hyperbolic fixed point is either linearly attractive or repulsive. No marginal direction.

Hartman-Grobman Theorem:

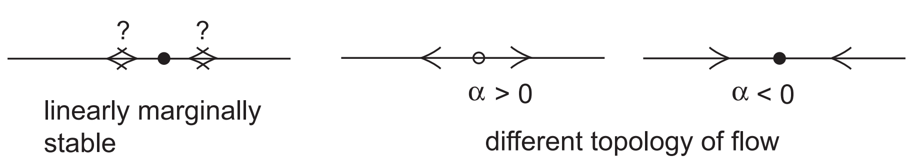
If $\underline{x}_0$ is a hyperbolic fixed point of $\dot{\underline{x}} = \underline{f}(\underline{x})$ then there exists a continuous invertible function $\underline{h}(\underline{x})$ that is defined on some neighborhood of $\underline{x}_0$ and maps all orbits of the nonlinear flow into those of the linear flow. The map can be chosen so that the parameterization of orbits by time is preserved.



Thus:

- For a hyperbolic fixed point $\underline{x}_0$ the linearization of the flow gives the **topology** of the nonlinear flow in a neighborhood of $\underline{x}_0$.
- If the fixed point is not hyperbolic, the linearization does not give sufficient information:

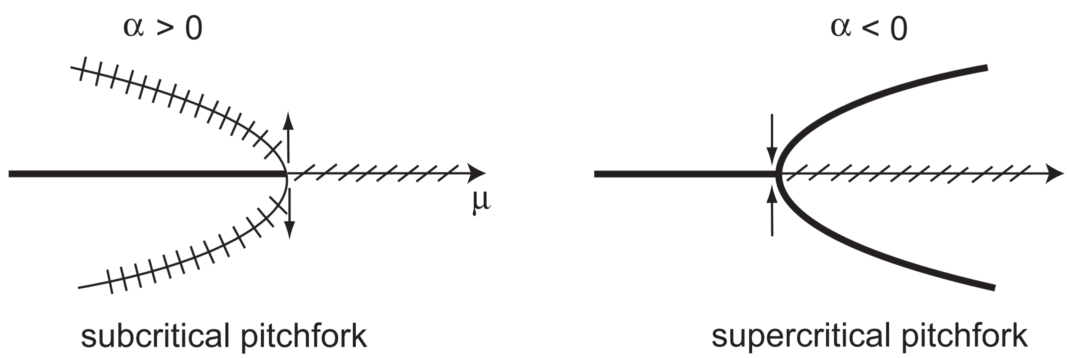
$$\dot{x} = \alpha x^3$$



- At any bifurcation the fixed point is not hyperbolic.

$$\dot{x} = \mu x + \alpha x^3$$

at $\mu = 0$ the linear systems are equal for all α.



- Away from the bifurcation point the nonlinear terms are negligible compared to the linear terms if one focuses on a sufficiently small neighborhood of the fixed point.

Note:

- the flow in the vicinity of a hyperbolic fixed point is structurally stable. This is not the case without hyperbolicity, e.g. for centers or fixed points undergoing bifurcations.

3.3.2 Phase Portraits

A phase portrait captures all relevant features of the phase plane.

Example:

$$\begin{aligned} \dot{x} &= f(x,y) = y \\ \dot{y} &= g(x,y) = x(1+y) - 1 \end{aligned}$$

1. Nullclines are lines along which the time derivative of one of the variables vanishes. They can only crossed parallel to the coordinate axis corresponding to the other variable.

$$f(x,y) = 0 \qquad \Rightarrow \quad y = 0$$
$$g(x,y) = 0 = x(1+y) - 1 \qquad \Rightarrow \quad y = \frac{1}{x} - 1$$

2. Fixed Points are at the intersections of the nullclines.

$$y = 0 \qquad x = 1$$

3. Linear stability of fixed point:

$$x(t) = 1 + \epsilon\, x_1(t) \qquad y = \epsilon y_1(t)$$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{y}_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

Eigenvalues:

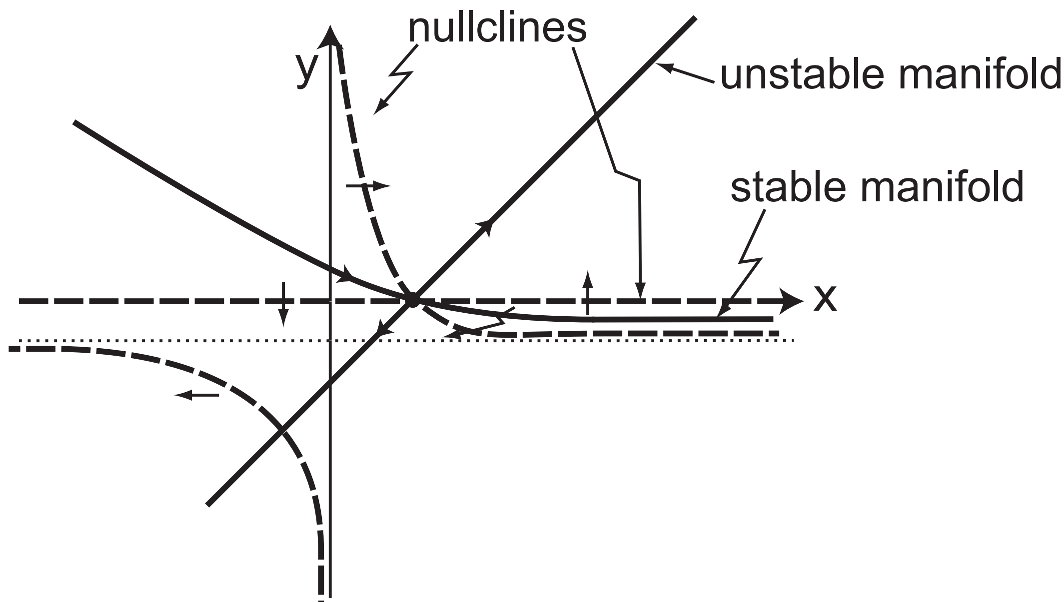
$$\lambda_{1,2} = \frac{1 \pm \sqrt{5}}{2} \qquad \text{saddle point}$$

Eigenvectors:

$$\begin{pmatrix} x_0^{(1,2)} \\ y_0^{(1,2)} \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \pm \sqrt{5} \end{pmatrix}$$

Note:

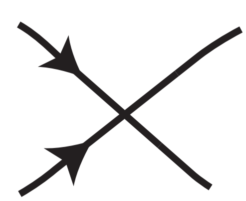
- This fixed point is hyperbolic ⇒ the eigenvectors of the linear stability analysis give the directions of the stable and unstable manifolds of the nonlinear flow.



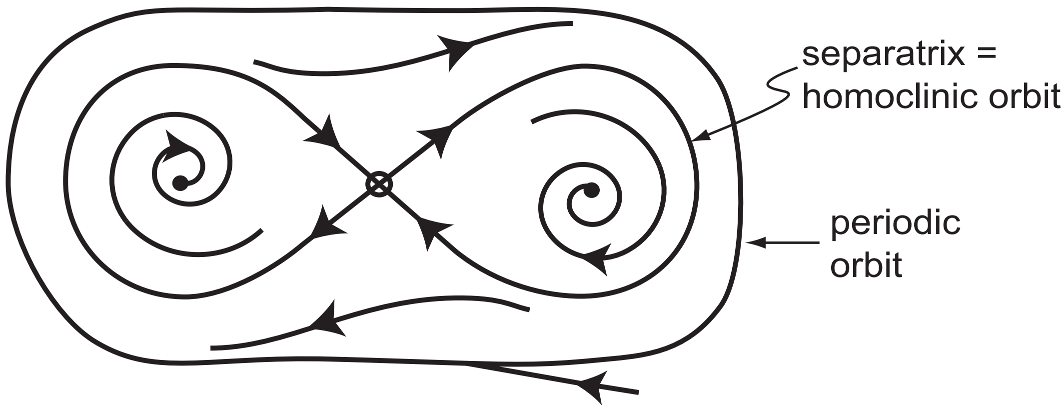
Notes:

- For $\dot{\underline{x}} = \underline{f}(\underline{x})$ the solutions are unique if all partial derivatives $\frac{\partial f_i}{\partial x_j}$ are continuous

  ⇒ orbits do not intersect. An intersection would imply a non-unique solution: starting with an initial condition at the intersection the system could go into two different directions. Thus, the lines denoting orbits can cross only at fixed points.



Phase portraits can be more complicated:



Phase portraits can contain

- nullclines
- fixed points with their stable/unstable manifolds
- periodic orbits
- separatrices: a separatrix separates basins of attraction of different attractors
- heteroclinic orbits: trajectories that connect two different fixed points. If two fixed points are connected by a heteroclinic orbit the unstable manifold of one fixed point is the stable manifold of the other.
- homoclinic orbits: a trajectory that returns to the same fixed point. In this case the unstable manifold conincides with the stable manifold of the fixed point.

3.3.3 Ruling out Persistent Dynamics

For what kind of systems can one rule out persistent dynamics like periodic orbits?

In the one-dimensional case we discussed already **Gradient Systems** (**Potential Systems**)

Thus, if

$$\dot{\underline{x}} = -\nabla V(\underline{x}) \qquad \text{i.e.} \qquad \dot{x}_i = -\frac{\partial V}{\partial x_i}$$

with $V \geq V_0$ for all $\underline{x}$ (bounded from below)
then

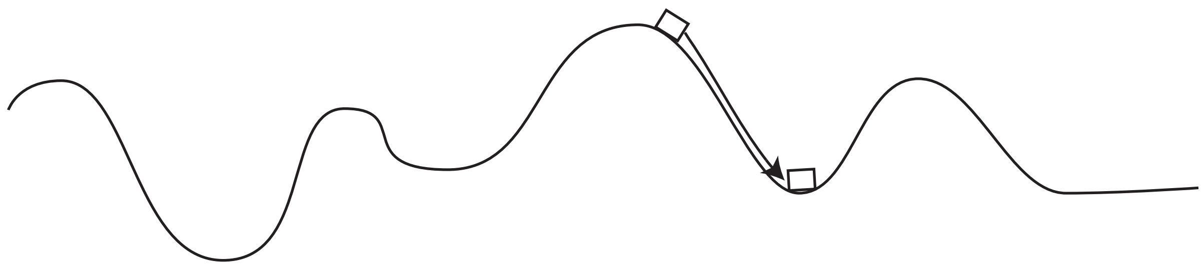
$$\frac{dV}{dt} = \sum_i \frac{\partial V}{\partial x_i}\dot{x}_i = -\sum_i \left(\frac{dx_i}{dt}\right)^2 \leq 0.$$

Thus, V eventually reaches a (local) minimum and

$$\frac{dV}{dt} = 0 \Leftrightarrow \dot{x}_i = 0 \qquad \text{for all } i$$

Thus, the system always goes to a fixed point.

Example: Mechanical overdamped particle in potential



ii) Lyapunov Functional

To rule out persistent dynamics one does not need $\dot{\underline{x}} = -\nabla V$.

More generally:

Assume there is a function $V(\underline{x})$ with $V(\underline{x}) > V_0$ for all $\underline{x} \neq \underline{x}_0$ where $\underline{x}_0$ is a fixed point

- if $\frac{dV}{dt} \leq 0$ for all $\underline{x} \neq \underline{x}_0$ in neighborhood $\mathcal{U}$ then $\underline{x}_0$ Lyapunov stable
  if $V(\underline{x}(t))$ is non-increasing in time, $\underline{x}(t)$ cannot escape.
- if $\frac{dV}{dt} < 0$ for all $\underline{x} \neq \underline{x}_0$ in $\mathcal{U}$ then $\underline{x}_0$ asymptotically stable
  if $V(\underline{x}(t))$ is strictly decreasing in time,$\underline{x}(t)$ must approach the fixed point.

Note:

- Such a $V(\underline{\mathbf{x}})$ is called a *Lyapunov function*.

Example:

a) damped particle in a bounded potential $\mathcal{U}(x)$

$$\ddot{x} + \beta\dot{x} = -\frac{d\mathcal{U}}{dx}$$

i.e.

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= -\beta v - \frac{d\mathcal{U}}{dx} \end{aligned}$$

Try the total energy

$$V = \frac{1}{2}\dot{x}^2 + \mathcal{U} = \frac{1}{2}v^2 + \mathcal{U}$$

$$\frac{dV}{dt} = v\dot{v} + \frac{d\mathcal{U}}{dx}\dot{x} = v(-\beta v - \frac{d\mathcal{U}}{dx}) + \frac{d\mathcal{U}}{dx}v = -\beta v^2 < 0 \qquad \text{for } v \neq 0$$

$\Rightarrow$ there are no periodic orbits and the system always ends up in a stable fixed point.

b)

$$\begin{aligned}\dot{x} &= -x + 4y \\ \dot{y} &= -x - y^3\end{aligned}$$

Simplest attempt: try a quadratic function that is bounded from below:

$$V = x^2 + ay^2 \qquad \text{with } a > 0.$$

The parameter a can be chosen as needed.

$$\begin{aligned}\frac{dV}{dt} &= 2x(-x + 4y) + 2\alpha y(-x - y^3) \\ &= \underbrace{-2x^2}_{\leq 0} + \underbrace{xy(8 - 2a)}_{\text{undetermined}} \underbrace{-2ay^4}_{\leq 0}\end{aligned}$$

$\Rightarrow$ choose $a = 4 \Rightarrow \frac{dV}{dt} < 0$ for $x \neq 0 \neq y$

$\Rightarrow (0, 0)$ asymptotically stable and no periodic orbits

Note:

- Potentials rule out persistent dynamics in **arbitrary dimensions**.

3.3.4 Poincaré-Bendixson Theorem: No Chaos in 2 Dimensions [8]

- How complex can the dynamics be in 2 dimensions?
- Can we guarantee a periodic orbit without explicitly calculating it?

Poincaré-Bendixson Theorem:

If

- R is a closed bounded subset of the plane
- $\dot{\underline{x}} = \underline{f}(\underline{x})$ with $\underline{f}(\underline{x})$ continuously differentiable on an open set containing R

then

> any orbit that remains in R for all t either converges to a fixed point or to a periodic orbit.

Simple Illustration:

[8] cf. Strogatz Ch.7

- in one dimension we had: no periodic orbits
  fixed point *divides* phase line into *left* and *right*
  $\Rightarrow$ cannot go back and forth
  $\Rightarrow$ no oscillatory approach to fixed point
  $\Rightarrow$ no persistent oscillations
  to get oscillations the trajectory would need to *spiral* into the fixed point: need 2 dimensions

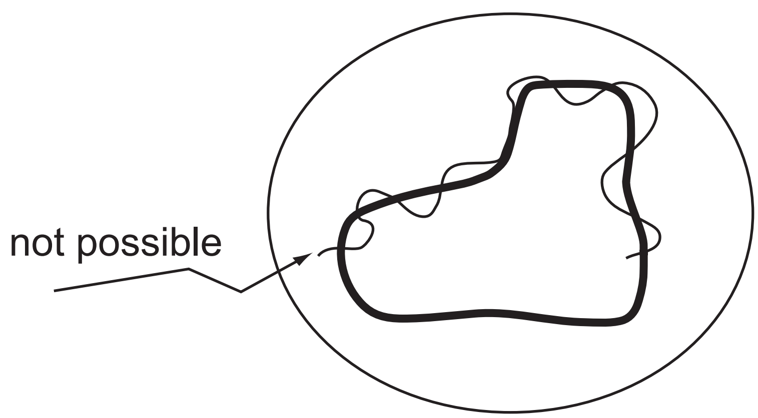
- in two dimensions:
  what is more "complicated" than periodic orbit?
  periodic orbit has single fundamental frequency ω

$$x(t) = A\cos\omega t + B\cos 2\omega t + C\cos 3\omega t + \ldots$$

  Can we have 2 incommensurate frequencies? I.e.

$$x(t) = A\cos\omega_1 t + B\cos\omega_2 t + \ldots \qquad \text{with} \qquad \frac{\omega_1}{\omega_2} \neq \frac{m}{n} \qquad \text{irrational}$$

  Consider the approach to a periodic orbit in two dimensions:



  The periodic orbit *divides* the phase plane into *inside* and *outside*.
  An oscillatory approach to the periodic orbit would require going from inside to outside and back. This is not possible without crossing the periodic orbit, which is not possible due to the uniqueness of the solution $\Rightarrow$ No second frequency.
  The system has to go to a fixed point or a periodic orbit.
  to get oscillatory approach the trajectory would have to *spiral* around the periodic orbit: need 3 dimensions.

Consequence of Poincaré-Bendixson:

- The only attractors of 2d-flows are fixed points or periodic orbits
- **No chaos in 2 dimensions.**

Example 2: Glycolysis Oscillations

Yeast cells break down sugar by glycolysis, which can proceed in an oscillatory fashion.

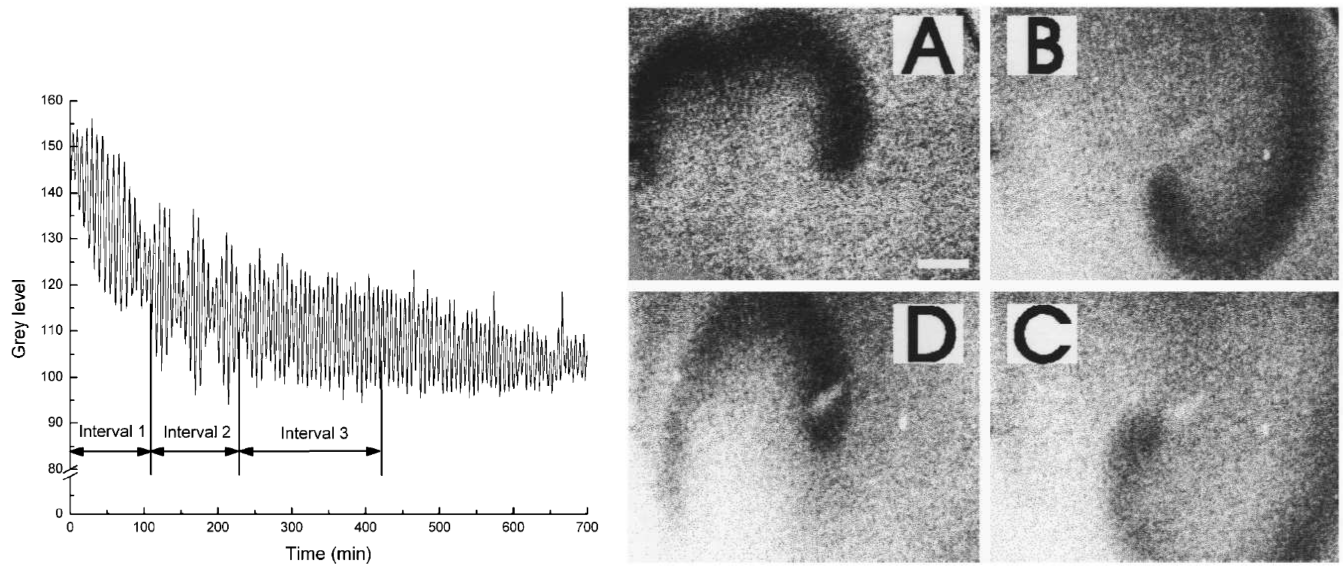


Figure 12: Glycolytic oscillations in an unstirred yeast extract leads to spiral waves [**?**, **?**].

Simple model[**?**, **?**]:

Phosphorylation of fructose-6-phosphate $F6P$ to fructose-6-diphosphate FDP

$$F6P \underbrace{\overset{PFK}{\rightarrow}}_{ATP \rightarrow ADP} FDP$$

This reaction is catalyzed by the enzyme phosphofructokinase PFK and accompanied by the conversion of adenosine triphosphate ATP into adenosine diphosphate ADP.

PFK is stimulated by binding with several ADP molecules, i.e. the phosphorylation of $F6P$ is enhanced by the presence of multiple ADP molecules.

$$\begin{array}{lrcl} \text{ADP adenosine diphosphate} & \dot{x} & = & -x + \overbrace{ay}^{\text{non-enhanced phosphorylation}} + x^2 y = f(x,y) \\ \text{F6P fructose-6-phosphate} & \dot{y} & = & b - ay - \underbrace{x^2 y}_{\text{enhanced phosphorylation}} = g(x,y) \end{array}$$

Are there parameter ranges for which can one guarantee the existence of a stable periodic orbit?

Phase portrait:

study **nullclines**: $\dot{x} = 0$ or $\dot{y} = 0$

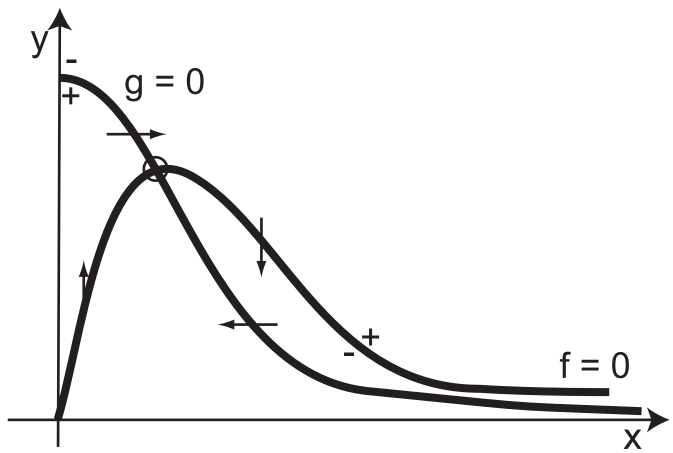
$$\begin{aligned} f &= 0 \qquad \Rightarrow y = \frac{x}{a + x^2} \\ g &= 0 \qquad \Rightarrow y = \frac{b}{a + x^2} \end{aligned}$$

$\Rightarrow$ fixed point at

$$y = \frac{x}{a+x^2} = \frac{b}{a+x^2}$$

$$\Rightarrow \qquad x = b \qquad \text{and} \qquad y = \frac{b}{a+b^2}$$

exists for all $b > 0$, $a > 0$



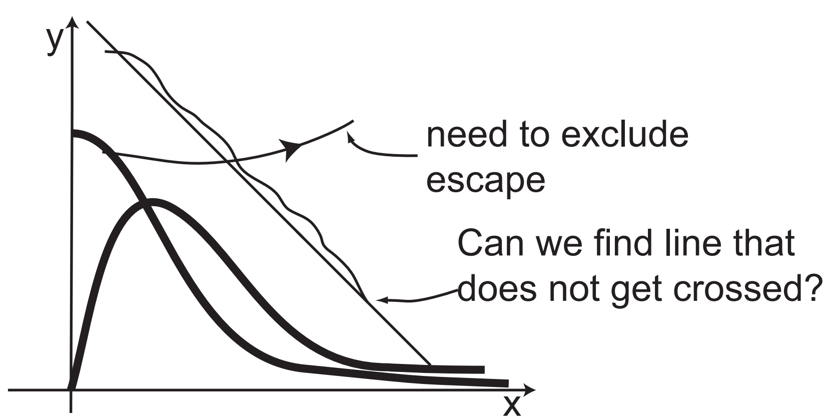
Nullclines show: spiraling motion

- to fixed point?
- to periodic orbit? which?
- to infinity?

To use the Poincaré-Bendixson theorem:

1. we need a trapping region $\mathcal{R}$
2. we need to exclude all fixed points from the trapping region

1) Trapping Region:



Consider large x and y to check the possibility of an escape

$$\left.\begin{array}{ccc}\dot{x} & \sim & x^2 y \\ \dot{y} & \sim & -x^2 y\end{array}\right\} \text{along the orbit one has: } \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = -\frac{-g}{f} = -\frac{x^2y + ay - b}{x^2y + ay - x}$$

Thus:

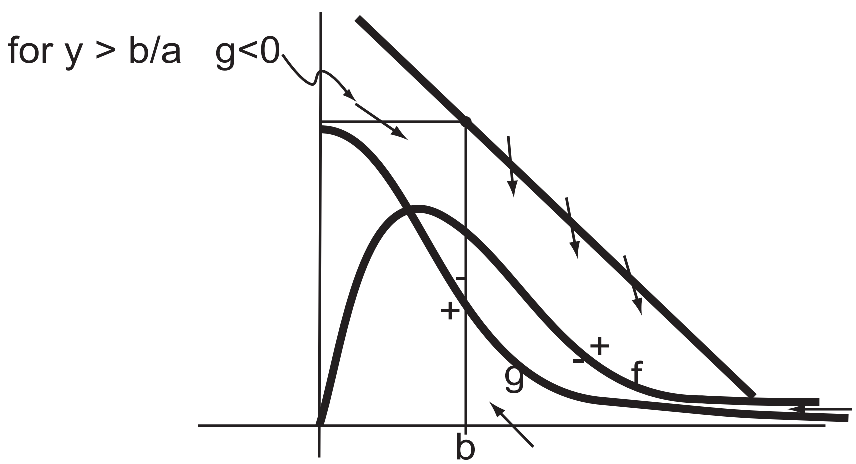
- for $x > b$ and (x, y) above the nullcline $f = 0$ we have $-g > f > 0$ and therefore $\frac{dy}{dx} < -1$:
  the trajectories enter the trapping region along $y = -x + C$
- below the nullcline $f = 0$ we have $\dot{x} < 0$:
  the trajectories enter the trapping region along $x = x_r$ with x_r being the x-coordinate of the intersection of the straignt line and $f = 0$

Alternatively one can compare $|\dot{x}|$ with $|\dot{y}|$ like this

$$\begin{aligned}\dot{x} - (-\dot{y}) &= -x + ay + x^2\, y + b - ay - x^2\, y \\ &= b - x\end{aligned}$$

$\Rightarrow$ for $x > b$ $\qquad |\dot{x}| < |\dot{y}|$
 $\Rightarrow$ flow inward along $y = -x + C$ for $x > b$ and C large enough

for $y > \frac{b}{a}$ we have $g < 0$ for any value of x
 $\Rightarrow$ flow inward for $y > b/a$



2) Fixed Points:

There is only a single fixed point $\left(b, \frac{b}{a+b^2}\right)$. We need to establish that the trajectories do not converge to that fixed point

The linear stability analysis shows that the fixed point is unstable for

$$1 - 2a - \sqrt{1 - 8a} < 2b^2 < 1 - 2a + \sqrt{1 - 8a}$$

$\Rightarrow$ limit cycle guaranteed for this range of b, which exists as long as $a \le \frac{1}{8}$

The instability at $2(b_H^{(1,2)})^2 = 1 - 2a \pm \sqrt{1 - 8a}$ is a Hopf bifurcation. Oscillations occur for $b_H^{(1)} < b < b_H^{(2)}$. No steady bifurcation is possible.

More generally: the trapping region needs to exclude a (possibly small or infinitesimal) neighborhood of the fixed point

- If the fixed point is linearly unstable, trajectories will not leave the trapping region and converge towards that fixed point.
- If the fixed point is marginally stable, nonlinear terms become relevant: need to establish crossing of the border like we did for the line $y = -x + C$ in the example.

3.4 Relaxation Oscillations

Class of systems for which one can see the periodic orbit relatively easily:

Fast-slow systems with N-shaped nullcline

Example:

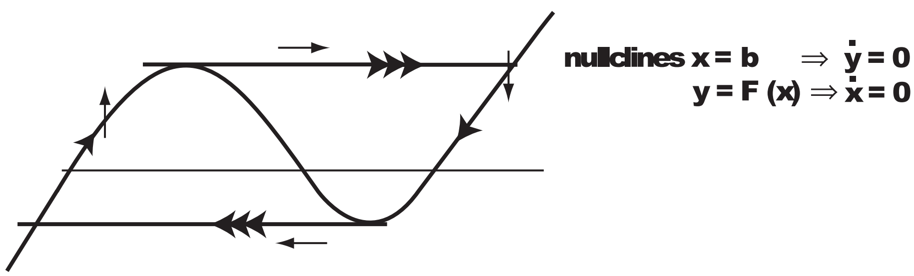
Consider for $\mu \gg 1$:

$$\begin{aligned} \dot{x} &= \mu\,(y - F(x)) \\ \dot{y} &= b - x \end{aligned}$$

with

$$F(x) = -x + \frac{1}{3}x^3$$

.



Nullcline $x = 0$ Is Missing

For $\mu \gg 1$ the variable x evolves much faster than y except near the nullcline $y = F(x)$:

- except near the nullcline $y = F(x)$ the vector field is essentially horizontal

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{b - x}{\mu\,(y - F(x))} \to 0 \qquad \text{for} \qquad \mu \to \infty$$

  the horizontal trajectories constitute two *fast* branches

- near the nullcline $y = F(x)$ the trajectory is pushed towards the nullcline: the nullcline $y = F(x)$ represents two *slow* branches

Fixed point:

$$x_0 = b \qquad y_0 = b - \frac{1}{3}b^3$$

Linear stability of the fixed point: expand around $(b, b - \frac{1}{3}b^3)$

$$x = x_0 + \epsilon x_1(t) \qquad y = y_0 + \epsilon y_1(t)$$

$$\begin{aligned}\epsilon\dot{x}_1 &= \mu\left(y_0 + \epsilon y_1 + x_0 + \epsilon x_1 - \frac{1}{3}\left(x_0 + \epsilon x_1\right)^3\right) \\ \epsilon\dot{y}_1 &= b - x_0 - \epsilon x_1.\end{aligned}$$

Collecting all terms at $\mathcal{O}(\epsilon)$ yields

$$\begin{pmatrix} \dot{x}_1 \\ \dot{y}_1 \end{pmatrix} = \mathbf{L}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

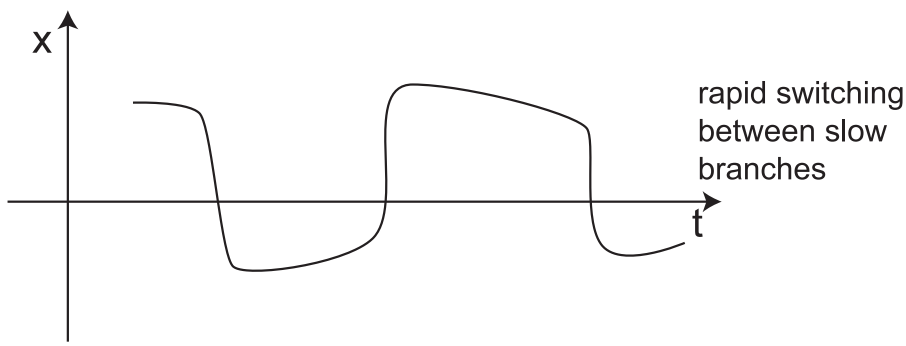
with

$$\mathbf{L} = \begin{pmatrix} \mu\left(1-b^2\right) & \mu \\ -1 & 0 \end{pmatrix}$$

In terms of the trace and the determinant of **L** the eigenvalues are given by

$$\lambda_{1,2} = \frac{1}{2}\left(trace\mathbf{L} \pm \sqrt{(trace\mathbf{L})^2 - 4det\mathbf{L}}\right) = \frac{1}{2}\left(\mu\left(1-b^2\right) \pm \sqrt{\mu^2\left(1-b\right)^2 - 4\mu}\right).$$

Thus, since $\mu > 0$, the fixed point is unstable with a complex pair of eigenvalues for $|b| < 1$ and stable otherwise. For these two values of b the fixed point is at either of the two extrema of $F(x)$.



The period of the periodic orbit is essentially determined by the time spent on the slow branches.

On the right slow branch (nullcline) (i.e. for $x > 0$) x changes essentially only because y changes. therefore express $\dot{x}$ in terms of $\dot{y}$,

$$y \sim F(x) \qquad \Rightarrow \qquad \dot{y} \sim \frac{dF}{dx}\,\dot{x}$$

And y changes according to

$$\dot{y} = g(x, y)$$

thus

$$\dot{x} = \frac{g(x, y)}{\frac{dF}{dx}} = \frac{g(x, F(x))}{\frac{dF}{dx}}$$

Consider the simpler, symmetric case $b=0$, for which the fixed point is at $(0,0)$,

$$T=\int dt \sim 2\int_{\mathcal{C}}\frac{dt}{dx}\,dx=2\int_{\mathcal{C}}\frac{1}{\dot{x}}\,dx=2\int_{\mathcal{C}}\frac{\frac{dF}{dx}}{-x}\,dx=2\int_{x_r}^{x_{min}}\frac{-1+x^2}{-x}dx$$

where $\mathcal{C}$ is the portion of the trajectory along the nullcline $y=F(x)$ from $x=x_r>0$ to $x=x_{min}$ with $x_{min}>0$ given by the minimum of $F(x)$:

$$\frac{dF}{dx}=0=-1+x^2 \qquad x_{min}=1 \qquad y_{min}=-\frac{2}{3} \qquad x_{max}=-1 \qquad y_{max}=\frac{2}{3}$$

The upper limit x_r is given by the condition

$$y_{max}\equiv\frac{2}{3}=F(x_r)=-x_r+\frac{1}{3}x_r^3$$

We know that $F(x_r)=y_{max}$ has a double zero at $x_{min}=-1$. The cubic can therefore be factorized easily to obtain $x_r=2$.

Thus

$$T\sim 2\left[\ln x-\frac{1}{2}x^2\right]\Big|_2^1=2\left\{-\ln 2+\frac{3}{2}\right\}$$

3.5 Weakly Nonlinear Oscillators

We would like to determine solutions for non-linear oscillators like

$$\ddot{x}+\beta\dot{x}+x+\alpha x^2\dot{x}+\gamma x^3=0.$$

Exact nonlinear solutions usually impossible to get.
To make *analytical* progress try to obtain *systematic approximate* solutions for
- the periodic orbits and
- the transients approaching periodic orbits.

3.5.1 Failure of Regular Perturbation Theory

Consider first a simple linear example to demonstrate the problem

$$\ddot{x}+2\epsilon\beta\dot{x}+(1+\epsilon\Omega)^2x=0 \qquad \text{with} \qquad \epsilon\ll 1$$

with some initial condition like $x(0)=0$, $\dot{x}(0)=1$.

Exact solution:

$$x_e=Ae^{\lambda t} \qquad \Rightarrow \qquad \lambda^2+2\epsilon\beta\lambda+(1+\epsilon\Omega)^2=0$$

$$\begin{aligned}
\lambda_{1,2} &= \frac{-2\epsilon\beta\pm\sqrt{4\epsilon^2\beta^2-4(1+\epsilon\Omega)^2}}{2}\\
&= -\epsilon\beta\pm i\sqrt{(1+\epsilon\Omega)^2-\epsilon^2\beta^2}\\
\Rightarrow & \\
x_{exact} &= e^{-\epsilon\beta t}\left(Ae^{i\omega t}+A^*e^{-i\omega t}\right)\\
\text{with} & \\
\omega &= \sqrt{(1+\epsilon\Omega)^2-\epsilon^2\beta^2}
\end{aligned}$$

Attempt perturbation solution using

$$x_a = x_0 + \epsilon x_1 + h.o.t$$

Insert

$$\begin{aligned} \frac{d^2}{dt^2}(x_0 + \epsilon x_1 + \ldots) &+ 2\epsilon\beta \frac{d}{dt}(x_0 + \epsilon x_1 + \ldots) \\ &+ (1+\epsilon\Omega)^2 (x_0 + \epsilon x_1 + \ldots) = 0 \end{aligned}$$

Collect orders in ϵ:
$\mathcal{O}(\epsilon^0)$:

$$\begin{aligned} \frac{d^2}{dt^2} x_0 + x_0 &= 0 \\ x_0 = Ae^{it} + A^* e^{-it} &= 2A_r \cos t - 2A_i \sin t \end{aligned}$$

$\mathcal{O}(\epsilon^1)$:

$$\frac{d^2}{dt^2} x_1 + 2\beta \frac{d}{dt} x_0 + 2\Omega x_0 + x_1 = 0$$

$$\frac{d^2}{dt^2} x_1 + x_1 = \underbrace{-2i\beta A e^{it} - 2\Omega A e^{it}}_{\sim \text{ resonant forcing}} + c.c. \equiv \alpha e^{it} + c.c.$$

This is a second-order constant-coefficient *inhomogeneous* differential equation:
General solution:

$$x_1(t) = x_h(t) + x_p(t)$$

with

$$\frac{d^2}{dt^2} x_h + x_h = 0 \qquad \Rightarrow x_h = A_1 e^{it} + c.c.$$

Try undetermined coefficients for the particular solution (since inhomogeneity is simple exponential function):

$$x_p = Be^{it} + c.c.$$

However:

$$\frac{d^2}{dt^2} Be^{it} + Be^{it} = 0 \qquad \Rightarrow \text{cannot balance inhomogeneity on r.h.s.}$$

Note:

- at $\mathcal{O}(\epsilon)$ the inhomogeneous term is forcing the oscillator x_1 at its resonance frequency: *resonant forcing*

We could now use the method of *variation of parameters* $x_p = B(t)e^{it}$ and reduce the order of the equation and solve the resulting first-order equation by integration

$$x_p = B(t)e^{it}$$

$$\ddot{x}_p = \ddot{B}e^{it} + 2i\dot{B}e^{it} - Be^{it} + c.c.$$

Thus,

$$\ddot{B}e^{it} + 2i\dot{B}e^{it} - Be^{it} + Be^{it} = \alpha e^{it}$$

with $V = \dot{B}$ we get

$$\dot{V} + 2iV = \alpha$$

and using an integrating factor

$$\frac{d}{dt}\left(e^{2it}V\right) = e^{2it}\alpha$$

$$V = e^{-2it}\int e^{it}\alpha dt = \frac{1}{2i}\alpha$$

and

$$\begin{aligned} B &= \frac{1}{2i}t\alpha + C \\ x_p(t) &= \frac{1}{2i}\left(-2i\beta - 2\Omega\right)te^{it}A + c.c. \end{aligned}$$

Alternatively, having some experience we can try directly the ansatz:

$$x_p = B\,t\,e^{it} + c.c.$$

Insert:

$$\begin{aligned} \frac{d^2}{dt^2}x_p + x_p &= \\ B\left(2ie^{it} - t\,e^{it} + t\,e^{it}\right) + c.c. &= -2i\beta Ae^{it} - 2\Omega Ae^{it} + c.c. \\ \Rightarrow B &= \frac{1}{2i}\left(-2i\beta - 2\Omega\right)A \end{aligned}$$

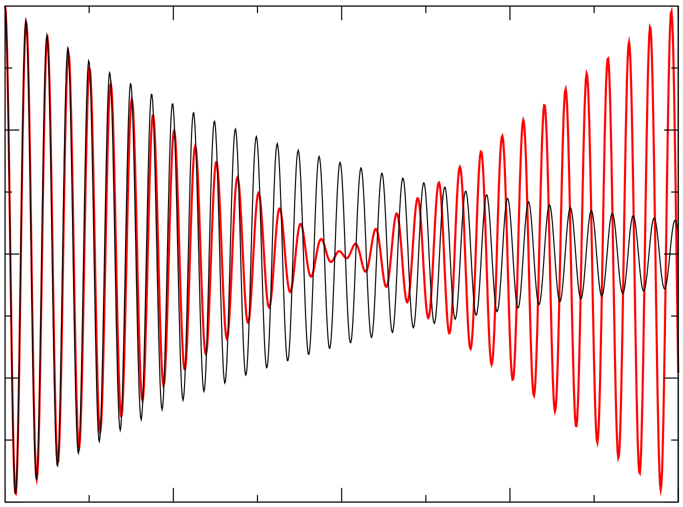
Put together

$$x_a(t) = x_0(t) + \epsilon x_1(t) = Ae^{it} + \epsilon\left(-\beta + i\Omega\right)\,te^{it}\,A + c.c. = \left(1 - \epsilon\beta t + i\epsilon\Omega t\right)Ae^{it} + c.c.$$

Notes:

- Initially, the oscillation amplitude of $x_a(t)$ is decaying like that of the exact solution.
- For larger times the oscillation amplitude of x_a grows linearly, while the exact solution $x_e(t)$ decays to 0.

Resonant forcing leads to (linear) growth without bounds: *secular terms* [9]



- for $t = \mathcal{O}(\epsilon^{-1})$ the perturbation ϵx_1 becomes as large as x_0: this contradicts the assumptions of the approach $\Rightarrow$ the perturbation approach breaks down for $t = \mathcal{O}\left(\epsilon^{-1}\right)$.

However: the approximation is indeed an expansion of the exact solution in ϵ:

$$x_e = \underbrace{e^{-\epsilon\beta t}}_{1-\epsilon\beta t+\mathcal{O}(\epsilon^2)} \left(Ae^{i\omega t} + c.c.\right)$$

with

$$\omega = \underbrace{\sqrt{(1+\epsilon\Omega)^2 - \epsilon^2\beta^2}}_{1+\epsilon\Omega+\mathcal{O}(\epsilon^2)}$$

$$x_e = Ae^{it} + \epsilon\left(-\beta t + i\Omega t\right)Ae^{it} + \mathcal{O}(\epsilon^2) + c.c.$$

Thus:

- The straightforward perturbation expansion captures
  - the slow growth/decay
  - the small change in frequency

  only initially. At later times it blows up
  It does not even capture the periodic solution for large times (for $\beta = 0$)

- Secular terms suggest what the true solution is doing.

- We need to expand in a more intelligent way that captures the change in frequency and the exponential decay for larger times.

[9]The term comes from perturbation calculations of planetary motion: their error accumulates over the ocurse of centuries (seculum).

3.5.2 Multiple Scales

Exact solution suggests that there are multiple time scales

$$x_{exact} = \mathcal{A}e^{-\epsilon\beta t + i\omega t} + c.c. = \mathcal{A}e^{-\epsilon\beta t + i(1+\Omega\epsilon)t} + c.c. = \underbrace{\mathcal{A}e^{-\beta\,\epsilon t + i\Omega\,\epsilon t}}_{A(\epsilon t)} e^{it} + c.c.$$

The fast oscillation with frequency $\omega_0 = 1$ has an amplitude that varies slowly with time since its argument changes only little as time progresses, $A = A(\epsilon t)$.

Introduce this slower time scale explicitly as a 'separate time',

$$T = \epsilon t$$

and let the function x depend on two time variables: $\hat{t} = t$ and T

$$x_e = x_e(\hat{t}, T)$$

Note:

- in this approach the two (or more times) are treated as essentially independent variables:

$$\frac{d}{dt}x(\hat{t},T) = \partial_{\hat{t}}x\,\frac{d\hat{t}}{dt} + \partial_T x\,\frac{dT}{dt} = \partial_{\hat{t}}x + \epsilon\partial_T x$$

$$\frac{d^2}{dt^2}x(\hat{t},T) = \frac{d}{dt}\left(\partial_{\hat{t}}x + \epsilon\partial_T x\right) = \frac{\partial^2}{\partial\hat{t}^2}x + 2\epsilon\frac{\partial^2}{\partial\hat{t}\partial T}x + \epsilon^2\frac{\partial^2}{\partial T^2}x$$

  the ordinary differential equation becomes a partial differential equation.

Try again the same linear problem:

Expand again

$$x_a = x_0(\hat{t},T) + \epsilon x_1(\hat{t},T) + \ldots$$

$$\left(\frac{\partial^2}{\partial\hat{t}^2} + 2\epsilon\frac{\partial^2}{\partial\hat{t}\partial T} + \epsilon^2\frac{\partial^2}{\partial T^2}\right)(x_0 + \epsilon x_1 + \cdots) \;+\; 2\epsilon\beta\left(\frac{\partial}{\partial\hat{t}} + \epsilon\frac{\partial}{\partial T}\right)(x_0 + \epsilon x_1 + \cdots)$$
$$+\;(1+\epsilon\Omega)^2\,(x_0 + \epsilon x_1 + \cdots) = 0$$

$\mathcal{O}(\epsilon^0)$:

$$\frac{d^2}{d\hat{t}^2}x_0 + x_0 = 0$$

$$x_0 = Ae^{i\hat{t}} + A^*e^{-i\hat{t}} = 2A_r\cos\hat{t} - 2A_i\sin\hat{t}$$

Note:

- Previously, A was a constant. Now we have two variables; the differential equation only involved $\hat{t}$. Therefore A cannot depend on $\hat{t}$ but it is allowed to depend on the slow time T: $A = A(T)$

$\mathcal{O}(\epsilon^1)$:

$$2\partial_{\hat{t}}\partial_T x_0 + \partial_{\hat{t}}^2 x_1 + 2\beta\partial_{\hat{t}} x_0 + 2\Omega x_0 + x_1 = 0.$$

$$\partial_{\hat{t}}^2 x_1 + x_1 = -2\left(i\frac{d}{dT}A + i\beta A + \Omega A\right)e^{i\hat{t}} + c.c.$$

Need to avoid secular terms $\Rightarrow$ require

$$\frac{d}{dT}A = -\beta A + i\Omega A \tag{3}$$

then no secular terms arise that would grow linearly in time.

Solution of the amplitude equation (3)

$$\begin{aligned} A &= \mathcal{A}e^{-\beta T + i\Omega T} \\ x_0 &= \mathcal{A}e^{-\beta T}e^{i\hat{t}+i\Omega T} + c.c. = \mathcal{A}e^{-\epsilon\beta t}e^{i(1+\epsilon\Omega)t} + c.c. \end{aligned}$$

Thus:

- Two-timing (multiple scales) avoids secular terms and gets frequency shift and slow damping correct to the order considered : no blow-up
- calculation easier in complex exponentials than using trig functions

Example: Duffing oscillator

$$\ddot{x} + x + \epsilon x^3 = 0$$

Ansatz:

$$\begin{aligned} x &= x_0(\hat{t},T) + \epsilon x_1(\hat{t},T) + \cdots \\ \left(\frac{d}{dt}\right)^2 &\rightarrow \partial_{\hat{t}}^2 + 2\epsilon\partial_{\hat{t}}\partial_T + O(\epsilon^2) \end{aligned}$$

$\mathcal{O}(\epsilon^0)$:

$$\partial_{\hat{t}}^2 x_0 + x_0 = 0 \qquad x_0 = Ae^{i\hat{t}} + A^* e^{-i\hat{t}}$$

$\mathcal{O}(\epsilon^1)$:

$$\partial_{\hat{t}}^2 x_1 + x_1 + \underbrace{2\partial_{\hat{t}}\partial_T x_0}_{2i\frac{dA}{dT}e^{i\hat{t}}+c.c.} + \underbrace{x_0^3}_{A^3e^{3i\hat{t}}+3|A|^2Ae^{i\hat{t}}+3|A|^2A^*e^{-i\hat{t}}+A^{*3}e^{-3i\hat{t}}} = 0$$

thus

$$\frac{\partial^2}{\partial\hat{t}^2}x_1 + x_1 = -\underbrace{e^{i\hat{t}}}_{\text{secular resonance term}}\left\{2i\frac{dA}{dT} + 3|A|^2A\right\} - e^{3i\hat{t}}A^3 + c.c.$$

require:

$$\frac{dA}{dT} = +\frac{3}{2}i|A|^2A \tag{4}$$

There is no issue with the term $A^3 e^{i3\hat{t}}$: use undetermined coefficient

$$x_1 = Be^{3i\hat{t}} + B^* e^{-3i\hat{t}}$$

and insert

$$-9Be^{3i\hat{t}} + Be^{3i\hat{t}} + c.c. = A^3 e^{3i\hat{t}} + c.c.$$

to obtain

$$B = -\frac{1}{8}A^3$$

and

$$x_1 = -\frac{1}{8}A^3 e^{3i\hat{t}} + c.c.$$

Note:

- the equation for $A(T)$ is also nonlinear. But it has a special form that make the solution easier

Separate into amplitude and phase

$$A(T) = R(T)e^{i\phi(T)}$$

$$\frac{d}{dT}R + iR\frac{d}{dT}\phi = \frac{3}{2}iR^3$$

Separating into real and imaginary part yields

$$\frac{dR}{dT} = 0 \qquad \frac{d\phi}{dT} = \frac{3}{2}R^2.$$

Thus, the amplitude decouples from the phase

$$\phi = \frac{3}{2}R^2 T$$

and

$$A = Re^{i\frac{3}{2}iR^2 T}$$

Putting everything together we get

$$x = R\,e^{i(1+\frac{3}{2}\epsilon R^2)t} - \epsilon\frac{1}{8}\left(R\,e^{i(1+\frac{3}{2}\epsilon R^2)t}\right)^3 + c.c. + \mathcal{O}(\epsilon^2)$$

Notes:

- The nonlinearity induces a frequency shift: $\omega = 1 + \frac{3}{2}\epsilon R^2$

  $\rightarrow$ soft and hard spring ($\epsilon \begin{smallmatrix}<\\>\end{smallmatrix} 0$)

$$\ddot{x} + (1 + \epsilon x^2)x = 0$$

- At order $\mathcal{O}(\epsilon^2)$ additional frequency shifts from secular terms in $x_0^2 x_1$

  $\Rightarrow$ approximate and exact solution get out of sync for $t \sim \mathcal{O}(\epsilon^{-2})$:

$$\cos\left((\omega + \epsilon\omega_1 + \underbrace{\epsilon^2\omega_2)t}_{\epsilon^2\omega_2 t \sim 2\pi \,\Rightarrow\, t=\mathcal{O}(\frac{1}{\epsilon^2})}\right)$$

  $\Rightarrow$ introduce additional slow time $\epsilon^2 t$

- To get a good solution at leading order ($\mathcal{O}(\epsilon^0)$) we needed to make sure that the $\mathcal{O}(\epsilon)$-term does not blow up. Beyond that one is often not very interested in the specific form of the $\mathcal{O}(\epsilon)$-term.

- Two-timing also very useful near bifurcation, where one time scale becomes very slow.

3.5.3 Hopf Bifurcation[10]

For complex eigenvalues we get stable or unstable spiral points: what kind of bifurcation does the transition represent?

Consider example

$$\begin{aligned} \dot{x} &= \mu x - y - x^3 && (5)\\ \dot{y} &= x + \mu y && (6) \end{aligned}$$

Linear stability of the fixed point $(0,0)$

$$\begin{pmatrix} \mu & -1 \\ 1 & \mu \end{pmatrix} \Rightarrow \lambda = \mu \pm i$$

Eigenvectors at the bifurcation point

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \pm i \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \qquad y_0 = \mp i x_0 \quad \mathbf{v} = \begin{pmatrix} 1 \\ \mp i \end{pmatrix}$$

At $\mu = 0$ the stable spiral turns into an unstable spiral and we may expect the orbit to saturate to a periodic orbit. We want to determine an approximation for this periodic orbita near the bifurcation point where the stability of the fixed changes, i.e. for $|\mu| \ll 1$, i.e. for small growth or decay rate.

Write

$$\mu = \epsilon^2 \mu_2.$$

A small growth rate means growth on a slow time scale: introduce a slow time via

$$T = \epsilon^2 t$$

[10] Strogatz Ch. 8.2

Expand now

$$\begin{pmatrix} x \\ y \end{pmatrix} = \epsilon \begin{pmatrix} x_1(\hat{t},T) \\ y_1(\hat{t},T) \end{pmatrix} + \epsilon^2 \begin{pmatrix} x_2(\hat{t},T) \\ y_2(\hat{t},T) \end{pmatrix} + \epsilon^3 \begin{pmatrix} x_3(\hat{t},T) \\ y_3(\hat{t},T) \end{pmatrix} + c.c.$$

Again we have

$$\frac{d}{dt}x(\hat{t},T) = \frac{\partial}{\partial\hat{t}}x(\hat{t},T) + \epsilon\frac{\partial x(\hat{t},T)}{\partial T}$$

Notes:

- At this point the scaling of $x = \mathcal{O}(\epsilon)$ and $y = \mathcal{O}(\epsilon)$ is a guess. Since the periodic orbit just came into existence we may assume that it is small; but this does not have to be the case.

Insert:

$\mathcal{O}(\epsilon)$:

$$\left.\begin{array}{l} \frac{d}{dt}x_1 = -y_1 \\ \frac{d}{dt}y_1 = x_1 \end{array}\right\} \qquad \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = A(T)\, e^{i\hat{t}} \underbrace{\begin{pmatrix} 1 \\ -i \end{pmatrix}}_{\text{eigenvector}} \underbrace{+ A^* e^{-i\hat{t}} \begin{pmatrix} 1 \\ +i \end{pmatrix}}_{\text{c.c.}}$$

with $A(T)$ yet undetermined. Our main goal is to determine an equation for $A(T)$.

$\mathcal{O}(\epsilon^2)$:

$$\begin{array}{rl} \frac{d}{dt}x_2 \quad +y_2 = & 0 \\ \frac{d}{dt}y_2 \quad -x_2 = & 0 \end{array}$$

A solution at this order is

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Note:

- one could keep a homogeneous solution

$$\begin{pmatrix} x_2(\hat{t},T) \\ y_2(\hat{t},T) \end{pmatrix} = A_1(T)\, e^{i\hat{t}} \begin{pmatrix} 1 \\ -i \end{pmatrix} + c.c.$$

  but this is not needed if we only want to determine the amplitude $A(T)$

$\mathcal{O}(\epsilon^3)$:

$$\begin{array}{rcl} \partial_{\hat{t}}x_3 + y_3 & = & -\partial_T x_1 + \mu_2 x_1 - x_1^3 \\ \partial_{\hat{t}}y_3 - x_3 & = & -\partial_T y_1 + \mu_2 y_1 \end{array}$$

With

$$x_1^3 = A^3 e^{3i\hat{t}} + 3\,|A|^2\, A e^{i\hat{t}} + 3\,|A|^2\, A^* e^{-i\hat{t}} + A^{*3} e^{-3i\hat{t}}$$

we can write this as

$$\begin{aligned} \partial_{\hat{t}} x_3 + y_3 &= I_{11} e^{i\hat{t}} + I_{13} e^{3i\hat{t}} + c.c. \\ \partial_{\hat{t}} y_3 - x_3 &= I_{21} e^{i\hat{t}} + I_{23} e^{3i\hat{t}} + c.c. \end{aligned}$$

with

$$I_{11} = -\frac{dA}{dT} + \mu_2 A - 3\left|A\right|^2 A \qquad I_{21} = -(-i)\frac{dA}{dT} + \mu_2(-i)A.$$

Ansatz with undetermined coefficients

$$\begin{pmatrix} x_3 \\ y_3 \end{pmatrix} = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} e^{i\hat{t}} + \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} e^{3i\hat{t}} + c.c.$$

Focus on terms $\propto e^{i\hat{t}}$

$$\begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \begin{pmatrix} I_{11} \\ I_{21} \end{pmatrix}$$

Try to solve for $B_{1,2}$

$$iB_1 + B_2 = I_{11} \qquad \rightarrow B_2 = I_{11} - iB_1$$

inserted

$$-B_1 + i\left(I_{11} - iB_1\right) = I_{21}$$

This equation has only a solution if the *solvability condition* is satisfied,

$$iI_{11} = I_{21} \tag{7}$$

In that case there are infinitely many solutions since B_1 is arbitrary and $B_2 = I_{11} - iB_1$.

Note:

- Mathematically, this solvability condition is a consequence of the Fredholm Alternative Theorem of linear algebra.

In our case the solvability condition (7) amounts to

$$i\left(-\frac{dA}{dT} + \mu_2 A - 3\left|A\right|^2 A\right) = -\frac{dA}{dT}\left(-i\right) + \mu_2 A\left(-i\right)$$

Thus

$$\frac{dA}{dT} = \mu_2 A - \frac{3}{2}\left|A\right|^2 A \tag{8}$$

Look for simple solutions: rewrite again using magnitude and phase

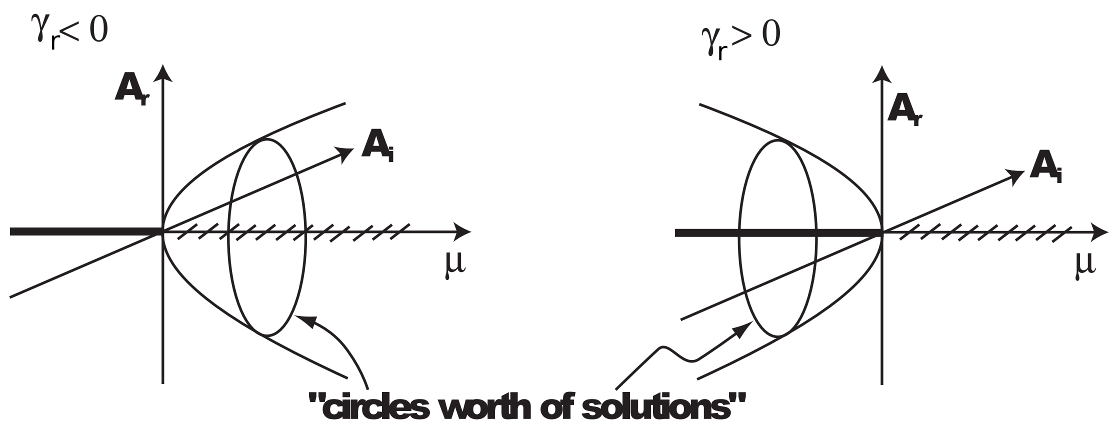
$$A(T) = R(T) e^{i\phi(T)}$$

$$\frac{dR}{dT} = \mu_2 R - \frac{3}{2} R^3 \qquad \frac{d\phi}{dT} = 0$$

Steady state, $\frac{dR}{dT} = 0$,

$$R_0 = \sqrt{\frac{2\mu_2}{3}} \qquad A = R_0 e^{i\phi_0} \quad \text{with } \phi_0 \text{ arbitrary}$$

Bifurcation diagrams:



Notes:

- Solutions exist for any phase ϕ_0: continuous family of solutions
- In this case the coefficients in (8) turned out to be real. In general they are complex: The **normal form** for the Hopf bifurcation and also for weakly nonlinear oscillators is given by
$$\frac{dA}{dT} = (\mu + i\Omega)\,A + (\gamma_r + i\gamma_i)\,|A|^2 A$$
(cf. (4) for the Duffing oscillator)
- The determinant of the linearization around the fixed point $(0,0)$ does not vanish $\Rightarrow$ in agreement with the implicit function theorem the number of fixed points does not change in a Hopf bifurcation.

3.6 1d-Bifurcations in 2d: Reduction of Dynamics

Higher-dimensional systems can undergo the same bifurcations as 1-dimensional systems.
$\Rightarrow$ can reduce dynamics to 1 dimension near the bifurcation.

3.6.1 Center-Manifold Theorem

Consider first a linear example: stable node

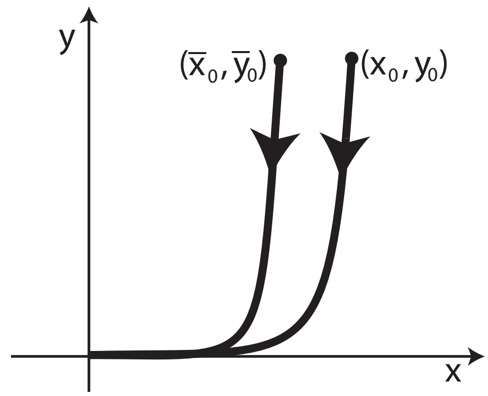
$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \mu & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

thus

$$\begin{aligned} \lambda_1 &= \mu \qquad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \lambda_2 &= -1 \qquad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}$$

Trajectories

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = x_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{\mu t} + y_0 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t} \qquad \Rightarrow \qquad y = y_0 \left(\frac{x}{x_0} \right)^{-\frac{1}{\mu}}$$



For small $|\mu|$

- for $\mu < 0$: $y \to 0$ extremely rapidly as $x \to 0$
- for $\mu > 0$: $y \to 0$ extremely rapidly as $x \to \infty$

Thus:

- after a short time any initial condition approaches the x-axis, which is in the direction of the eigenvector $\mathbf{v}_1$ with $\lambda_1 < 0$.
- after the decay of initial transients the dynamics become effectively one-dimensional and is along the direction of eigenvector $\mathbf{v}_2$ with $|\lambda_2| \ll 1$

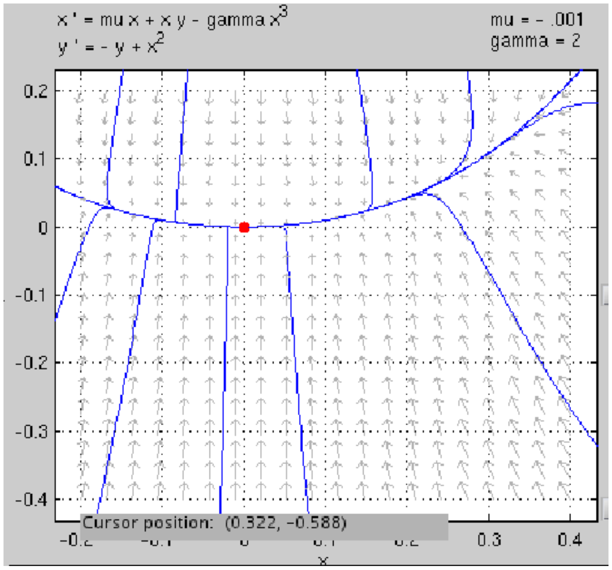
Can this also work for nonlinear systems? Add nonlinearities to our example[11].

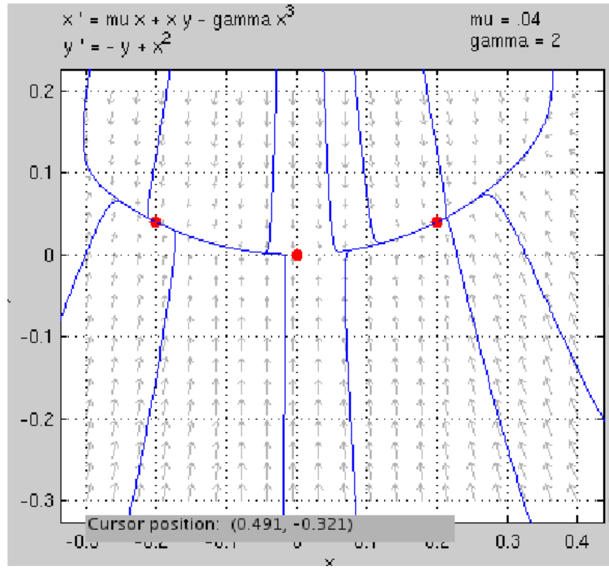
$$\dot{x} = \mu x + xy - \gamma x^3 \tag{9}$$

$$\dot{y} = -y + x^2 \tag{10}$$

If it was not for the term involving y the equation for x would describe a pitch-fork bifurcation. What effect does the y have?

[11] Demo: $\gamma = 2$, window $[-.4$ .4 -.4 .4] $\mu = -0.1, -0.01, -0.01, +0.01$. find also unstable fixed point (0,0) and determine its stable/unstable manifold.





Notes:

- For small $|\mu|$:
  - rapid compression in the direction of the eigenvector corresponding to the stable eigenvalue: stable eigenspace.
  - all trajectories converge to a line ('manifold') that is **not** given by the eigenvector v_1 corresponding to the small (vanishing) eigenvalue (center eigenspace). But the manifold is *tangent* to the center eigenspace.
  - a pitch-fork bifurcation occurs on the center manifold as μ goes through 0.

To discuss the dynamics 3 types of eigenvectors and eigenspaces need to be distinguished:

- the stable eigenspace is spanned by the eigenvectors with $Re(\lambda_i^{(s)}) < 0$
  $E^{(s)} = \{\underline{x} \,|\, \underline{x} = \sum \alpha_i \underline{v}_i^{(s)}, Re\left(\lambda_i^{(s)}\right) < 0\}$
  rapid contraction
- center eigenspace: $E^{(c)} = \{\underline{x} \,|\, \underline{x} = \sum \alpha_i \underline{v}_i^{(c)}\}, Re(\lambda_i^{(c)}) = 0\}$
  slow dynamics
- unstable eigenspace: $E^{(u)} = \{\underline{x} \,|\, \underline{x} = \sum \alpha_i \underline{v}_i^{(u)}, Re(\lambda_i^{(u)}) > 0\}$
  rapid expansion

Eigenspaces of the linearization of $(0,0)$ in (9,10):

$\mu < 0:$	$E^{(s)} = R^2$	$E^{(c)}$ empty	$E^{(u)}$ empty
$\mu = 0:$	$E^{(s)}$ = y-axis	$E^{(c)}$ = x-axis	$E^{(u)}$ empty
$\mu > 0:$	$E^{(s)}$ = y-axis	$E^{(c)}$ empty	$E^{(u)}$ = x-axis

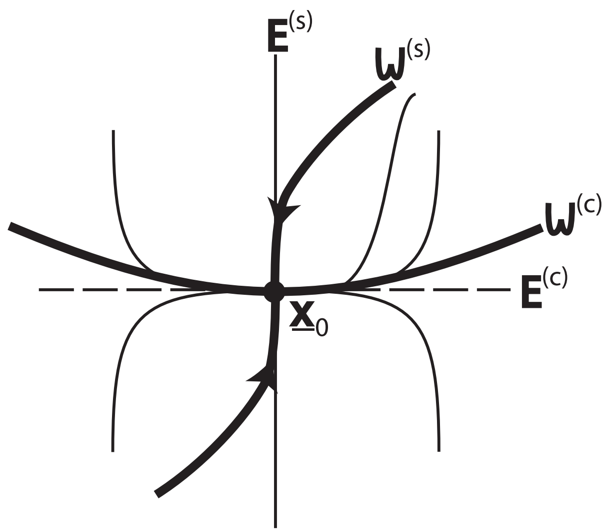
Goal:

- obtain a description of higher-dimensional system in terms of these *slow low-dimensional dynamics*

Extension to nonlinear systems:

Center Manifold Theorem:

- For a fixed point $\underline{x}_0$ with eigenspaces $E^{(s,u,c)}$ there exist stable, unstable, and center manifolds $W^{(s,u,c)}$ such that $W^{(s,u,c)}$ are tangent to $E^{(s,u,c)}$ at $\underline{x}_0$, respectively.
- The manifolds $W^{(s,u,c)}$ are invariant under the flow. $W^{(s)}$ and $W^{(u)}$ are unique. $W^{(c)}$ need not be unique.



Notes:

- To get a mathematically justified separation of the fast decay of the initial transient and the slow evolution thereafter we need an infinite ratio between the respective time scales: we need a center manifold. On it the dynamics are infinitely slower than on the stable manifold.
- To have a center eigenspace and a center manifold we need to be at a bifurcation point.

3.6.2 Reduction to Dynamics on the Center Manifold

We exploit the separation of time scales between the stable manifold and the center manifold and use again multiple time scales.

We introduce a slow time T, which is associated with the slow evolution on the center manifold and reflects the small growth rate stemming from the eigenvalue that has a vanishing real part at the bifurcation point.

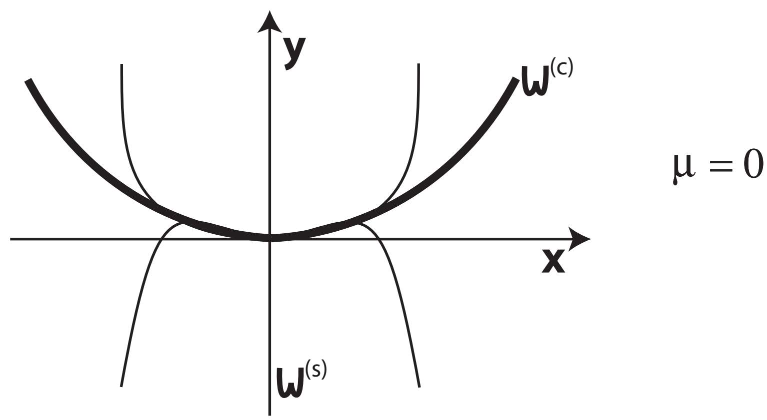
Example from before:

$$\dot{x} = \mu x + xy - \gamma x^3 \tag{11}$$
$$\dot{y} = -y + x^2 \tag{12}$$

Eigenvalues of the linearization around $(0,0)$

$$\lambda_1 = \mu \qquad \lambda_2 = -1$$

Bifurcation occurs at $\mu = 0$.



Near the bifurcation point expand μ

$$\mu = \epsilon^2 \mu_2 \qquad T = \epsilon^2 t$$

Since the bifurcation is a steady bifurcation there is no fast time scale (which was associated with oscillations for the Hopf bifurcation):

$$x = x(T) \qquad y = y(T) \qquad \frac{d}{dt} = \epsilon^2 \frac{\partial}{\partial T}$$

Near the fixed point we can also expand x and y

$$x = \epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3 + \ldots$$
$$y = \epsilon y_1 + \epsilon^2 y_2 + \epsilon^3 y_3 + \ldots$$

Note:

- At this point the scaling of x and y is a guess based on previous experience.
- Eqs.(11,12) are odd in x and even in y. Since the eigenvector associated with $\lambda = \mu$ is in the x-direction one may expect a pitch-fork bifurcation due to the reflection symmetry. This provides guidance for the choice of the scaling of x, y.

Insert the expansion:

$\mathcal{O}(\epsilon)$:

$$\begin{aligned} 0 &= 0 \\ 0 &= -y_1 \end{aligned}$$

thus:

$$y_1 = 0$$

$\mathcal{O}(\epsilon^2)$:

$$\begin{aligned} 0 &= x_1 y_1 \\ 0 &= -y_2 + x_1^2 \end{aligned}$$

thus:

$$y_2 = x_1^2$$

$\mathcal{O}(\epsilon^3)$:

$$\begin{aligned} \frac{\partial x_1}{\partial T} &= \mu_2 x_1 + x_1 y_2 - \gamma x_1^3 \\ \frac{\partial y_1}{\partial T} &= -y_3 + 2x_1 x_2 \end{aligned}$$

thus:

$$y_3 = 2x_1 x_2$$

$$\frac{\partial x_1}{\partial T} = \mu_2 x_1 + (1-\gamma)\, x_1^3 \tag{13}$$

Notes:

- The system undergoes a pitch-fork bifurcation at $\mu = 0$
  - bifurcation is supercritical for $\gamma > 1$
  - bifurcation is subcritical for $\gamma < 1$
- The coefficient of the linear term $\mu_2 x_1$ of the amplitude equation (13) is given by the growth rate λ_1 of the relevant mode (to leading order)
- The center manifold $W^{(c)}$ is given to leading order in x by

$$\begin{aligned} y &= \epsilon y_1 + \epsilon^2 y_2 + \epsilon^3 y_3 + \ldots \\ &= \epsilon^2 x_1^2 + 2\epsilon^3 x_1 x_2 + \ldots \\ &= x^2 + \mathcal{O}(x^3). \end{aligned}$$

As expected, the center manifold is tangent to the center eigenspace spanned by $\mathbf{v}_1 = (1,0)$

In terms of the eigenvectors the solution can be written as

$$\left(\begin{array}{c} x \\ y \end{array}\right) = \epsilon x_1(T) \underbrace{\left(\begin{array}{c} 1 \\ 0 \end{array}\right)}_{\mathbf{v}_1} + \epsilon^2 \left\{ x_1^2(T) \underbrace{\left(\begin{array}{c} 0 \\ 1 \end{array}\right)}_{\mathbf{v}_2} + x_2(T) \underbrace{\left(\begin{array}{c} 1 \\ 0 \end{array}\right)}_{\mathbf{v}_1} \right\} + \mathcal{O}\left(\epsilon^3\right)$$

Note:

- For a general problem

$$\dot{\mathbf{u}} \quad = \quad \mathbf{L}\,\mathbf{u} + \mathbf{N}(\mathbf{u}),$$

  in which $\mathbf{L}$ is a linear operator, i.e. a matrix, and $\mathbf{N}$ contains all the nonlinear terms, one expands the solution efficiently as

$$\mathbf{u} = \epsilon A(T)\mathbf{v_1} + \epsilon^2 \mathbf{u}_2(T) + \ldots$$

  where $\mathbf{v}_1$ is the eigenvector for eigenvalue $\lambda_1 = 0$ of $\mathbf{L}$

$$\mathbf{L}\mathbf{v}_1 = 0.$$

Example:

$$\left(\begin{array}{c} \dot{x} \\ \dot{y} \end{array}\right) = \underbrace{\left(\begin{array}{cc} 1+\mu & -1 \\ 2 & -2 \end{array}\right)}_{\mathbf{L}} \left(\begin{array}{c} x \\ y \end{array}\right) + \left(\begin{array}{c} x^2 \\ 0 \end{array}\right) \tag{14}$$

Linearization around the fixed point $(0,0)$:

eigenvalues and eigenvectors of $\mathbf{L}$

$$trace\,(\mathbf{L}) = -1 + \mu \qquad det\,(\mathbf{L}) = -2\mu$$

$$2\lambda_{1,2} = trace(\mathbf{L}) \pm \sqrt{trace(\mathbf{L})^2 - 4\,det(\mathbf{L})}$$

Possible bifurcations:

- No Hopf bifurcation: it would require $\mu = 1$ to make $trace(\mathbf{L}) = 0$, which makes $det(\mathbf{L})$ negative

- Steady bifurcation for $det(\mathbf{L}) = 0$, i.e. $\mu = 0$

$$\lambda_{1,2} = \frac{1}{2}\left(-1 + \mu \pm \sqrt{(-1+\mu)^2 + 8\mu}\right) = \frac{1}{2}\left(-1 + \mu \pm \sqrt{1 + 6\mu + \mu^2}\right) = \left\{ \begin{array}{c} 2\mu + \mathcal{O}(\mu^2) \\ -1 + \mathcal{O}(\mu) \end{array} \right.$$

  Growth rate positive for $\mu > 0$.
  Eigenvector $\mathbf{v}_1$ at the bifurcation point $\mu = 0$

$$\left(\begin{array}{cc} 1 & -1 \\ 2 & -2 \end{array}\right) \mathbf{v}_1 = 0 \qquad \Rightarrow \qquad \mathbf{v}_1 = \left(\begin{array}{c} 1 \\ 1 \end{array}\right)$$

What kind of bifurcation should we expect?

- $(0,0)$ is a fixed point for all values of μ
- no reflection symmetry due to the term x^2

Therefore we expect a transcritical bifurcation, which is described by an equation of the form

$$\frac{dA}{dT} = a(\mu)A + bA^2$$

with $a(\mu)$ corresponding to the growth rate of the relevant mode: $a(\mu) \sim \lambda_1 \sim \mu$

For our expansion this suggests

$$A \sim a(\mu) \sim \mu$$

Expansion

$$\begin{pmatrix} x \\ y \end{pmatrix} = \epsilon A(T) \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{\mathbf{v}_1} + \epsilon^2 \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} + \ldots$$

with scaling

$$\mu = \epsilon\mu_1 \qquad T = \epsilon t$$

Insert into (14)

$\mathcal{O}(\epsilon)$:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix} A(T)\,\mathbf{v}_1$$

$\mathcal{O}(\epsilon^2)$:

$$-\begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = -\frac{dA}{dT}\mathbf{v}_1 + \begin{pmatrix} \mu_1 x_1 \\ 0 \end{pmatrix} + \begin{pmatrix} x_1^2 \\ 0 \end{pmatrix}$$

We could now again start solving for x_2 and y_2. We expect that a solvability condition will arise.

The solvability condition can be obtained more directly without solving for x_2 and y_2. The matrix **L** has also a left zero-eigenvector $\mathbf{v}^+$ (row eigenvector rather than column eigenvector)

$$\begin{pmatrix} x_0^+, & y_0^+ \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix} = 0 \qquad \Rightarrow \qquad \mathbf{v}_1^+ = \begin{pmatrix} x_0^+, & y_0^+ \end{pmatrix} = \begin{pmatrix} 2, & -1 \end{pmatrix}$$

Note:

- the left eigenvector $\mathbf{v}_1^+$ is not the transpose of the right eigenvector $\mathbf{v}_1^t$
- if the matrix is symmetric then $\mathbf{v}^+ = \mathbf{v}^t$

Multiply the equation at $\mathcal{O}(\epsilon^2)$ by $\mathbf{v}_1^+ = \left(\begin{array}{cc} x_0^+, & y_0^+ \end{array} \right)$

LHS:

$$-\left(\begin{array}{cc} x_0^+, & y_0^+ \end{array} \right) \left(\begin{array}{cc} 1 & -1 \\ 2 & -2 \end{array} \right) \left(\begin{array}{c} x_2 \\ y_2 \end{array} \right) = 0$$

for any x_2 and y_2.

Thus, RHS yields

$$\left(\begin{array}{cc} x_0^+, & y_0^+ \end{array} \right) \left\{ -\frac{dA}{dT} \mathbf{v}_1 + \left(\begin{array}{c} \mu_1 x_1 \\ 0 \end{array} \right) + \left(\begin{array}{c} x_1^2 \\ 0 \end{array} \right) \right\} = 0 \tag{15}$$

$$-\frac{dA}{dT} \left(\begin{array}{cc} 2, & -1 \end{array} \right) \left(\begin{array}{c} 1 \\ 1 \end{array} \right) + 2\left[\mu_1 x_1 + x_1^2 \right] = 0$$

with $x_1 = A$

$$\frac{dA}{dT} = 2\mu_1 A + 2A^2$$

Note:

- This reduction to the center manifold works for systems of arbitrary dimension.
- If the linearization has multiple 0 eigenvalues the center manifold has as many dimensions as there are vanishing eigenvalues and one obtains as many solvability conditions.

Note:

- The solvability condition (15) expresses the *Fredholm Alternative theorem* from linear algebra:
  If the homogeneous linear system of equations
  $$\mathbf{A}\mathbf{x} = 0$$
  has a non-trivial solution $\mathbf{x} \neq 0$, then the inhomogeneous system of equations
  $$\mathbf{A}\mathbf{x} = \mathbf{b}$$
  has a solution if and only if the inhomogeneity satisfies
  $$\mathbf{v} \cdot \mathbf{b} = 0$$
  where $\mathbf{v}$ is a zero-eigenvector of $\mathbf{A}^t$, i.e.
  $$\mathbf{A}^t \mathbf{v} = 0.$$
  Proof: consider the transposed equation
  $$\mathbf{x}^t \mathbf{A}^t = \mathbf{b}^t$$
  and multiply it by $\mathbf{v}$
  $$\mathbf{x}^t \underbrace{\mathbf{A}^t \mathbf{v}}_{=0} = \mathbf{b}^t \mathbf{v} = 0 \qquad \text{i.e.} \qquad \mathbf{b} \cdot \mathbf{v} = 0$$

Formally:

Consider the system

$$\dot{\mathbf{u}} = \mathbf{L}\,\mathbf{u} + \mathbf{N}(\mathbf{u})$$

where **L** is a linear operator, i.e. a matrix, and **N** contains all the nonlinear terms.

Analogous to Hopf: expand for small amplitudes A in the 'direction' of the critical eigenvector $\mathbf{v}_1$of the linearized operator $\mathbf{L}_0$

$$\mathbf{L}_0\mathbf{v}_1 = 0$$

Expand the **u**

$$\mathbf{u} = \epsilon^{\beta} A(T)\,\mathbf{v}_1 + \epsilon^{2\beta}\mathbf{u}_2(T) + \ldots$$

Introduce a slow time

$$T = \epsilon^{\alpha} t$$

and expand the control parameter

$$\mu = \mu_0 + \epsilon^{\gamma}\mu_{\gamma}$$

Since the control parameter variation should appear at the same order as the solvability condition, which in turn should include the slow time derivative one typically has $\gamma = \alpha$.

Since **L** is singular, i.e. the homogeneous linear equation has a non-trivial solution, the higher-order equations can only be solved if a solvability condition is satisfied. This solvability condition can be seen to arise since **L** also has a left-eigenvector with vanishing eigenvalue

$$\mathbf{v}^{+}\mathbf{L} = \mathbf{0}$$

Thus at higher orders one has

$\mathcal{O}(\epsilon^{2\beta})$:

$$-\mathbf{L}_0\mathbf{u}_2 = \mathbf{L}_1\mathbf{u}_1 + \mathbf{N}_2(\mathbf{u}_1)$$

where $\mathbf{L}_1\mathbf{u}_1$ contains terms from expanding the control parameter in ϵ and also from the slow time derivative. Multiply this equation from the left with $\mathbf{v}^{+}$ to obtain the condition

$$0 = -\mathbf{v}^{+}\mathbf{L}_0\mathbf{u}_2 = \mathbf{v}^{+}\left\{\mathbf{L}_1\mathbf{u}_1 + \mathbf{N}_2(\mathbf{u}_1)\right\}$$

3.6.3 Reduction to Center Manifold without Multiple Scales

For $W^{(c)}$ to exist need to be at bifurcation point: $\mu = 0$

$$E^{(c)} = \{(x,0)\}, \qquad E^{(s)} = \{(0,y)\}$$

$\Rightarrow$ write $\underline{x} = (x,y)$ with $y = h(x)$

insert into o.d.e.:

$$\dot{y} = \frac{dh}{dx}\dot{x} = \frac{dh}{dx}(xy - \gamma x^3) \overbrace{=}^{!} -y + x^2 = -h(x) + x^2$$

Thus:

- obtain nonlinear differential equation for $h(x)$
- $W^{(c)}$ tangent to $E^{(c)} \Rightarrow h(x)$ is strictly nonlinear
- local analysis $\Rightarrow$ expand $h(x)$ for small x

Expansion

$$h = h_2x^2 + h_3x^3 + h_4x^4 + \cdots$$

inserted

$$(2h_2x + 3h_3x^2 + \cdots)\{x(h_2x^2 + h_3x^3) - \gamma x^3\} = \\ \underbrace{\overset{!}{=}} -h_2x^2 - h_3x^3 - h_4x^4 + x^2$$

collect:

$$\begin{aligned} \mathcal{O}(x^2) &: \quad 0 = -h_2 + 1 \Rightarrow h_2 = 1 \\ \mathcal{O}(x^3) &: \quad 0 = h_3 \Rightarrow h_3 = 0 \\ \mathcal{O}(x^4) &: \quad 2h_2(h_2 - \gamma) = -h_4 \\ & \quad\;\; h_4 = 2(\gamma - 1) \end{aligned}$$

Thus:

$$y = h(x) = x^2 + 2(\gamma - 1)x^4 + \mathcal{O}(x^5)$$

$$\dot{x} = x(x^2 + 2(\gamma - 1)x^4 + \cdots) - \gamma x^3$$

Evolution equation on center manifold:

$$\dot{x} = (1 - \gamma)x^3 + 2(\gamma - 1)x^5 + \cdots$$

More generally: we want also description for $0 \neq |\mu| \ll 1$

To use center manifold theorem consider **suspended system**

$$\begin{aligned} \dot{\mu} &= 0 \\ \dot{x} &= \mu x + xy - \gamma x^3 \\ \dot{y} &= -y + x^2 \end{aligned}$$

Thus:

- μx is now a nonlinear term
- dynamics in μ-direction is trivial:
  value of μ is simply given by initial condiiton

Now:

$$E^{(c)} = \{(\mu, x, 0)\} \quad * .3in E^{(s)} = \{(0,0,y)\}$$

$$\Rightarrow y = h(\mu, x) \qquad \text{for} \qquad (\mu, x, y) \in W^{(c)}$$

Expand $h(\mu, x)$ in μ and x:
to keep relevant terms in expansion guess relationship $x \Leftrightarrow \mu$ from expected equation on $W^{(c)}$

Symmetries:
Reflections: $(\mu, x, y) \to (\mu, -x, y)$

$\Rightarrow$ expect

$$\begin{aligned} m\dot{x} &= f(\mu, x) \qquad \text{with } f \text{ odd in } x \\ &= a\mu x + bx^3 + \cdots \end{aligned}$$

$\Rightarrow$ expect $\mu \sim \mathcal{O}(x^2)$, h even in x

$$\text{Expand } h(\mu, x) = \underbrace{h_{20}\mu^2}_{\text{higher order}} + \underbrace{h_{11}\mu x}_{\text{wrong symmetry}} + h_{02}x^2 + [h_{12}\mu x^2 + h_{04}x^4] + \ldots$$

Inserted:

$$\begin{aligned} \dot{y} = \frac{dh}{dx}\dot{x} + \frac{dh}{d\mu}\underbrace{\dot{\mu}}_{0} &= (h_{11}\mu + 2h_{02}x + 2h_{12}\mu x + 4h_{04}x^3 + \ldots)(\mu x + x(h_{02}x^2 + \ldots) - \gamma x^3) \\ &= -(h_{20}\mu^2 + h_{11}\mu x + h_{02}x^2 + h_{12}\mu x^2 + \cdots) + x^2 \end{aligned}$$

$$\begin{aligned} \mathcal{O}(\mu^2 x^0): &\quad -h_{20} &&= 0 \\ \mathcal{O}(\mu^1 x^1): &\quad -h_{11} &&= 0 \\ \mathcal{O}(\mu^0 x^2): &\quad 0 &&= -h_{02} + 1 \Rightarrow h_{02} = 1 \\ \mathcal{O}(\mu^1 x^2): &\quad 2h_{02}(1 + h_{10}) &&= -h_{12} \\ \Rightarrow &\quad h_{12} &&= -2 \\ \mathcal{O}(x^4): &\quad -2h_{02}\gamma + 2h_{02}^2 &&= -h_{04} \\ &\quad h_{04} &&= 2(1-\gamma) \end{aligned}$$

$$\begin{aligned} my &= x^2 - 2\mu x^2 + 2(1-\gamma)x^4 \\ \dot{x} &= \mu x + x(x^2 - 2\mu x^2 + 2(1-\gamma)x^4) - \gamma x^3 \end{aligned}$$

Evolution on center manifold

$$\dot{x} = \mu x - (\gamma - 1 + 2\mu)x^3 + \left[2(1-\gamma)x^5 + \ldots\right]$$

Thus:

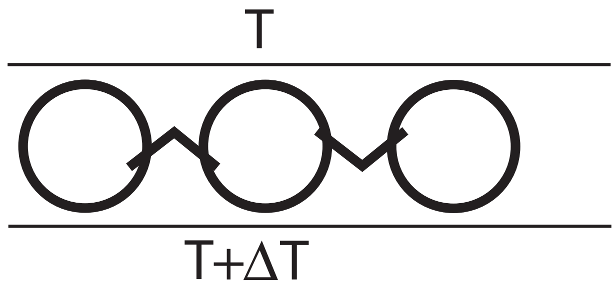
- For $\gamma > 1$ supercritical pitchfork bifurcation
  For $\gamma < 1$ subcritical pitchfork bifurcation

3.7 Global Bifurcations

4 Chaos

4.1 Lorenz Model

Lorenz considered a minimal model for thermal convection



In two dimensions the fluid velocity v of an incompressible fluid can be expressed in terms of a stream function $(v_x, v_z) = (-\partial_z\psi, \partial_x\psi)$.

The stream function ψ and the temperature T satisfy the coupled Navier-Stokes equation and the heat equation.

The stream function was approximated as

$$\psi = 2\sqrt{6}\,X(t)\,\cos\pi z \sin\left(\frac{\pi}{\sqrt{2}}x\right)$$

The temperature profile of the layer was approximated as

$$T(x,z,t) = \underbrace{-rz}_{\text{basic profile}} + \underbrace{9\pi^3\sqrt{3}\,Y(t)\,\cos\pi z\cos\left(\frac{\pi}{\sqrt{2}}x\right)}_{\text{critical mode}} + \underbrace{\frac{27\pi^3}{4}Z(t)\,\sin 2\pi z}_{\text{harmonic mode}}$$

The Rayleigh number r characterizes the temperature difference across the layer.

To obtain differential equations for the three amplitude $X(t)$, $Y(T)$, $Z(t)$ this ansatz was inserted into the Navier-Stokes equations. Keeping only terms of the form used in the ansatz yields then

$$\begin{aligned}\dot{X} &= -\sigma(X-Y)\\ \dot{Y} &= rX - Y - XZ\\ \dot{Z} &= XY - bZ\end{aligned}$$

Notes:

- The model constitutes a severe truncation of a Galerkin expansion for free-slip boundary conditions. It may be expected to give reasonable results for weak convection. But, in contrast to the center-manifold reduction, this procedure does not represent a systematic expansion.

- The Prandtl number σ is given by the ratio of viscosity to thermal diffusivity.
- The parameter b is related to the wavenumber of the convection pattern.

Demos: Very nice Java programs by M. Cross (Caltech) at
`http://www.cmp.caltech.edu/%7emcc/Chaos_Course/Lesson1/Demos.html`

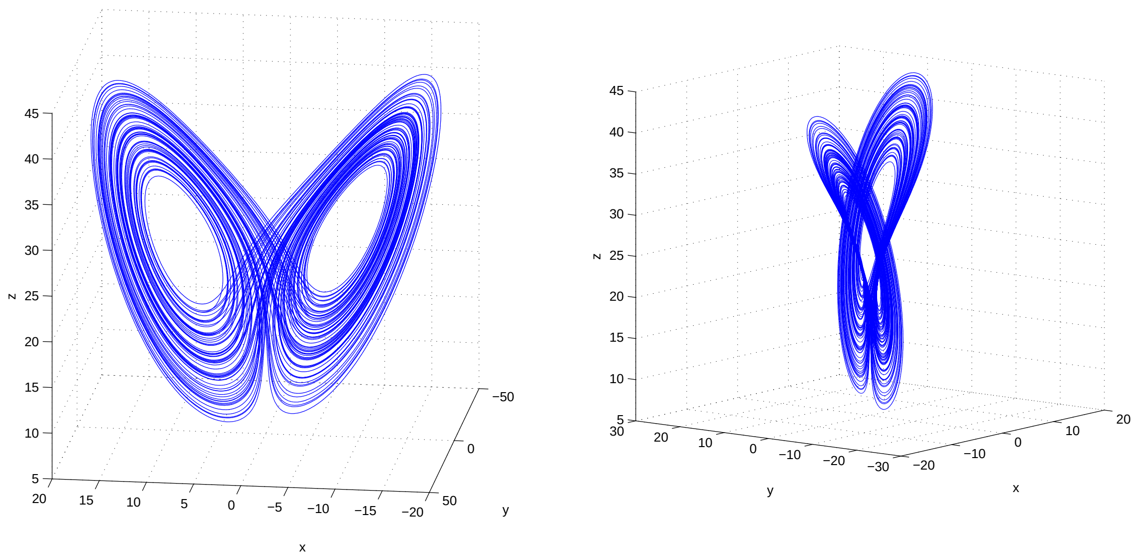
Demo: **Lorenz Attractor**

[12] increase r ($= a$ in Cross program; $\sigma = c$):

- 0.5 origin is stable fixed point (use initial condition $X_0 = 4.85$, $Y_0 = 5$, $Z_0 = 23.5$ and set $trans = 0$ (i.e. plot all transients, plot X and Z)
- 1.2 origin is unstable, new fixed point (symmetry-related fixed points, in fact)
- 4 this initial condition goes to the other fixed point
- 10 fixed point is clearly a stable spiral point.
- 24 the fixed point is still linearly stable spiral point
- 24.4 still attracting for this initial condition
- 24.8 fixed point unstable spiraling outward, but growth still slow. Hopf bifurcation
- 25 transition to strange attractor.
  transitions occur at: $r = 1$, $r_H = 24.739$
- 3d plot of attractor
- $X(t)$ (choose variable =0 in $x - ax$)

For $r > r_H$ one gets an attractor that looks very different than the attractors we had before (fixed points and periodic orbits): it is a *strange attractor.*

[12] go to list of topics first; (if program does not plot right away: make sure speed is below 500. may have to reload java program or click at Lorenz

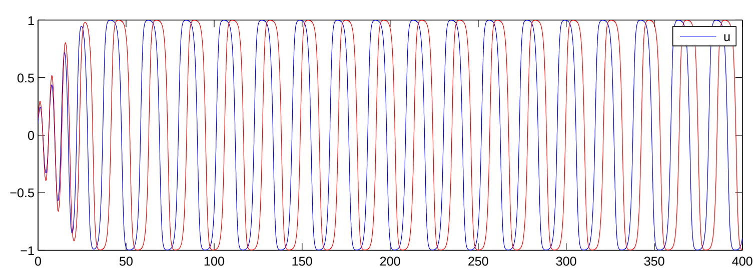


Demo: **Sensitive dependence on initial conditions**

first simulation of Hopf oscillator from the homework: tb_multi.

For stable periodic orbit:

- different initial conditions evolve to periodic orbits that are shifted with respect to each other in time
- the phase shift reaches a constant after a transient: the distance between the two oscillations does not grow or shrink
  $\Rightarrow$ small perturbations in the initial conditions lead only to small chanages in the solution at any later time.



For Lorenz attractor simulation with $x_0 = 2$ $y_0 = 5$ $z_0 = 20$ and $z_0 = 20 + \Delta z$
$r(= a) = 28$ $\sigma(= c) = 10$ $b = 8/3$ $\Delta z = 10^{-3}$ 10^{-5} 10^{-7}
$x - z$ plot: the two trajectories separate ever further as time progresses.
count the number of periods over which the two trajectories are sort of in sync

- $\Delta z = 10^{-3}$ ca 15
- $\Delta z = 10^{-5}$ca 22

- $\Delta z = 10^{-7}$ ca. 26
- $\Delta z = 10^{-9}$ ca 35
- $\Delta z = 10^{-11}$ ca 44

looks logarithmic in the initial difference: ca 4 periods per decadee

Note:

- even extremely small perturbations in the initial conditions can lead to large changes in the solution after some time, which is not even very long

4.1.1 Simple Properties of the Lorenz Model

i) Reflection symmetry:

The equations are *equivariant* under

$$(X, Y, Z) \to (-X, -Y, Z)$$

i.e. the left- and right-hand sides of the equations are transformed in the same way under this operation: the equations for $\dot{X}$ and $\dot{Y}$ change sign, while that for $\dot{Z}$ stays the same. This symmetry is reflected in the appearance of two symmetrically related fixed points near $r = 1$.

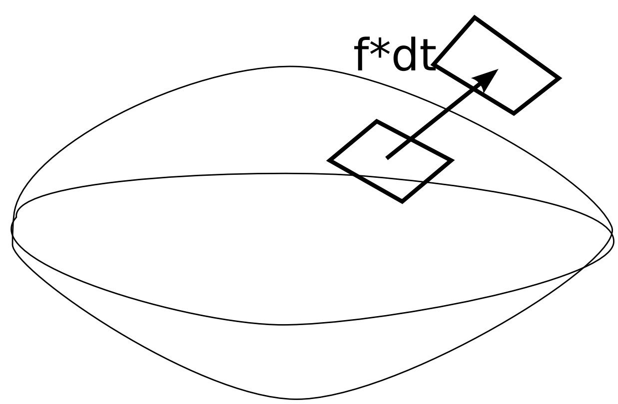
ii) Volume contraction

The Lorenz system is *dissipative*, i.e. volumes in phase space shrink under the evolution of the system.

Consider general dynamical system in 3 dimensions

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

Consider a volume $V(t)$ in phase space with the closed surface $S(t)$



During a time interval dt

- each point $\mathbf{x}$ on the surface moves from $\mathbf{x}(t)$ to $\mathbf{x}(t+dt) = \mathbf{x}(t) + \mathbf{f}(\mathbf{x}(t))\mathbf{dt}$
- any patch dS of the surface sweeps out a volume $dS\,\mathbf{n}\cdot\mathbf{f}\,dt$

The volume $V(t)$ changes therefore by

$$dV = dt \int_S \mathbf{n}\cdot\mathbf{f}\,dS$$

Using the divergence theorem

$$\frac{dV}{dt} = \int_S \mathbf{n}\cdot\mathbf{f}\,dS = \int_V \nabla\cdot\mathbf{f}\,dV$$

Note:

- This expression holds in arbitrary dimensions.

For the Lorenz equations we have

$$\begin{aligned} \nabla\cdot\mathbf{f} &= \frac{\partial}{\partial X}\left\{-\sigma\left(X-Y\right)\right\} + \frac{\partial}{\partial Y}\left\{rX - Y - ZX\right\} + \frac{\partial}{\partial Z}\left\{b\left(XY - Z\right)\right\} \\ &= -\sigma - 1 - b < 0 \end{aligned}$$

Thus:

- *all* volumes in phase space shrink under the evolution of the Lorenz equations
- since $\nabla\cdot\mathbf{f}$ is constant for the Lorenz equations all volumes decrease exponentially and with the same rate

Note:

- since $\frac{dV}{dt} < 0$ everywhere there can be no fixed points with only unstable directions: all unstable fixed points have to have at least one stable direction (saddles)

iii) Fixed Points and Bifurcations

The origin $(0,0,0)$ is a fixed point.

Linear stability

$$\begin{pmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{pmatrix} = \begin{pmatrix} -\sigma\left(X-Y\right) \\ rX - Y \\ -bZ \end{pmatrix} = \begin{pmatrix} -\sigma & +\sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

Eigenvalues

$$\lambda_3 = -b$$

For $\lambda_{1,2}$

$$trace = -\left(\sigma + 1\right) \qquad det = \sigma\left(1 - r\right)$$

- Since $trace < 0$ this fixed point cannot undergo a Hopf bifurcation
- Real eigenvalue goes through 0 for $r = 1$: steady bifurcation
  A $r = 1$

$$\lambda_1 = 0 \qquad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

The center eigenspace is spanned by $\mathbf{v}_1$. The combined reflection $x \to -x$ and $y \to -y$ flips also the sign of $\mathbf{v}_1$. This symmetry suggests that the bifurcation is a pitch-fork bifurcation. It creates the fixed points $(\pm X_0, \pm Y_0, Z_0)$ with

$$Y_0 = X_0 \qquad Z_0 = \frac{1}{b} X_0^2 = r - 1$$

One can show that this fixed point is stable for

$$1 < r < r_H = \frac{\sigma\,(\sigma + b + 3)}{\sigma - b - 1} \qquad \text{for } \sigma - b - 1 > 0$$

and undergoes a Hopf bifurcation at $r = r_H$.

Notes:

- For the Lyapunov function $V(X,Y,Z) = \frac{1}{\sigma}X^2 + Y^2 + Z^2$ one can show that $\frac{dV}{dt} < 0$ if $(X,Y,Z) \neq (0,0,0) \to (0,0,0)$ is globally attractive for $r < 1$.
- for the standard parameter set $r_H = 24.737$
- the numerical simulations indicate that the Hopf bifurcation is subcritical
- the three fixed points $(0,0,0)$ and $(\pm X_0, \pm Y_0, Z_0)$ are the only fixed points of the Lorenz equations

For $r > r_H$ we have so far

- no stable fixed points
- no stable small-amplitude periodic orbit
- trajectories remain confined to some region
- phase space volume decreases monotonically
- instability of the origin pushes trajectories apart along the x-axis
- trajectories pushed down along z-axis.

4.1.2 Lyapunov Exponents

Characterize the sensitive dependence on initial conditions

Consider the dynamical system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

and two trajectories that start very close to each other at $t = 0$

$$\mathbf{x}(t; \mathbf{x}_0) \qquad \mathbf{x}(t; \mathbf{x}_0 + \delta\mathbf{x}_0)$$

At later times the two trajectories are separated by $\Delta x(t)$,

$$\Delta\mathbf{x}(t) = \mathbf{x}(t; \mathbf{x}_0 + \delta\mathbf{x}_0) - \mathbf{x}(t; \mathbf{x}_0)$$

Measure the distance between these trajectories

$$\|\Delta\mathbf{x}(t)\| = \|\mathbf{x}(t; \mathbf{x}_0 + \delta\mathbf{x}_0) - \mathbf{x}(t; \mathbf{x}_0)\|$$

If $\Delta\mathbf{x}(t)$ is very small one can linearize the differential equation around $\mathbf{x}(t; \mathbf{x}_0, t_0)$ at the time t

$$\begin{aligned} \frac{d}{dt}\Delta\mathbf{x}(t) &= \mathbf{f}\left(\mathbf{x}(t; \mathbf{x}_0 + \delta\mathbf{x}_0)\right) - \mathbf{f}\left(\mathbf{x}(t; \mathbf{x}_0)\right) \\ &= \mathbf{f}\left(\mathbf{x}(t; \mathbf{x}_0) + \Delta\mathbf{x}(t)\right) - \mathbf{f}\left(\mathbf{x}(t; \mathbf{x}_0)\right) \\ &\approx \mathbf{J}\left(\mathbf{x}(t; \mathbf{x}_0)\right)\, \boldsymbol{\Delta}\mathbf{x}(t) \end{aligned}$$

where $\mathbf{J}\left(\mathbf{x}(t; \mathbf{x}_0)\right)$ is the Jacobian of $\mathbf{f}(\mathbf{x})$ at time t and position $\mathbf{x}(t)$.

Notes:

- If the Jaocbian was constant in time one would expect $\Delta\mathbf{x}$ to grow or decay exponentially
- In general the Jacobian depends on time through the position $\mathbf{x}(t)$ on the attractor
- Over long times $\mathbf{x}(t)$ the explores the whole attractor and one could imagine that the growth of $\Delta\mathbf{x}$ is determined by something like an 'average Jacobian'.

Numerically one finds in the limit of long times and small initial distance

$$\|\Delta\mathbf{x}(t)\| \sim \|\Delta\mathbf{x}(0)\|\, e^{\lambda t}$$

Motivated by this observation one defines

$$\lambda \equiv \lim_{t\to\infty} \lim_{\|\Delta\mathbf{x}(0)\|\to 0} \frac{1}{t} \ln\left(\frac{\|\Delta\mathbf{x}(t)\|}{\|\Delta\mathbf{x}(0)\|}\right)$$

Notes:

- to determine λ one considers first the limit of an infinitesimal perturbation, $\Delta\mathbf{x}(0) \to 0$, in order to stay in the regime in which the linearization is o.k., and then one studies how the perturbation evolves for long times

- λ is a Lyapunov exponent of the system
  - more precisely, this λ is the largest Lyapunov exponent, which dominates $\|\Delta \mathbf{x}(t)\|$ for large t
  - in an N-dimensional sytem there are N Lyapunov exponents, corresponding to the N dimensions of $\Delta \mathbf{x}(0)$
- in principle, λ depends on the specific trajectory $\mathbf{x}(t; \mathbf{x}_0)$ and one needs to average over multiple trajectories on the attractor
- for systems with time-independent coefficients: time translation symmetry
  - if $\mathbf{x}(t)$ is a solution if $\mathbf{x}(t + \Delta t)$ is a solution for any Δt.
  - perturbations along the attractor do not grow or shrink: $\lambda = 0$ (cf. evolution of perturbations for stable periodic orbits)
- the Lorenz system has 3 Lyapunov exponents. For the strange attractor one has

$$\lambda_1 > 0 \qquad \lambda_2 = 0 \qquad \lambda_3 < 0$$

  The volume contraction is determined by the sum of the three Lyapunov exponents

Time Horizon:

The exponential growth of the difference between nearby trajectories limits predictions severely

Assume we can measure the initial condition with a precision $\delta_0 = \|\Delta \mathbf{x}(0)\|$. If we need to make a prediction with an accuracy δ_{max}, i.e. we require $\|\Delta \mathbf{x}(t)\| < \delta_{max}$, then we can predict the system up to a time t_h

$$\delta_{max} = \|\Delta \mathbf{x}(t_h)\| = \|\Delta \mathbf{x}(0)\| \, e^{\lambda t_h}$$

i.e.

$$t_h(\delta_0) = \frac{1}{\lambda} \ln \left(\frac{\delta_{max}}{\delta_0} \right)$$

Note:

- the time horizon t_h grows only logarithmically with the precision δ_0 of our knowledge of the initial condition. This matches our simulations where we found that each increase in the precision by a factor of 10 increased the time over which the two trajectories stayed close to each other only by roughly 4 periods.
  to increase the prediction time from 22 periods to 44 periods we had to increase the precision by a factor of 10^6.

Characterization of Chaos:

Chaos is aperiodic, long-term behavior in a deterministic system that exhibits sensitive dependence on initial conditions.

- aperiodic: trajectories do not settle down to a fixed point, periodic orbit, or quasi-periodic orbit
- sensitive dependence: distances between trajectories grow exponentially fast: positive Lyapunov exponent

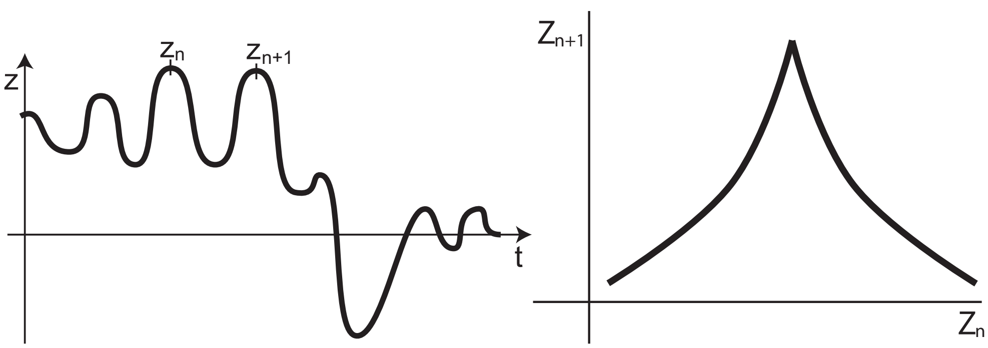
Quasi-periodic motion:

Consider two uncoupled harmonic oscillators, e.g. mass-spring system in the x-direction to which a pendulum is attached that swings only in the $y-z$-plane.

- if both have the same frequency the pendulum tip traces out a circle
- if the frequencies are different the trace is a Lissajous figure
  - for rational ratios of the two frequencies the traces close on themselves: periodic motion
  - for irrational frequency ratios the figures never close: quasi-periodic motion

Question: Can one understand the complex dynamics of the Lorenz equations and similar systems in simpler models yet?

Lorenz reduced the three-dimensional flow to a one-dimensional iterated map by asking: can we predict the next maximum of the variable z, z_{n+1}, if we only know the previous maximum z_n. He therefore plotted z_{n+1} vs. z_n and obtained a map.



Demo:

- $Z(t)$ (x-ax =0 y-ax = 3)
- keep only maxima of Z and plot Z_{n+1}vs Z_n. (demo max map)

Notes:

- The line in that map is actually not a line, but has finite thickness
  For the Lorenz model the line thickness is, however, small ⇒ knowledge of Z_n is sufficient to predict Z_{n+1} quite reliably.
- the reduction to a map is only approximate:
  the original ode's can also be solved backward
  the map can, however, not be iterated backward since $f^{-1}(z)$ is multiple valued

4.2 One-Dimensional Maps

Consider maps as dynamical systems

$$x_{n+1} = f(x_n)$$

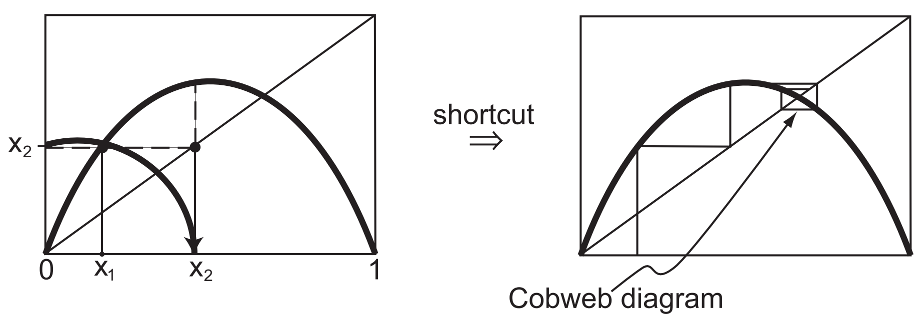
Example: logistic map

$$x_{n+1} = ax_n(1 - x_n) \qquad 0 \leq x_n \leq 1$$

Notes:

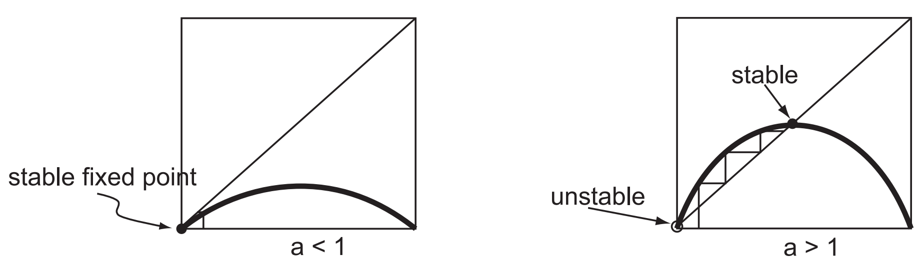
- this map could be thought of as a (very poor) numerical solution of the logistic differential equation (using forward Euler with large time step).
- the logistic map has a maximum at $x = \frac{1}{2}$. It maps $[0, 1]$ onto $[0, \frac{1}{4}a]$. For $0 \leq a \leq 4$ the x_n remain in the interval $[0, 1]$.

Graphical iteration via the cobweb diagram:



$x^{(0)} \equiv 0$ is a fixed point for all values of a

Vary a:



For $a = 1$ an additional fixed point appears

$$x = ax - ax^2 \qquad \Rightarrow \qquad 0 = x\,(a - 1 - ax)$$
$$x^{(1)} = \frac{a-1}{a}$$

Linear stability analysis:

linearize around a fixed point x_f

$$\begin{aligned} x_n &= x_f + \epsilon\Delta x_n \\ x_f + \epsilon\Delta x_{n+1} &= f(x_f + \epsilon\Delta x_n) = f(x_f) + \epsilon\Delta x_n f'(x_f) \\ \Rightarrow \Delta x_{n+1} &= \Delta x_n f'(x_f) \end{aligned}$$

$\Rightarrow$ $|\Delta x_n|$ grows for $|f'(x_f)| > 1$ $|\Delta x_n|$ decays for $|f'(x_f)| < 1$

Thus:

- The stability limits for maps are given by $|f'(x_f)| = 1$
- For comparison: for one-dimensional flows the stability limit is given by $f'(x_f) = 0$.
- In the maxima map of the Lorenz model $|f'(z)| > 1$ for all z: the fixed point (=periodic orbit) is unstable

As expected from the graph: at $a = 1$ the fixed point $x^{(0)}$ becomes linearly unstable.

Stability of the fixed point $x^{(1)} = \frac{a-1}{a}$:

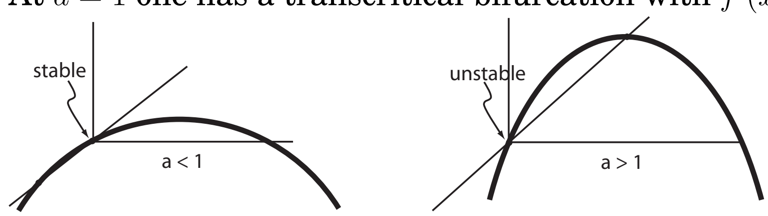
$$f'(x^{(1)}) = a - 2ax^{(1)} = a - 2(a-1) = 2 - a$$

Thus

$$|f'(x_1)| < 1 \qquad \text{for} \qquad 1 < a < 3$$

Note:

- At $a = 1$ one has a transcritical bifurcation with $f'(x^{(1)}) = +1$.



- At $a = 3$ one has $f'(x^{(1)}) = -1$

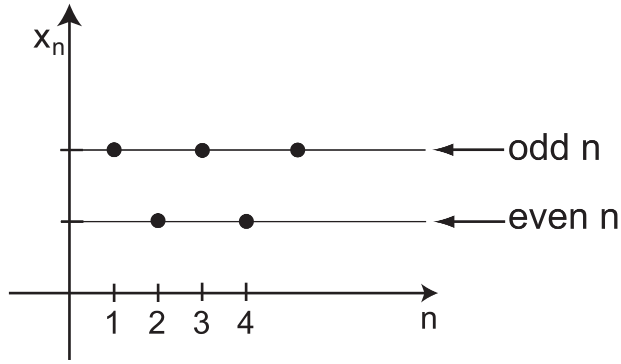
$$\Delta x_{n+1} = -\Delta x_n$$

suggests that the solution jumps back and forth

Demo: what happens at the bifurcation at $a = 3$? [13]

$\Rightarrow$ converges to period-2 solution

[13] Cross Demo 1: take $a = 3.1$, initial condition $x = 0.68$



[14]

Determine period-2 solution:
Period 2: fixed point under second iterate of $f(x)$

$$\begin{aligned} x_{n+2} &= f(x_{n+1}) = f\left(f(x_n)\right) \equiv f^{(2)}(x_n) \\ &= a x_{n+1}(1-x_{n+1}) = a\left(a\,x_n(1-x_n)\right)\left(1-a\,x_n(1-x_n)\right) \end{aligned}$$

Fixed point of $f^{(2)}: x_{n+2} = x_n$

$$x^{(2)} = f^{(2)}(x^{(2)})$$

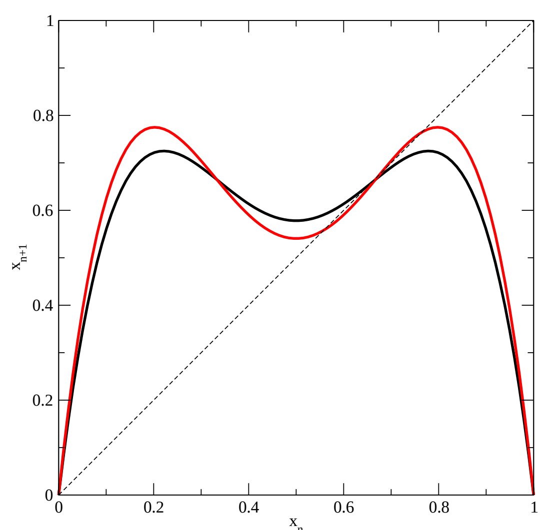
Fixed points of the first iterate $f(x)$ itself are also fixed points of the second iterate. Therefore the fixed point condition can be factored as

$$\underbrace{-x(xa+1-a)}_{\text{known fixed points}} \quad (a^2x^2 - a(1+a)x + 1 + a) = 0$$

Thus one gets two new fixed points of the second iterate $f^{(2)}$

$$x^{(2)}_{1,2} = \frac{1}{2a}\left\{1 + a \pm \sqrt{a^2 - 2a - 3}\right\}$$

They arise in a pitch-fork bifurcation and exist for $a > 3$. [15]
The fixed points of $f^{(2)}$ correspond to a period-2 orbit alternating $x_1^{(2)} \leftrightarrow x_2^{(2)}$



[14]Demo: logmap.m for plotting x_n as a function of n
[15]Cross Demo: use second iterate to see pitch-fork bifurcation

Stability of the period-2 orbit $x_{1,2}^{(2)}$: consider the stability of the fixed points of the second iterate

$$\begin{aligned}
x_{1,2}^{(2)} + \Delta x_{n+2} &= f\left(f(x_{1,2}^{(2)} + \Delta x_n)\right) \\
\Delta x_{n+2} &= \underbrace{\frac{d}{dx}\left(f\left(f(x)\right)\right)\Big|_{x_{1,2}^{(2)}}}_{\lambda} \Delta x_n \\
\lambda &= f'\left(f(x_{1,2}^{(2)})\right) f'(x_{1,2}^{(2)})
\end{aligned}$$

With

$$f(x_{1,2}^{(2)}) = x_{2,1}^{(2)}$$

we get

$$\begin{aligned}
\lambda &= f'\left(x_1^{(2)}\right) f'\left(x_2^{(2)}\right) = a^2\left(1 - 2x_1^{(2)}\right)\left(1 - 2x_2^{(2)}\right) \\
&= a^2\left(1 - 2\left(x_1^{(2)} + x_2^{(2)}\right) + 4x_1^{(2)}x_2^{(2)}\right) \\
&= a^2 - 2a\,(1+a) + (1+a)^2 - \left(a^2 - 2a - 3\right) \\
&= -a^2 + 2a + 4
\end{aligned}$$

The stability limits are given by

$$\begin{aligned}
\lambda_1 &= +1 \qquad \Rightarrow \qquad a = 3 \\
\lambda_2 &= -1 \qquad \Rightarrow \qquad a = 1 + \sqrt{6} \approx 3.4495
\end{aligned}$$

Notes:

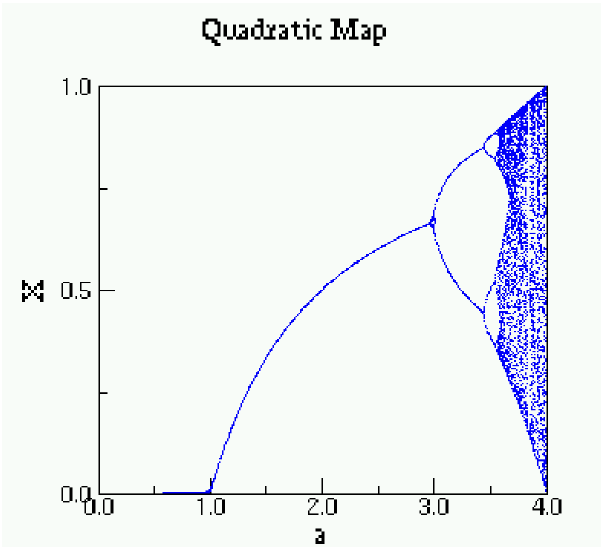
- At $a = 3$ the fixed point of $f^{(1)}(x)$ undergoes a period-doubling bifurcation and the eigenvalue of its linearization is $\lambda^{(1)} = -1$. The eigenvalue of the linearization of $f^{(2)}(x)$ at $a = 3$ is

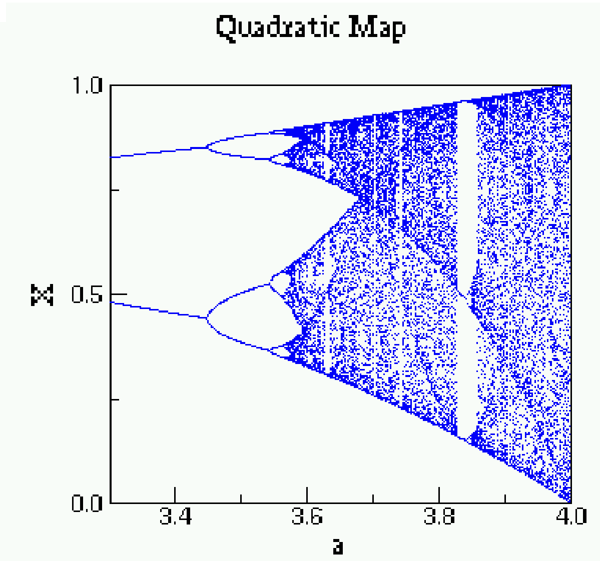
$$\lambda^{(2)} = 1 = f'\left(\underbrace{f(x^{(1)})}_{x^{(1)}}\right) f'\left(x^{(1)}\right) = \lambda^{(1)}\lambda^{(1)}$$

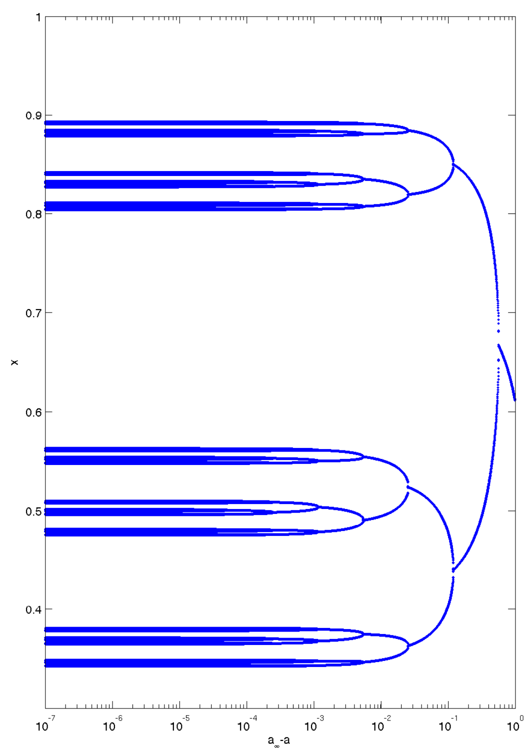
- For $a = 1+\sqrt{6}$ the fixed point of $f^{(2)}$ becomes unstable in a period-doubling bifurcation: birth of a period-4 orbit.

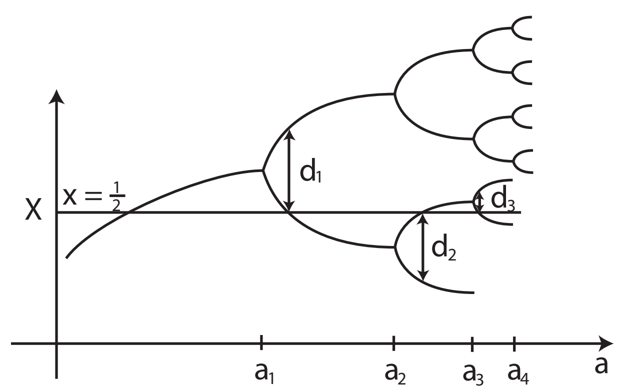
[16]

[16]Demo: succcessive bifurcations, $a = 0..4$, $a = 3.3..4$
zoom in with left mouse zoom out in stopped plot with mouse outside figure. Cross Demo 3









Period-Doubling Cascade:

- There is an infinite number of period-doubling bifurcations, which accumulate at

$$a_\infty = 3.569945672\ldots$$

- With each bifurcation the period doubles: at a_∞ the period is infinite and dynamics are not periodic any more.

- The distance between successive bifurcations becomes smaller with each bifurcation. Their ratio approaches a fixed value

$$\lim_{n\to\infty} \frac{a_n - a_{n-1}}{a_{n+1} - a_n} = \delta = 4.6692016091029909...$$

  The constant δ is called the Feigenbaum constant, after M. Feigenbaum who discovered the scaling and showed that it is universal for all maps with a single maximum. In terms of a_∞ this scaling can be written as

$$a_n = a_\infty + Ae^{-\delta' n}$$

  with

$$\delta = \frac{1 - e^{\delta'}}{e^{-\delta'} - 1} = e^{\delta'} \qquad \delta' = 1.54098809542$$

- The width of the bifurcation also gets smaller in each bifurcation. The ratio of the widths, measured when one branch intersects $x_m = \frac{1}{2}$, also approaches a fixed value

$$\lim_{n\to\infty} \frac{d_n}{d_{n+1}} = \alpha = -2.5029...$$

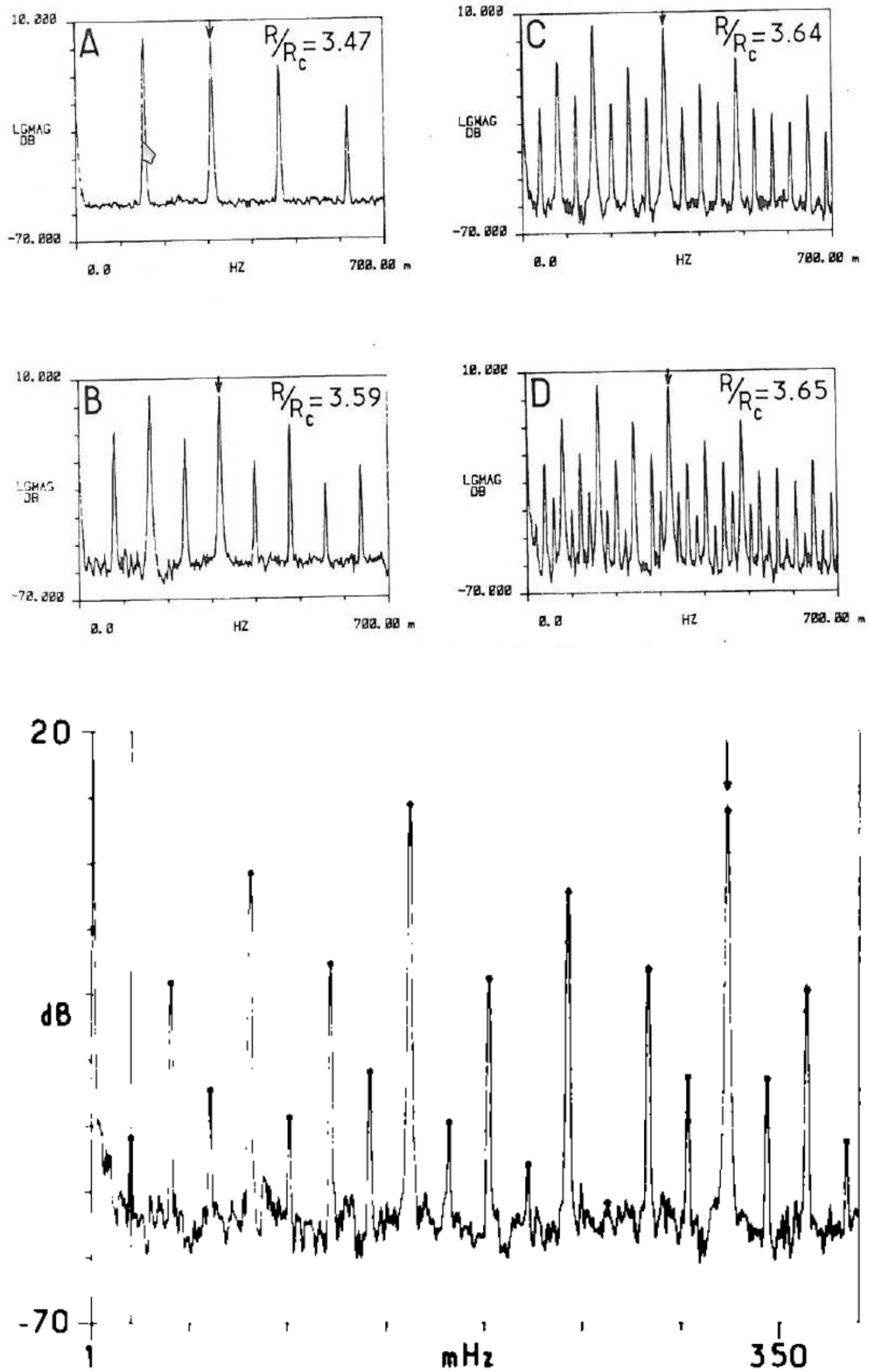
  The minus sign indicates that successive bifurcations are on opposite sides of the mid point $x_m = \frac{1}{2}$.

- The approach to fixed fixed ratios suggests self-similarity as a_∞ is approached: zooming in yields a picture that looks essentially the same again, the same structures repeat on smaller scales

- The self-similarity can be described with a *renormalization theory*. It shows that period-doubling cascades in *all* maps with a quadratic maximum have the same *universal behavior*. i.e. they all have the same values of the Feigenbaum constant δ and of α.

Among the early experimental observations of the period-doubling cascade to chaos and the measurement of the Feigenbaum constant were experiments on Rayleigh-Benard convection of mercury [**?**]:

$$\delta = 4.4 \pm 0.1$$



a) b)

Figure 13: Period doubling cascade observed in convection of mercury with a horizontal magnetic field. The arrow in b) marks the fundamental frequency in the power spectrum. Period 32 is just about visible in b). [?, ?]

17

[17]Demo: logmap_scan.m $a_{min=3.5}$, $a_{max} = a_{inf} - .1e - 10$, 10^6 points , $a_\infty = 3.569945672$. Self-similarity zoom in $a - a_{\rm inf} = \mathcal{O}(10^{-6})$

Chaotic Dynamics:

For $a > a_\infty$ the dynamics are chaotic and exhibit sensitive dependence on initial conditions.

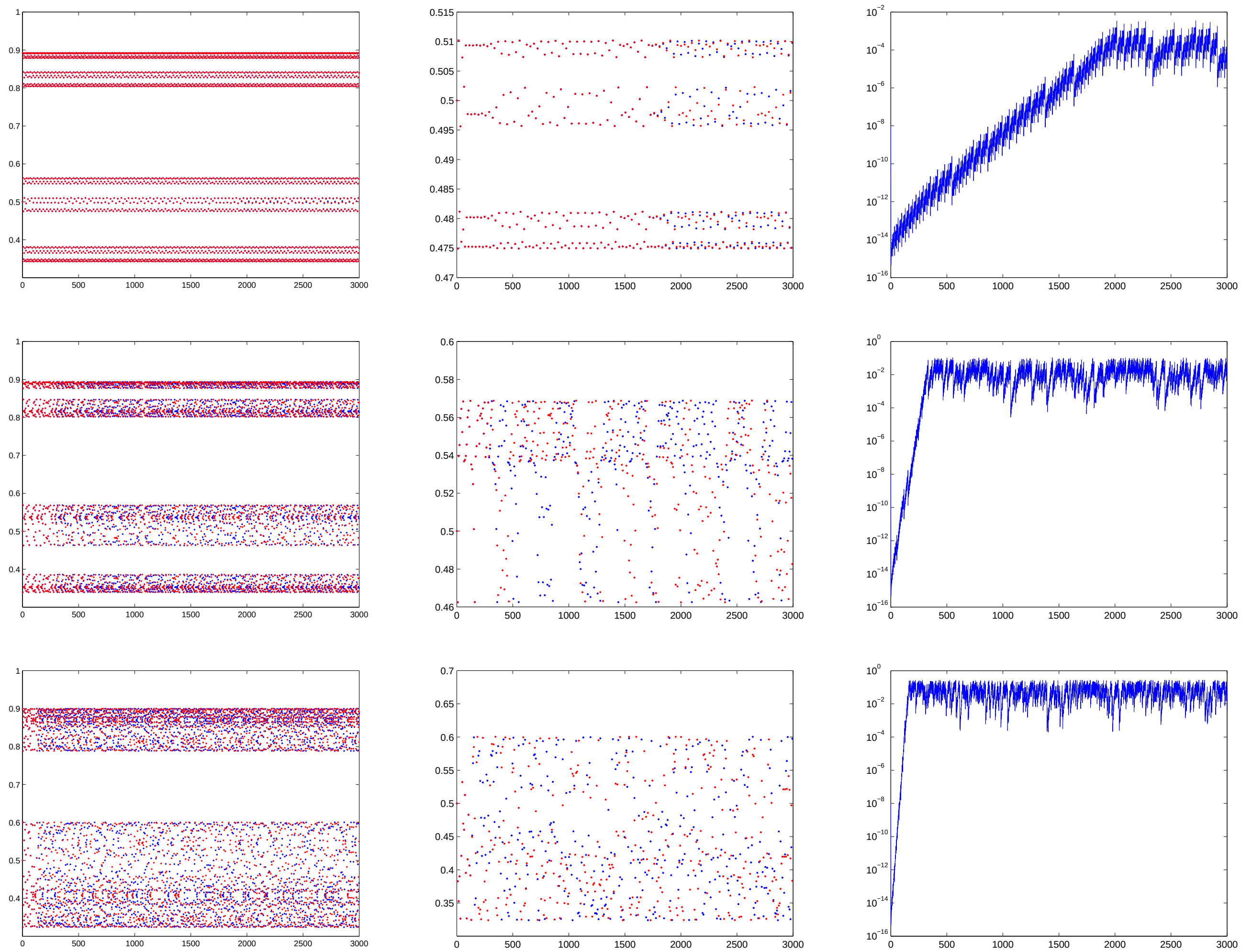


Figure 14: Sensitive dependence on initial conditions. Left panel: orbit for two slightly different initial conditions, $x_0 = 0.5$ (red) and $x_0 = 0.5 + 10^{-8}$ (blue). Middle panel: zoomed in showing the orbits for every 8th iteration. Right panel: growth of the difference between orbits. a) $a = 3.569994$: chaotic period-32 band with the period 32 still clearly visible. b) $a = 3.5753$: close to the merging of the period-8 and period-4 band. c) $a = 3.6$: chaotic period-2 band.

In analogy to the Lyapunov exponents for flows introduce Lyapunov exponents for maps

$$\lambda = \lim_{n\to\infty} \lim_{\Delta x_0 \to 0} \frac{1}{n} \ln \left| \frac{\Delta x_n}{\Delta x_0} \right|$$

Use

$$\begin{aligned}
\Delta x_n &= f^{(n)}\left(x_0+\Delta x_0\right)-f^{(n)}\left(x_0\right) \\
&= \left.\frac{df^{(n)}}{dx}\right|_{x_0} \Delta x_0=\prod_{j=1}^{n} f'\left(x_j\right) \Delta x_0
\end{aligned}$$

with

$$x_j=f^{(j)}\left(x_0\right)$$

Thus

$$\lambda=\lim_{n\rightarrow\infty}\frac{1}{n}\ln\left|\prod_{j=1}^{n} f'\left(x_j\right)\right|=\lim_{n\rightarrow\infty}\frac{1}{n}\sum_{j=1}^{n}\ln\left|f'(x_j)\right|$$

For a periodic orbit with period p we get

$$\lambda=\frac{1}{p}\ln\left|\prod_{j=1}^{p} f'\left(x_j\right)\right|=\frac{1}{p}\ln\left|\left.\frac{df^{(n)}}{dx}\right|_{x_0}\right|$$

Superstable orbits:

If $\frac{df^{(n)}(x)}{dx}=0$ then $\lambda\rightarrow-\infty$: small perturbations of such orbits decay extremely fast.

Consider for simplicity a superstable fixed point x_{ss}

$$\begin{aligned}
\Delta x_n &= \underbrace{f'(x_{ss})}_{0}\,\Delta x_{n-1}+\frac{1}{2}f''(x_{ss})\,\Delta x_{n-1}^2+\mathcal{O}\left(\Delta x_n^3\right) \\
&= \frac{1}{2}f''(x_{ss})\,\left(\frac{1}{2}f''(x_{ss})\,\Delta x_{n-2}^2\right)^2=\frac{1}{2}f'(x_{ss})\,\left[\frac{1}{2}f''(x_{ss})\,\left(\frac{1}{2}f''(x_{ss})\Delta x_{n-3}^2\right)^2\right]^2 \\
&= \left(\frac{1}{2}f''(x_{ss})\right)^{\sum_{j=0}^{n-1}2^j}\left(\Delta x_0\right)^{(2^n)}=\left(\frac{1}{2}f''(x_{ss})\right)^{2^n-1}\left(\Delta x_0\right)^{(2^n)} \\
&= \left(\frac{1}{2}f''(x_{ss})\right)^{-1}\left(\frac{1}{2}f''(x_{ss})\,\Delta x_0\right)^{(2^n)}
\end{aligned}$$

This decay is much faster than exponential. For comparison

$$\frac{\Delta x_{n+1}}{\Delta x_n}=\begin{cases} c & \text{for exponential decay to stable fixed point} \\ \propto \Delta x_n & \text{for convergence to superstable fixed point}\end{cases}$$

e.g.

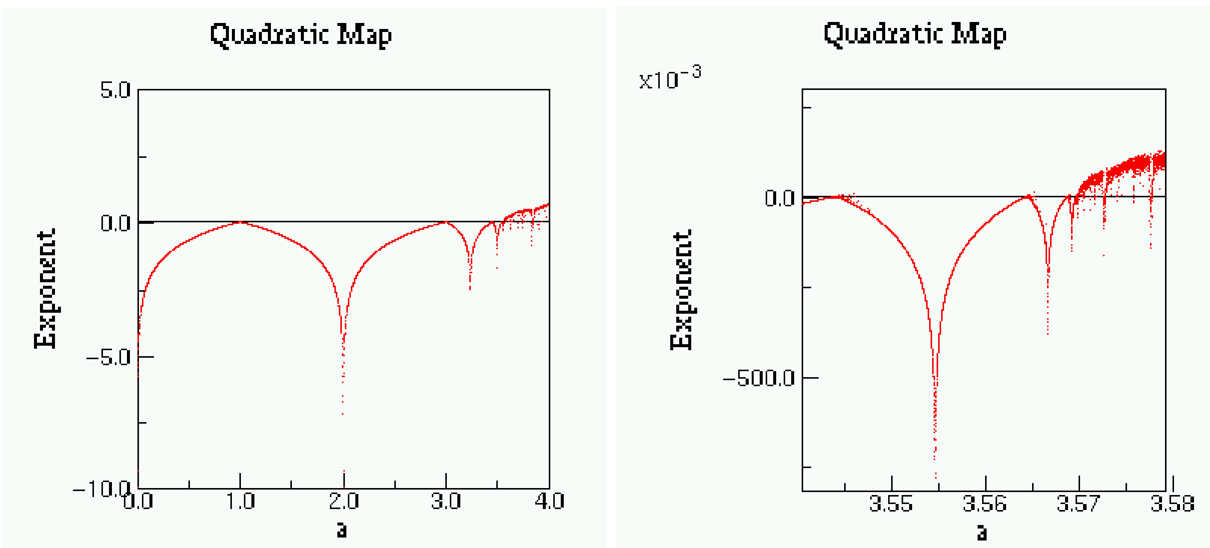
10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	for exponential decay
10^{-1}	10^{-2}	10^{-4}	10^{-8}	10^{-16}	10^{-32}	decay at superstable fixed point

Note:

- when a periodic orbit first arises its Lyapunov exponent is $f'(x)=+1$
- when a periodic orbit undergoes a period-doubling bifurcation $f'(x)=-1$

- between these two bifurcations $f'(x)$ has to go through 0, $\lambda \to -\infty$: the periodic orbit is superstable
- At the period-doubling bifurcations perturbations grow or decay very slowly $\to$ it takes a long time for the orbit to approach the attractor. Measuring the bifurcation points precisely is therefore computationaly quite expensive. To characterize the period-doubling cascade it is computationally more efficient to mark the superstable points rather than the bifurcation points (cf. homework)

[18]



Origin of the sensitive dependence on initial conditions:

Consider $a = 4$:

Notes:

- The interval $[0, \frac{1}{2}]$ is *stretched* all the way to $[0, 1]$
- The interval $[\frac{1}{2}, 1]$ is also stretched in length to $[0, 1]$, but it is at the same time *folded* back
- Separation of trajectories requires stretching and folding because the trajectories are stretched and at the same time confined to a finite interval.

The stretching suggests a positive Lyapunov exponent. A very rough hand-waving guess gives

$$\Delta x_{n+1} \sim 2\Delta x_n \sim 2^{n+1}\Delta x_0$$

leading to

$$\lambda \sim \lim_{n\to\infty} \frac{1}{2} \ln 2^{n+1} \to \ln 2.$$

[18] Demo 4 by Cross: plot Lyapunov exponent

For $a = 4$ one can actually give an explicit exact solution of the iteration for any initial condition. Rewrite the iteration

$$x_{n+1} = 4x_n(1 - x_n)$$

in terms of a new variable θ_n defined via

$$x_n = \sin^2\theta_n \qquad x_{n+1} = \sin^2\theta_{n+1}.$$

Then

$$\begin{aligned}\sin^2\theta_{n+1} &= 4\sin^2\theta_n\,(\underbrace{1-\sin^2\theta_n}_{\cos^2\theta_n}) = \\ &= (2\sin\theta_n\cos\theta_n)^2 = \sin^2(2\theta_n)\end{aligned}$$

Thus, in terms of the variable θ the dynamics are simple

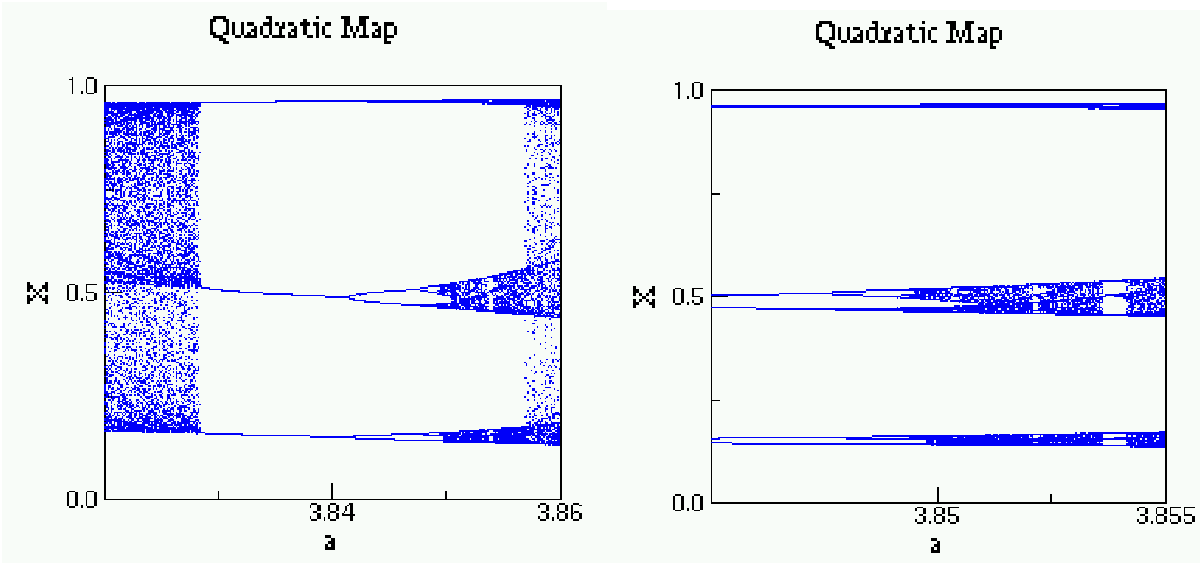
$$\begin{aligned}\theta_{n+1} &= 2\theta_n \\ \Rightarrow \qquad \theta_n &= 2^n\theta_0 \\ x_n &= \sin^2(2^n\theta_0) = \sin^2\left(2^n \arcsin\sqrt{x_0}\right)\end{aligned}$$

The analytical solution allows to compute the Lyapunov exponent exactly as well

$$\begin{aligned}\lambda &= \lim_{n\to\infty}\frac{1}{n}\ln\left|\left.\frac{df^{(n)}(x)}{dx}\right|_{x_0}\right| \\ &= \lim_{n\to\infty}\frac{1}{n}\ln\left|\left. 2\sin(2^n\theta_0)\cos(2^n\theta_0)\,2^n\,\frac{d}{dx}\arcsin\sqrt{x}\right|_{x_0}\right| \\ &= \ln 2 + \lim_{n\to\infty}\left(\frac{1}{n}\mathcal{O}(1)\right) = \ln 2\end{aligned}$$

'Chaotic' regime, periodic windows:
$a = 3.825 \to 3.85 \to 3.86$
$a = 3.83 \to 3.82$
[19]

[19] transient points: 1024 points:256 (to capture periods up to 256) ; intermittency before window 3: a=3.82841 and a=3.828415



Periodic windows in the chaotic regime[20]

- Tangent bifurcation (saddle-node bifurcation) of $f^{(3)}(x)$ at $1+\sqrt{8}=3.8284$
  A stable and an unstable fixed point of $f^{(3)}(x)$ are created, corresponding to a stable and an unstable period-3 orbit.
- Just before the saddle-node bifurcation the 'ghost' of the period-3 orbit traps the orbit for a long time $\Rightarrow$ intermittent behvaior: long durations of near period-3 behavior, which are interrruped by chaotic excursions.
- The period-3 orbit also undergoes a period-doubling cascade.
- There are also windows with period 5, 7, 9, 11, etc. and each window has its own period-doubling cascade.

[20] use logmap.m demo to show time series $a=3.828415$ has long laminar regions $a=3.82837$ plot the function $f^{(3)}$ in the plot: use Cross Demo 1: set compose=3. plot cobweb diagram.

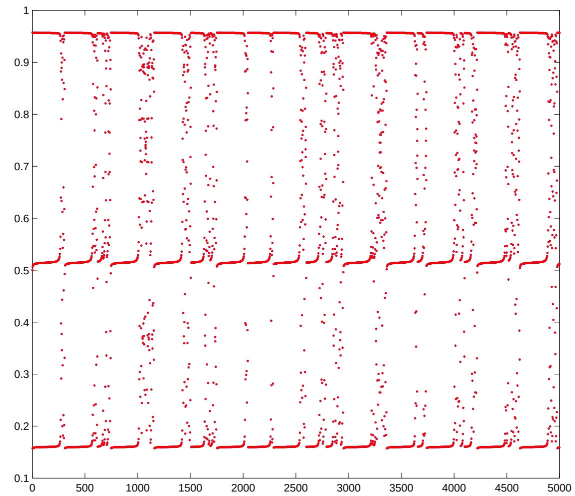


Figure 15: Intermittency for $a = 3.82841$ reflecting the 'ghost' of the period-3 orbit that is created via a saddle-node biufrcation (tangent bifurcation) at slightly larger a.

4.3 Strange Attractors and Fractal Dimensions

In the Lorenz system and the logistic map we saw that in the chaotic regime the trajectories converged onto a complicated set, a *strange attractor*.

We still need to define an attractor more precisely:

A set A is called an attractor if

1. A is an invariant set: any trajectory that starts in A remains in A for all times.
2. A attracts an open set of initial conditions: there exists an open set U containing A such that any trajectory starting in U converges to A for $t \to \infty$.
3. A is minimal, i.e. there is no proper subset of A that satisfies conditions 1 and 2.

Note:

- The largest open set U satisfying condition 2 is the basin of attraction of the attractor A.

The attractors of the Lorenz system and the logistic map give the impression of a complex geometry: they are *strange* attractors. How can we characterize their geometry?[21]

Consider yet another system since both the Lorenz system and the logistic map have drawbacks

[21] Despite their striking geometric properties strange attractors are often defined via the property of sensitive dependence on initial conditions.

- Lorenz system is complicated since it is three-dimensional:
  for chaotic flows this is the minimal dimension, however.
- The logistic map is not invertible, i.e. the dynamics cannot be run backwards in time. Chaotic one-dimensional maps have to be non-invertible, since they need to include stretching and folding. The folding introduces the non-invertibility.

Two-dimensional, invertible map: Henon map

$$\begin{aligned} x_{n+1} &= y_n + 1 - a x_n^2 \\ y_{n+1} &= b x_n \end{aligned}$$

The map can be thought of as composed of the following steps

1. stretch and fold a rectangle into a parabolic shape
$$x \to x \qquad y \to y + 1 - a x^2$$
2. compress in the x-direction
$$x \to bx \qquad y \to y$$
3. reflection about the diagonal to orient the object again along the original rectangle
$$x \to y \qquad y \to x$$

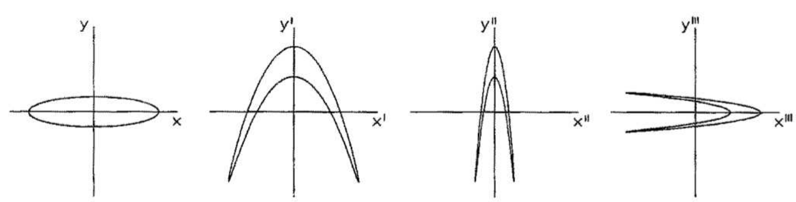


Figure 16: Henon map decomposed

4.

Notes:

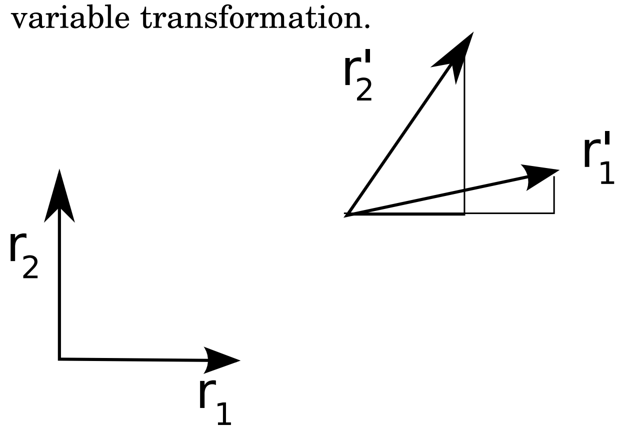
- The Henon map is indeed invertible

$$\begin{aligned} x_n &= \frac{1}{b} y_{n+1} \\ y_n &= x_{n+1} - 1 + a x_n^2 = x_{n+1} - 1 + \frac{a}{b} y_{n+1}^2 \end{aligned}$$

- The Henon map is dissipative
  Consider infinitesimal area $dx\,dy$ and its mapping under a variable transformation

$$(x, y) \to (u(x, y), v(x, y))$$

from multi-variable integration we know that the rectangle with area $dx\,dy$ is transformed into a parallelogram with area $|\det \mathbf{J}|\,dx\,dy$ where $\mathbf{J}(x, y)$ is the Jacobian of the variable transformation.



The area of the parallelogram generated by $\mathbf{r}'_1$ and $\mathbf{r}'_2$ is given by

$$|\mathbf{r}'_1 \times \mathbf{r}'_2| = |r_{1x}r_{2y} - r_{1y}r_{2x}| = \left| \frac{\partial u}{\partial x}\Delta x \frac{\partial v}{\partial y}\Delta y - \frac{\partial u}{\partial y}\Delta y \frac{\partial v}{\partial x}\Delta x \right| = |\det \mathbf{J}|\,\Delta x \Delta y$$

An iteration of the Henon map can be thought of as a variable transformation $\Rightarrow$ the change in area in the phase plane is determined by $|\det \mathbf{J}|$

$$\det \mathbf{J} = \left| \begin{array}{cc} -2ax_n & 1 \\ b & 0 \end{array} \right| = -b$$

Thus: for $|b| < 1$ the Henon map is dissipative everywhere in the phase plane.

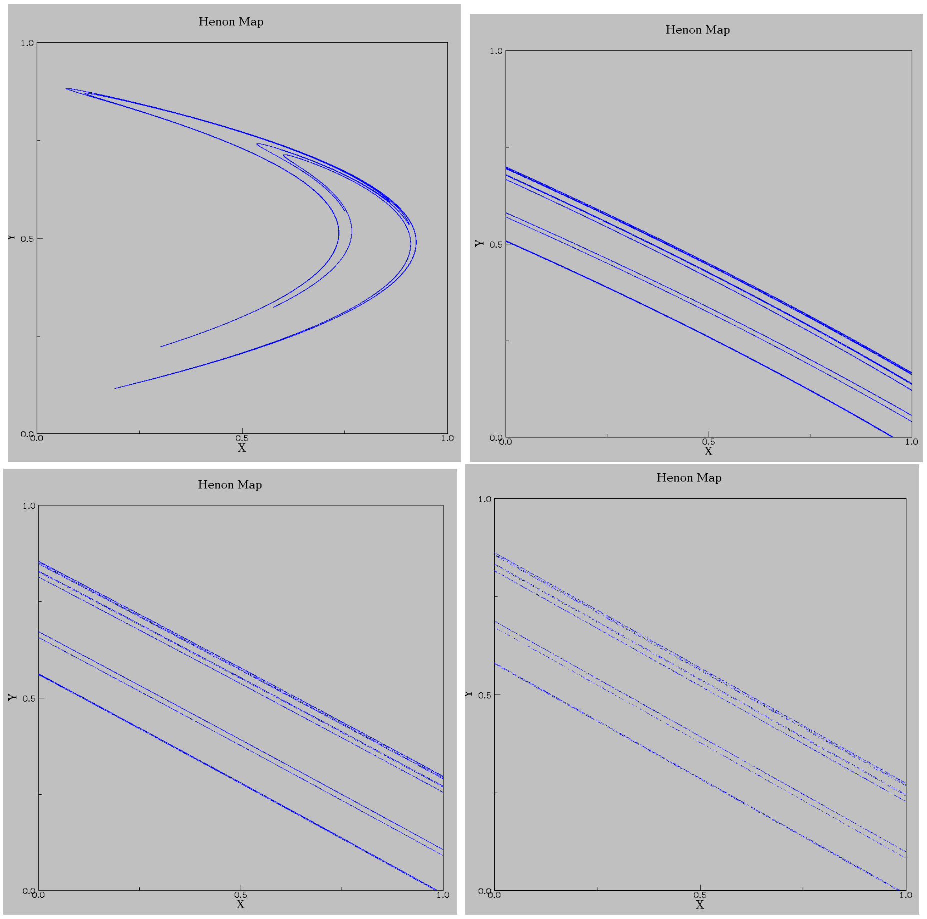


Figure 17: Self-similarity in the Henon map for $a = 1.4$ and $b = 0.3$. Axes $[xmin, xmax, ymin, ymax]$ are [0,1,0,1], [0.69,0.91,0.13,0.2], [0.77,0.80,0.16,0.17], and [0.7829,0.7875,0.1629,0.1643].

The Henon attractor exhibits self-similarity: zooming into the upper three-layered sheet of the three sheets reveals again three sheets comprised of 1, 2, and 3 sheets each. How do we characterize such structures?

Consider the following construction of a set $\mathcal{S}$ that is somewhat similar to the cross section of the Henon attractor

1. Take the interval [0,1]
2. cut out the middle third
3. from each remaining subinterval cut out the middle third
4. repeat step 3 ad infinitum

Note:

- the resulting set $\mathcal{S}$ is called the Cantor set after Georg Cantor, the developer of set theory

Question:

- how many pieces make up the set $\mathcal{S}$?
- can one count them?
- what is its cardinality?

Countable and uncountable sets:

- a set is countable if there is a 1-to-1 correspondence between each of its elements and the natural numbers

Examples

1. The rational numbers are countable
   each rational number $\frac{p}{q}$ can be represented as an element in a matrix **Q**

$$\begin{pmatrix} q\backslash p & 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & \frac{1}{1} & \frac{2}{1} & \ldots & & & \\ 2 & \frac{1}{2} & \frac{2}{2} & \frac{3}{2} & \ldots & & \\ 3 & \frac{1}{3} & \ldots & & & & \\ 4 & \frac{1}{4} & \ldots & & & & \\ 5 & \frac{1}{5} & \frac{2}{5} & & & & \end{pmatrix}$$

   then count $Q_{11} \rightarrow Q_{21} \rightarrow Q_{12} \rightarrow Q_{31} \rightarrow Q_{22} \rightarrow Q_{13} \rightarrow Q_{41} \rightarrow \ldots$

2. The real numbers in $[0, 1]$ are not countable
   Cantor showed this by contradiction using his 'diagonal argument':
   Assume the real numbers were countable. Then they could be listed in sequence

$$\begin{array}{l} 0.x_{11}\, x_{12}\, x_{13}\, \ldots \\ 0.x_{21}\, x_{22}\, x_{23}\, \ldots \\ 0.x_{31}\, x_{32}\, x_{33}\, \ldots \\ 0.x_{41}\, x_{42}\, x_{43}\, \ldots \\ \qquad\qquad\quad \ldots \\ \qquad\qquad\quad \ldots \end{array}$$

   with $x_{ij} \in \mathbb{R}$.
   Consider now the number

$$0.\bar{x}_{11}\, \bar{x}_{22}\, \bar{x}_{33}\, \ldots$$

   where $\bar{x}_{jj} \neq x_{jj}$ for all j. This number is clearly a realy number but it is not in the list because it differs from each number in the list by at least one digit. The enumerated list of numbers is therefore not complete.

3. What is the cardinality of the Cantor set?
   Use again Cantor's diagonal argument as in the case of the real numbers: each piece can be labeled by a number $q \in \{0, 1, 2\}$ whose digit at the l^{th} position states whether at the l^{th} level of the Cantor construction this piece ends up in the left, middle or right third subdivision of the interval
   All points in $\mathcal{S}$ are then given by a number whose digits are taken from $\{0, 2\}$, since the middle piece is always taken out[22].
   If the Cantor set is countable then the elements can be listed,

$$\begin{aligned} x_1 &= x_{11}\, x_{12}\, x_{13} \ldots \\ x_2 &= x_{21}\, x_{22}\, x_{23} \ldots \\ x_3 &= x_{31}\, x_{32}\, x_{33} \ldots \end{aligned}$$

   Again, the number $\bar{x}_{11}\, \bar{x}_{22}\, \bar{x}_{33} \ldots$ is not contained in the list, but it is in $\mathcal{S}$.
   Therefore $\mathcal{S}$ is not countable.

What is the total length ('measure') of the Cantor set $\mathcal{S}$?

In each step a third of the remaining length is removed:

$$L_j = \frac{2}{3} L_{j-1} = \left(\frac{2}{3}\right)^j \qquad \Rightarrow \qquad \lim_{j\to\infty} L_j = \lim_{j\to\infty} \left(\frac{2}{3}\right)^j = 0$$

Thus, the Cantor set

- is uncountable (like the real numbers, but unlike the rationals)
- has zero measure (like the rational numbers, but unlike the real numbers)

Dimensions

As one goes down the levels the number of pieces in the Cantor set increases and diverges: the number of pieces diverges as one increases the spatial resolution

Compare with the usual shapes:

Square:

- the square can be covered by 4 squares of half the linear dimension
- as the linear dimension of the pieces is reduced by a factor r the number m required to construct the square increase by a factor m

$$m = r^d \qquad \text{with} \qquad d = \frac{\ln m}{\ln r}$$

Apply this approach to the Cantor set:

- in each step the linear dimension of the pieces decreases by a factor of 3

[22] It is like a binary representation.

- in each step the number of elements goes up by a factor of 2

$$d_{Cantor}^{(sim)} = \frac{\ln 2}{\ln 3} \approx 0.63$$

Notes:

- this definition of the dimension relies on the self-similarity of the structure: *similarity dimension*.

Example:

von Koch[23] curve:

Recursive construction:

- replace each straight line segment of length l by 4 segments of length $l/3$

Similarity dimension:

- in each step linear dimension of the pieces decreases by a factor of 3
- in each step the number of pieces increases by a factor of 4

$$d_{Koch}^{(sim)} = \frac{\ln 4}{\ln 3} \approx 1.26$$

For sets that are not self-similar generalize the procedure: cover the set with elements of size ϵ where $\epsilon \to 0$, i.e. reduce the linear dimension by a factor of $r = \frac{1}{\epsilon}$,

$$d^{(box)} = \lim_{\epsilon \to 0} \frac{\ln N(\epsilon)}{\ln\left(\frac{1}{\epsilon}\right)}$$

Note:

- This defines the *box counting dimension.*

Apply to the Cantor set:

At each level choose $\epsilon = \left(\frac{1}{3}\right)^n$, then one needs 2^n 'boxes' to cover the Cantor set

$$d^{(box)} = \lim_{n \to \infty} \frac{\ln 2^n}{\ln \frac{1}{\left(\frac{1}{3}\right)^n}} = \lim_{n \to \infty} \frac{n \ln 2}{n \ln 3} = \frac{\ln 2}{\ln 3}$$

Example: Attractor of the Henon map

[23] von Koch (1870–1924)

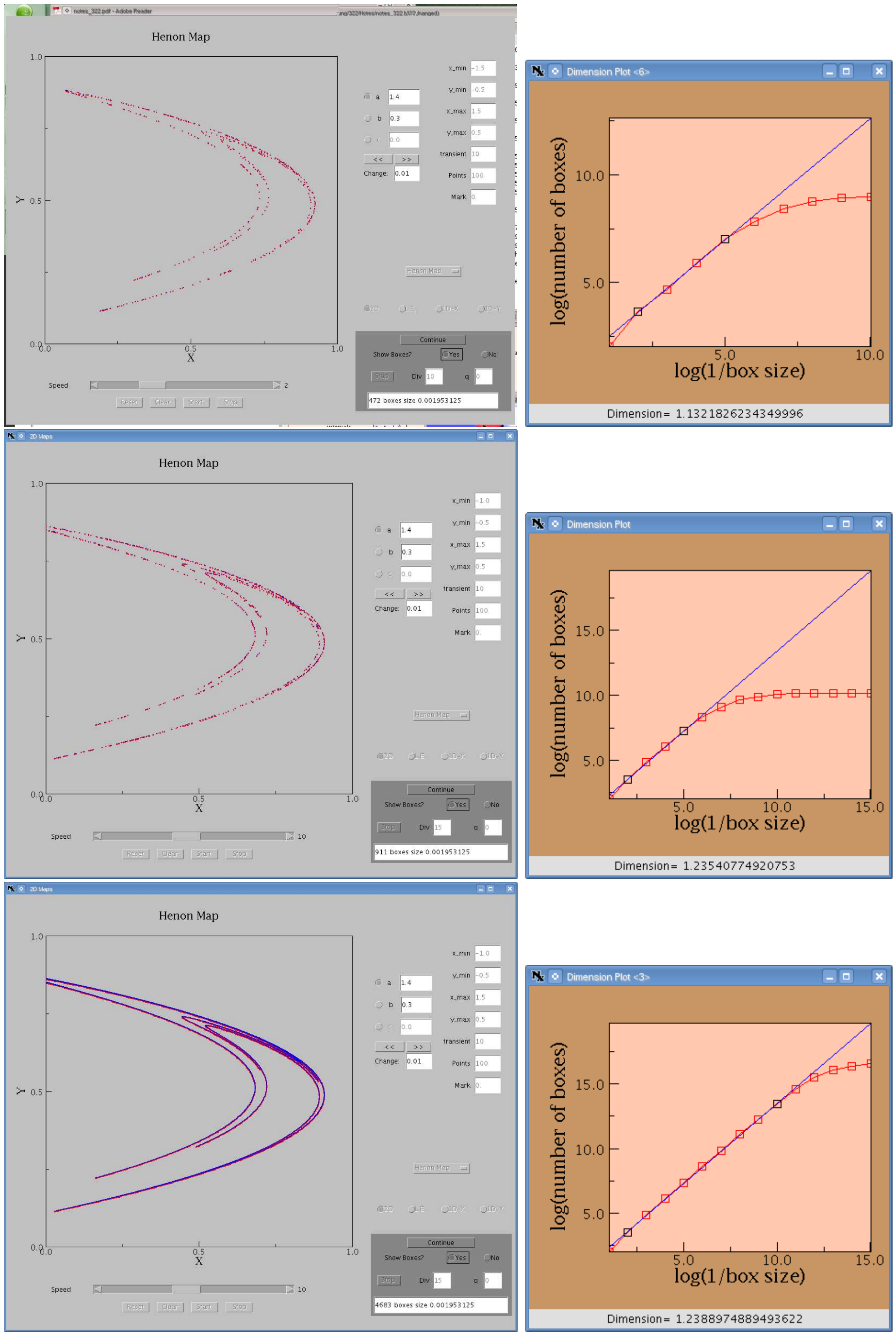


Figure 18: Box dimension of Henon attractor for 500, 1,000, and 100,000 points. Left panels; attractor with boxes, right panels: scaling of the number of boxes with the box size.

The box dimension does not depend on how many points of the trajectory are inside a given box as long as there is at least one point.

Correlation Dimension:

For a given point r on the attractor determine the number of other points on the attractor that fall within a ball of size ϵ of r

$$N_{\mathbf{r}}(\epsilon) \propto \epsilon^{d_{\mathbf{r}}}$$

$d_{\mathbf{r}}$ is the pointwise dimension of r.

The correlation dimension is obtained by averaging $N_{\mathbf{r}}(\epsilon)$ over all (sufficiently many) r of the attractor

$$C(\epsilon) = \frac{1}{\mathcal{N}} \sum_{i=1}^{\mathcal{N}} N_{\mathbf{r}_i}(\epsilon)$$

with $\mathcal{N}$ the number of attractor points r included in the average.

Expecting

$$C(\epsilon) \propto \epsilon^{d_c}$$

define

$$d_c = \frac{d \ln C(\epsilon)}{d \ln (\epsilon)}$$

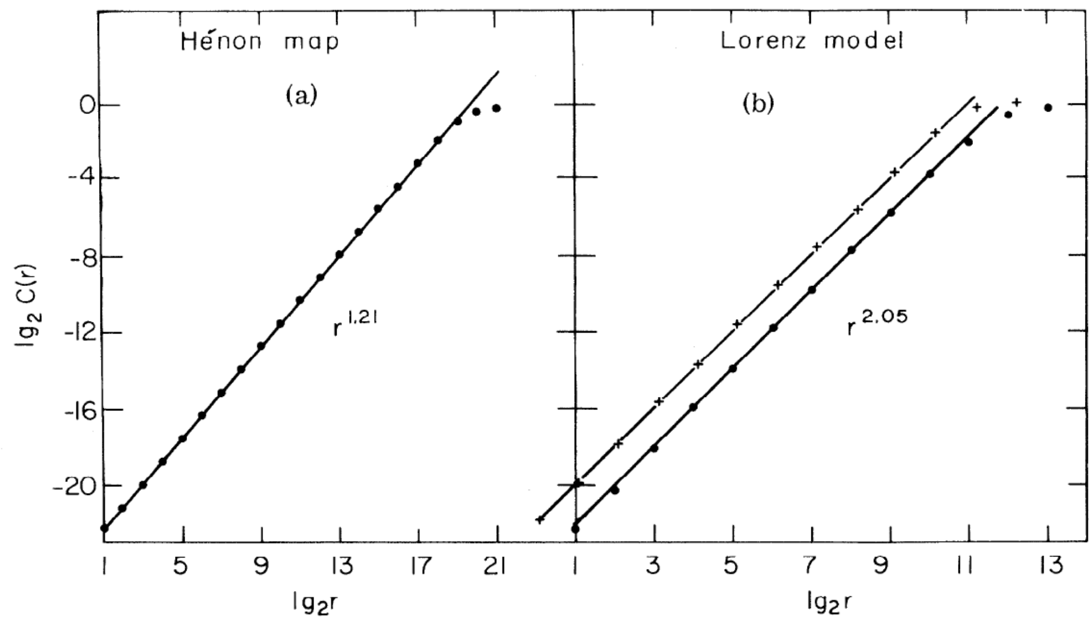


Figure 19: Correlation dimension for Henon map and Lorenz equations [?].

Note:

- Dynamics do enter correlation dimension: where are the points dense, where not, i.e. where in phase space is the system more often?
- one can show $d_c \leq d_b$
  but usually $d_c \sim d_b$

Note:

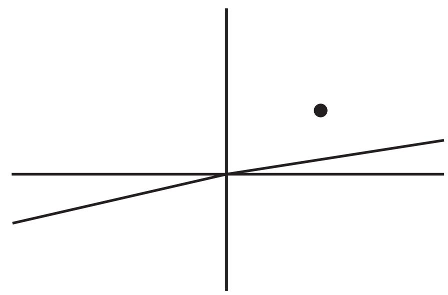
- there are further dimensions along these lines:
  whole spectrum of dimensions generated by weighing the probability of finding points in a small ball with different powers

Lyapunov Dimension:

Include dynamics explicitly in the definition of the dimension

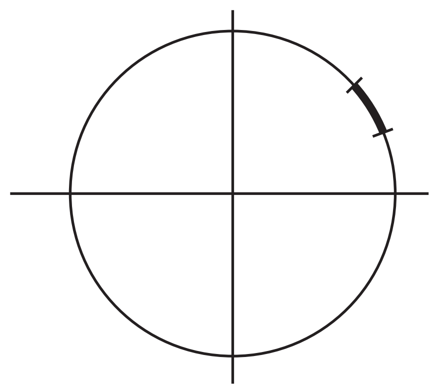
Consider dimension of a box that neither grows nor shrinks under the dynamics.

Point attractor



any box with $d \geq 1$ shrinks to a point: $d_L = 0$

Periodic orbit



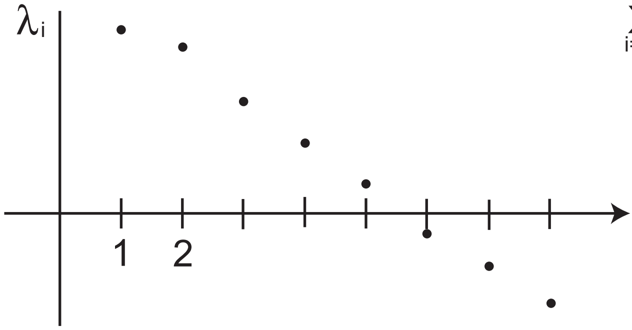
line segments along the attractor are transported along the orbit without volume change (on average), but area covering the width of the attractor shrinks to a line: $d_L = 1$

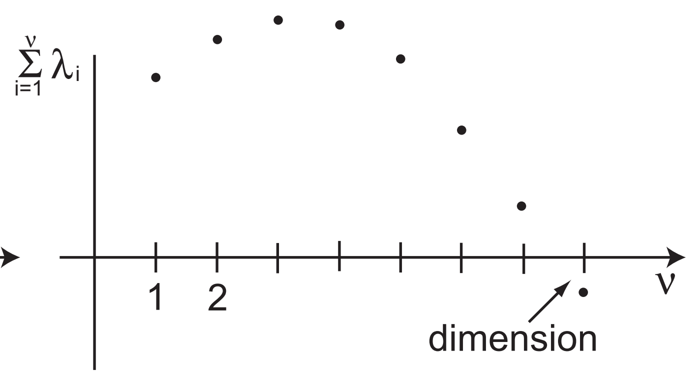
Growth of a ν-dimensional volume in phase space is given by the expansions in the ν directions

$$V(t) = L_1 e^{\lambda_1 t}\, L_2 e^{\lambda_2 t}\, L_3 e^{\lambda_3 t}\, \ldots L_\nu e^{\lambda_\nu t}$$

for $V = const.$ we need

$$\sum_{i=1}^{\nu} \lambda_i = 0$$





Arrange eigenvalues in decreasing order: $\lambda_i \geq \lambda_{i+1}$

Consider

$$f(n) = \sum_{i=1}^{n} \lambda_i$$

Interpolate linearly the zero of $f(n)$:

$$f(d_L) = 0 \approx f(\nu) + \frac{f(\nu+1) - f(\nu)}{\nu + 1 - \nu}(d_L - \nu) \qquad \Rightarrow \qquad d_L = \nu - \frac{f(\nu)}{f(\nu+1) - f(\nu)}$$

Thus, choose ν to satisfy $\sum_{i=1}^{\nu} \lambda_i > 0$ but $\sum_{i=1}^{\nu+1} \lambda_i < 0$. Then the Lyapunov dimension is defined by

$$d_L = \nu + \frac{1}{|\lambda_{\nu+1}|} \sum_{i=1}^{\nu} \lambda_i$$

Note:

- d_L gives a measure of how many degrees of freedom are "active"

4.4 Experimental Data: Attractor Reconstruction and Poincare Section

Experimentally, one typically cannot monitor all or even a large fraction of the relevant dynamical variables. How can one obtain relevant information about the attractor?

Given only the time series for a single dynamical variable $x(t)$ one can reconstruct a representation of the attractor using *time-delayed coordinates*:

- plot $x(t)$ vs $x(t-\tau)$,$x(t-2\tau)$, ..., $x(-N\tau)$ etc.
- estimate the dimension (e.g. correlation dimension) of the resulting attractor as a function of the number N of delays
- if the dimension saturates above some value of N one has found a sufficient embedding dimension for the attractor

Example: Lorenz system

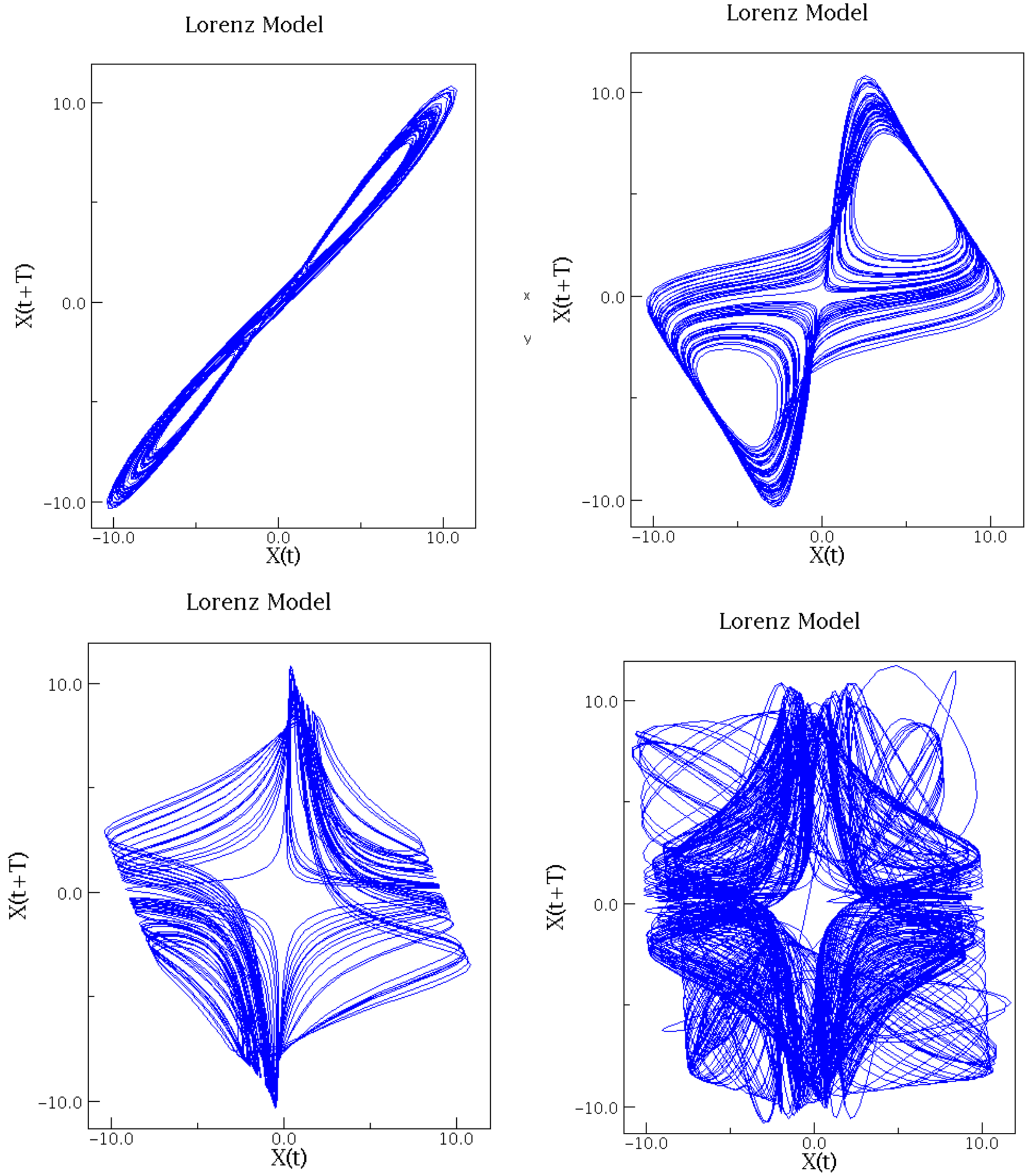


Figure 20: Reconstruction of the Lorenz attractor using delayed coordinates. a) $\tau = 1$, b) $\tau = 10$, c) $\tau = 20$, d) $\tau = 100$.

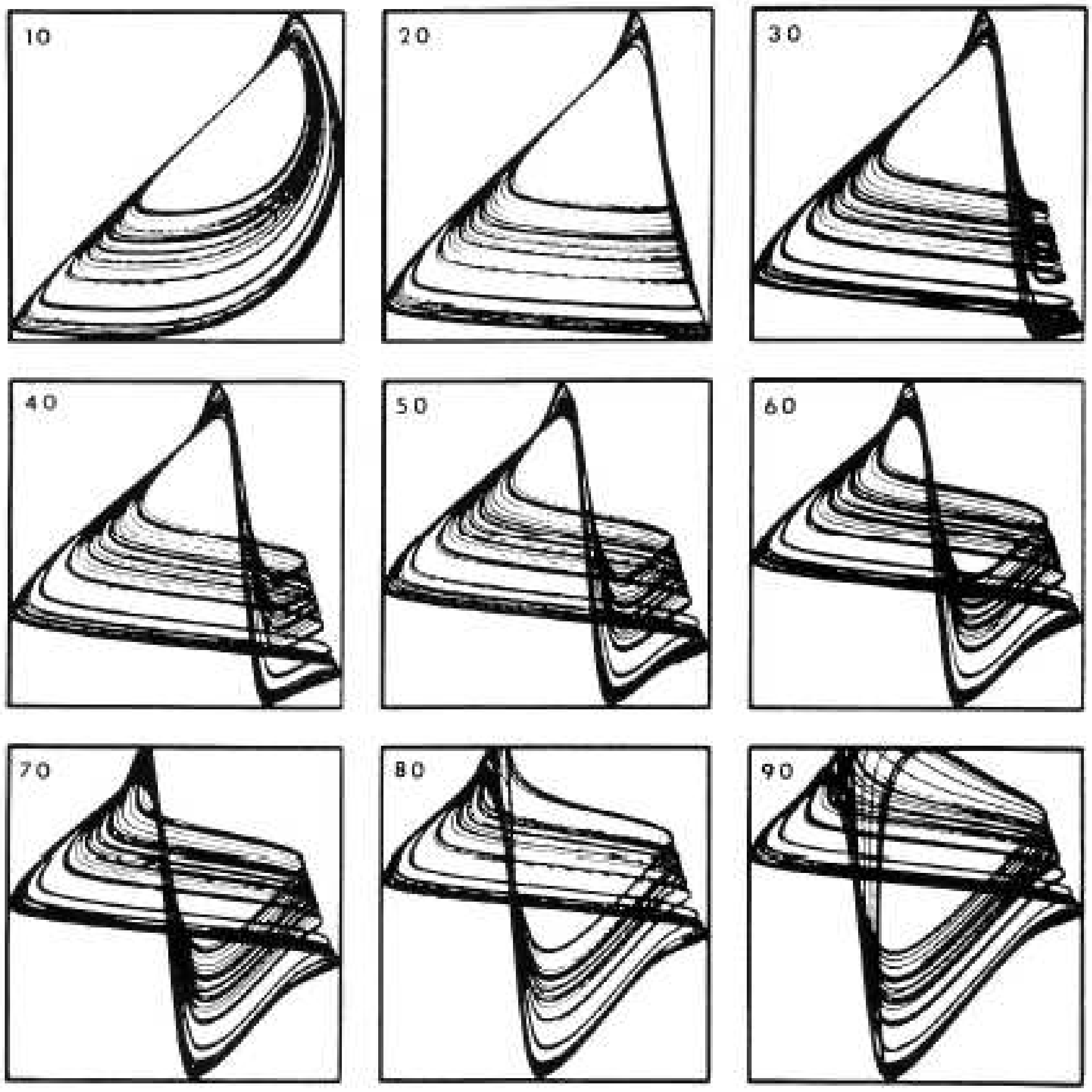


Figure 21: Attractor reconstruction for the Belousov-Zhabotinsky reaction. The delay used is indicated in each panel [**?**].

Note:

- The geometric shape of the attractor reconstruction depends on the delay τ . its dimension dimension does not.
- For very small τ the delayed variable $x(t-\tau)$ is strongly correlated with $x(t)$ $\Rightarrow$they do not provide independent information about the attractor.
- For very large τ the two variables are completely uncorrelated: they do not provide insight into the attractor
- Optimal delay at intermediate values: first minimum of the mutual information between the two variables [**?**]

From the attractor one may be able to obtain a description in terms of a map on the Poincare section, which generates a cross-section of the attractor:
For an attractor embedded in 3 dimensions mark all locations where the trajectory crosses a two-dimensional manifold in one direction. If that cross-section can be parameterized sufficiently well by a single variable one may obtain an iterated map for the dynamics.

a) b)

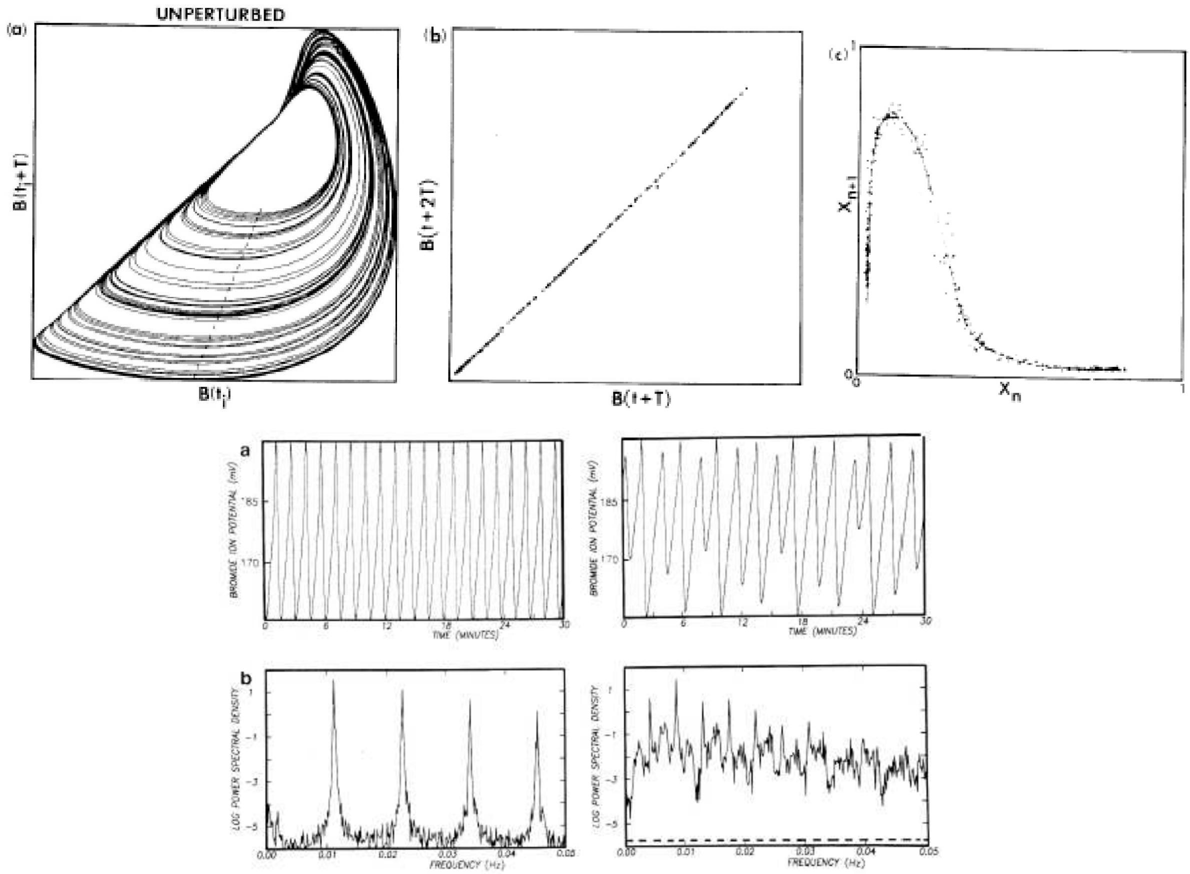


Figure 22: Belousov-Zhabotinsky rection. a) Poincare section (dashed line in upper panel) yields a thin line (middle panel). The dynamics on that Poincare section is well captured by an (almost) one-dimensional iterated map on that Poincare section. The unimodal map suggests the appearance of a period-doubling cascade. The spectrum (b) shows a periodic oscillation and a chaotic oscillation that still reflectes an approximate period 4 [?].