Methods of Nonlinear Analysis

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Contents

[1] For a simpler start-up example see Notes for 322.

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Books

Here is a list of books that are of interest for this class. Unfortunately only two of them are available online. None of them are necessary, however.

- *Nonlinear oscillations, dynamical systems, and bifurcations of vector fields*
  John Guckenheimer, Philip Holmes. Applied mathematical sciences (Springer-Verlag, New York Inc.) ; v. 42 . 519.05 A652 v.42
- *Introduction to applied nonlinear dynamical systems and chaos*
  Stephen Wiggins. Texts in applied mathematics New York : Springer-Verlag Creation Date ©1990. 519.05 A652 v.73
- *Pattern formation : an introduction to methods*
  Rebecca B. Hoyle, Cambridge University Press Creation Date 2006. Q172.5.C45 H69 2006
- *Spatio-Temporal Pattern Formation : With Examples from Physics, Chemistry, and Materials Science*
  Daniel Walgraef. Edition 1st ed. 1997.
  https://link-springer-com.turing.library.northwestern.edu/book/10.1007%2F978-1-4612-1850-0
- *New Trends in Nonlinear Dynamics and Pattern-Forming Phenomena The Geometry of Nonequilibrium*
  Editors Pierre Coullet, Patrick Huerre
  https://link-springer-com.turing.library.northwestern.edu/book/10.1007%2F978-1-4684-7479-4
- *Practical Bifurcation and Stabiliity Analysis,*
  R. Seydel, Springer (2009). [available on Canvas]

This class overlaps to some extent with our undergraduate 322 *Applied Nonlinear Dynamics*. Parts of the lecture notes for that class are included in the notes here as Appendix. The current full version of those notes are available on Canvas under Files/Lecture Notes.

A good book for the material of 322 is

- *Nonlinear dynamics and chaos (with applications to physics, biology chemistry, and engineering).*
  Stephen Strogatz.

Strogatz' Lectures for a class he taught at Cornell are on Video:

https://www.youtube.com/watch?v=ycJEoqmQvwg&list=PLbN57C5Zdl6j_qJA-pARJnKsmROzPnO9V&index=1

1 Introduction

Nonlinear equations arise in all kinds of systems:

- Mechanics: even simple pendulum
- Chemical systems:
  - Examples of oscillatory reactions: Belousov-Zhabotinsky, Briggs-Rauscher
  - Flames in combustion →
    `https://people.esam.northwestern.edu/~riecke/Vorlesungen/412/1999/flames.html`
- Fluid dynamics:
  - Taylor vortex flow →
    `https://people.esam.northwestern.edu/~riecke/research/TVF/tvf.overview.html`

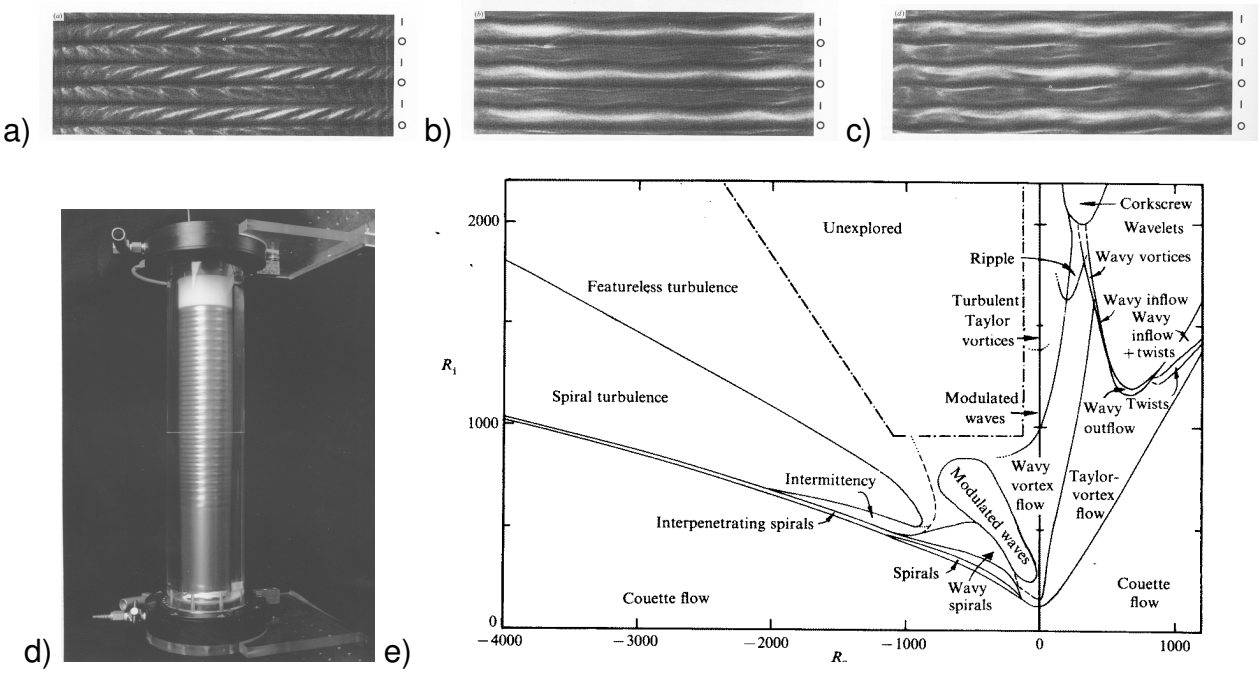


Figure 1: Taylor vortex flow exhibits a bewildering multitude of qualitatively different behaviors when the rotation rates are changed. a) Twisted Taylor vortices, b) wave inflow vortices, c) wavelets, d) Taylor vortices in cylinders with ramped radius (vortices appear where the gap is wider), e) phase diagram of states obtained depending on inner and outer cylinder rotation rates (R_i vertical axis, R_o horizontal axis). (Andereck et al., 1986).

  - Rayleigh-Bénard convection →
    `https://people.esam.northwestern.edu/~riecke/Vorlesungen/412/1999/rb.html`
- Crystal growth:
  - directional solidification →
    `https://people.esam.northwestern.edu/~riecke/Vorlesungen/412/1999/dirsol.html`

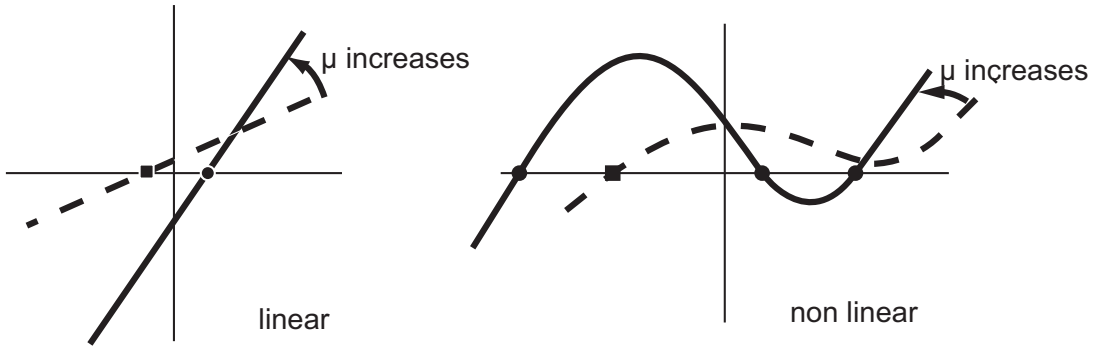
Characteristics of Nonlinear Systems:

- **Qualitative** changes in behavior and non-smooth dependence on parameters:
  - Rayleigh-Benard convection: heat transport
  - Taylor vortex flow: torque, transitions between different types of states
- Multiplicity of solutions: hysteresis
  - rolls vs. spiral-defect chaos convection
- Chaotic dynamics
  - many frequencies, coexisting (unstable) periodic solutions

Simple illustration: **linear vs. nonlinear**

Consider

$$f(v, \mu) = 0$$



Linear:

- for all values of the control parameter μ (essentially) always 1 unique solution
- quantitative but **no qualitative** change

Nonlinear:

- # of solutions can change with changing μ: solutions appear and disappear
- **quantitative** and **qualitative** changes

Nonlinear equations are difficult to solve

Example:

i) **Linear** reaction-diffusion equation

$$\partial_t v = D\Delta v + av$$

Fourier expansion

$$v(x,t) = \sum_{n=-\infty}^{\infty} v_n(t)\, e^{i\frac{2\pi}{L}nx}$$

Eigenmodes: each v_n satisfies:

$$\frac{dv_n}{dt} = -D(\frac{2\pi n}{L})^2 v_n + av_n \qquad \rightarrow \qquad v_n(t) = v_n(0)\, e^{-D(\frac{2\pi n}{L})^2 t + at}$$

Different modes do **not** interact

ii) **Nonlinear** reaction-diffusion equation: no superposition, different modes **do interact!**

$$\partial_t v = D\Delta v + \underbrace{v^2}_{\sum v_n v_m e^{i\frac{2\pi}{L}(n+m)x}}$$

v^2 generates new wave numbers: couples n & m to $n+m$ and to $n-m$

Any interaction between different objects (A and B) implies nonlinearity:
evolution of A depends on state of A and that of B

$\rightarrow$ Cannot build general solution from a set of basic solutions by simply adding them

$\rightarrow$ in general: cannot find exact solutions: **HARD**.

$\rightarrow$ typically *numerical* solution is required

Numerical Solution:

- confirms the model/basic equations:
  of great interest if model has not been established, e.g., chemical oscillations, heart muscle
- gives quantitative details for **specific** values of system parameters:
  these details may not be accessible in experiments: 3d fluid flow, turbulent, chemical concentrations of each species, temporal evolution of the state of ion channels in neurons ...

To get **insight** into the system the most interesting points are the *transition points*:

- *qualitatively new features* of the solutions arise

Qualitative Analysis

- Change in behavior as system parameters are changed
  transitions between **qualitatively** different states
- Analytical techniques for transitions
  approximations near transition points:
  reduction in the dimension of the dynamical system
- Visualization: **geometry of dynamics**, phase space
- Overview of **all** possible behaviors

1.1 Central Tool: Separation of Time Scales

A key feature that allows the reduction in the dimension of a dynamical system is a separation in time scales.

Consider first again scalar case

$$\frac{d}{dt}v = f(v)$$

Two fixed points appear when the function $f(v)$ touches the v-axis: $f(v_{1,2}) = 0$ and v_1 is close to v_2.

The new solutions arise in a *bifurcation.*

Two simplifications:

- Near the minimum f can be *expanded* in low-order polynomial
- For smooth f this implies f is small: v **evolves *slowly*.**

Consider now the general case: many interacting modes

At the bifurcation

- only one or few modes evolve slowly
- the remaing modes are in principle fast, but they follow the evolution of the slow modes

Thus:

Near the bifurcation

- the fast modes can be eliminated 'adiabatically'.
- the system evolves on the 'center manifold', which has much lower dimension than the full system

Time-scale separation can arise from a number of different causes:

1. Bifurcations
2. Conservation laws:
   long-wave dynamics is slow
   e.g., mass conservation leads to slow diffusive relaxation of long-wave density fluctuations, Navier-Stokes equations.
3. Broken continuous symmetries.
   e.g., solitons in nonlinear optics can have arbitrary amplitude:
   they break a continuous symmetry, non-conserving perturbations lead to slow evolution of the amplitude.

4. Weak interaction between objects
   separation of time scales as distance between objects goes to infinity.

Note:

Separation of time scales is at the core of many analytical approaches for the analysis of nonlinear systems.

2 Linear Systems

Before plunging into nonlinear system, consider under what conditions linear systems are sufficient/insufficient to obtain a qualitative picture of the dynamics of a nonlinear system.

The most simple situation to consider is a fixed point and the dynamics in its vicinity: what does the flow in the vicinity of a fixed point look like? Under what conditions will the linearization of the system around that fixed point give a qualitatively good approximation of the full system?

Consider the general linear system

$$\dot{\mathbf{x}} = \mathbf{L}\,\mathbf{x} \qquad\qquad \mathbf{x}(0) = \mathbf{x}_0$$

Formal solution

$$\mathbf{x}(t) = e^{\mathbf{L}\,t}\mathbf{x}_0$$

where the matrix exponential is defined via the Taylor series

$$e^{\mathbf{L}t} = 1 + \mathbf{L}\,t + \frac{1}{2}\mathbf{L}^2 t^2 + \dots$$

In general one can find a similarity transform $\mathbf{S}$ such that the transformed $\mathbf{L}$ is comprised of blocks

$$\mathbf{S}^{-1}\,\mathbf{L}\,\mathbf{S} = \begin{pmatrix} \dots & \dots & 0 & 0 & 0 \\ \dots & \dots & 0 & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & \dots \\ 0 & 0 & 0 & \dots & \dots \end{pmatrix}$$

that each are associated with a different eigenvalue λ_i and are either diagonal,

$$\begin{pmatrix} \dots & \dots \\ \dots & \dots \end{pmatrix} = \begin{pmatrix} \lambda_i & 0 \\ 0 & \lambda_i \end{pmatrix},$$

or have Jordan normal form,

$$\begin{pmatrix} \dots & \dots \\ \dots & \dots \end{pmatrix} = \begin{pmatrix} \lambda_j & 1 \\ 0 & \lambda_j \end{pmatrix}.$$

Notes:

- If no eigenvalues are repeated, then $\mathbf{S}^{-1}\mathbf{L}\mathbf{S}$ is diagonal.
  - The columns of $\mathbf{S}$ are the eigenvectors of $\mathbf{L}$:
    Consider
    $$\mathbf{S}^{-1}\mathbf{L}\mathbf{S} \begin{pmatrix} 0 \\ \dots \\ 0 \\ 1 \\ 0 \\ \dots \\ 0 \end{pmatrix} = \lambda_i \begin{pmatrix} 0 \\ \dots \\ 0 \\ 1 \\ 0 \\ \dots \\ 0 \end{pmatrix}$$
    then
    $$\Rightarrow \mathbf{L}\ \mathbf{S} \underbrace{\begin{pmatrix} 0 \\ \dots \\ 0 \\ 1 \\ 0 \\ \dots \\ 0 \end{pmatrix}}_{\mathbf{v}^{(i)}} = \lambda_i \mathbf{S} \underbrace{\begin{pmatrix} 0 \\ \dots \\ 0 \\ 1 \\ 0 \\ \dots \\ 0 \end{pmatrix}}_{v^{(i)}}$$
    and we have
    $$\mathbf{L}\,\mathbf{v}^{(i)} = \lambda_i\,\mathbf{v}^{(i)}$$
  - The dynamics in the eigendirections $\mathbf{v}^{(i)}$ is given by simple exponentials
    $$\begin{aligned} e^{\mathbf{L}t}\,\mathbf{v}^{(i)} &= \{1 + \mathbf{L}\,t + \frac{1}{2}(\mathbf{L}t)^2 + \cdots\}\,\mathbf{v}^{(i)} = \\ &= \{1 + \lambda_i t + \frac{1}{2}\lambda_i^2 t^2 + \cdots\}\,\mathbf{v}^{(i)} = \\ &= e^{\lambda_i t}\,\mathbf{v}^{(i)}\,. \end{aligned}$$
    Thus, a solution that starts with an initial condition that is along an eigenvector continues in that direction. The vector space spanned by that eigenvector is *invariant* under the flow.
  - The general solution is given by
    $$\mathbf{x}(t) = e^{\lambda_1 t}\mathbf{v}^{(1)}A_1 + e^{\lambda_2 t}\mathbf{v}^{(2)}A_2 + \dots$$
    with $\mathbf{x}_0 = A_1\mathbf{v}^{(1)} + A_2\mathbf{v}^{(2)} + \dots$
  - for complex eigenvalues
    $$\begin{aligned} \lambda &= \sigma \pm i\omega \\ \mathbf{x}(t) &= e^{\sigma t}\left(Ae^{i\omega t}\mathbf{v} + A^* e^{-i\omega t}\mathbf{v}^*\right) \end{aligned}$$
    since $\mathbf{x}(t)$ is real.
- For degenerate (repeated) eigenvalues the dynamics can be somewhat more complicated, see later.

The eigenvectors associated with each eigenvalue λ_i define linear subspaces E_{λ_i}:

For a real eigenvalue λ that is m-times repeated

$$E_\lambda = \{\mathbf{v} \in \mathbb{R}^n \mid (\mathbf{L} - \lambda)^m \mathbf{v} = 0\}\,.$$

For a complex eigenvalue λ complex that is m-times repeated[2]

$$E_\lambda = \{\mathbf{v} \in \mathbb{R}^n \mid (\mathbf{L} - \lambda)^m (\mathbf{L} - \lambda^*)^m \mathbf{v} = 0\}\,.$$

Phase space is spanned by these eigenspaces E_λ

$$\mathbb{R}^n = \underbrace{E_s}_{Re(\lambda)<0\ \text{stable}} \oplus \underbrace{E_c}_{Re(\lambda)=0\ \text{center}} \oplus \underbrace{E_u}_{Re(\lambda)>0\ \text{unstable}}$$

The eigenspaces $E_{s,c,u}$ are *invariant* under the dynamics of the linear system: the linear flow cannot enter or leave these spaces:

$$\mathbf{v}(0) \in E_\alpha \ \rightarrow\ \mathbf{v}(t) \in E_\alpha \text{ for all } t \qquad \alpha = s,\ c,\ u\,.$$

With nonlinearities these *linear* eigenspaces would not be invariant, but they help define curved *invariant manifolds:*

Definition: Stable/unstable manifolds $W^{(s,u)}$ of a fixed point x_0:

$$\begin{aligned} W^{(s)} &= \{\mathbf{y} \in \mathbb{R}^n \,|\, \mathbf{x}(t=0) = \mathbf{y} \Rightarrow \mathbf{x}(t) \to \mathbf{x}_0 \text{ for } t \to +\infty\} \\ W^{(u)} &= \{\mathbf{y} \in \mathbb{R}^n \,|\, \mathbf{x}(t=0) = \mathbf{y} \Rightarrow \mathbf{x}(t) \to \mathbf{x}_0 \text{ for } t \to -\infty\} \end{aligned}$$

Note:

- In the linear case the stable and unstable manifolds are given by E_s and E_u.

To obtain an overview of the dynamics in phase space, we are interested in the trajectories (orbits) in phase space, which are parametrized by the time t. Consider as a simple two-dimensional example a diagonal $\mathbf{L}$ with two distinct real eigenvalues $\lambda_{1,2}$,

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{array}{l} x = e^{\lambda_1 t} x_0 \\ y = e^{\lambda_2 t} y_0 \end{array}$$

Solving for the exponential one gets

$$e^t = (\frac{x}{x_0})^{1/\lambda_1}\,,$$

[2]Consider $\mathbf{v} = \mathbf{w} \pm \mathbf{w}^*$ with $\mathbf{A}\mathbf{w} = \lambda\mathbf{w}$ and $\mathbf{A}\mathbf{w}^* = \lambda^*\mathbf{w}^*$. Then

$$(\mathbf{A} - \lambda)\,(\mathbf{A} - \lambda^*)\,(\mathbf{w} \pm \mathbf{w}^*) = (\lambda - \lambda)\,(\lambda - \lambda^*)\,\mathbf{w} \pm (\lambda^* - \lambda)\,(\lambda^* - \lambda^*)\,\mathbf{w}^* = 0$$

and $\frac{1}{2}\,(\mathbf{w} + \mathbf{w}^*)$ and $\frac{1}{2i}\,(\mathbf{w} - \mathbf{w}^*)$ are both real.

which yields

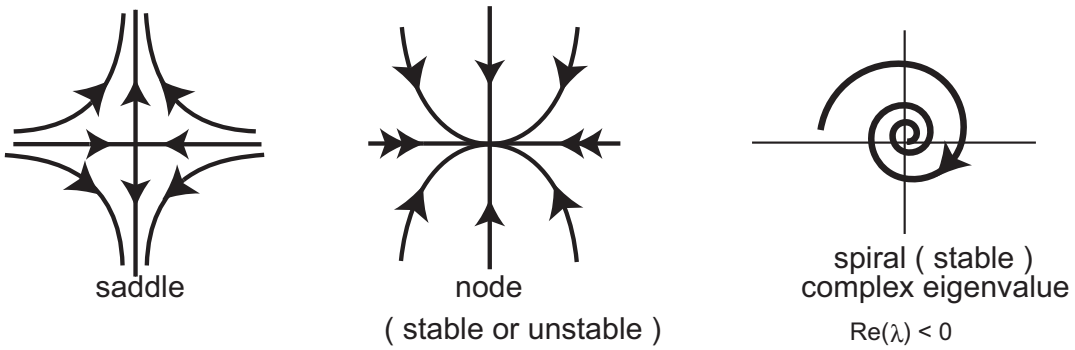
$$y(t) = \left(\left(\frac{x}{x_0} \right)^{1/\lambda_1} \right)^{\lambda_2} y_0 = y_0 \left(\frac{x}{x_0} \right)^{\frac{\lambda_2}{\lambda_1}} .$$

Thus,

$$y(t) = C\, x(t)^{\frac{\lambda_2}{\lambda_1}} .$$

Possible Phase Portraits:

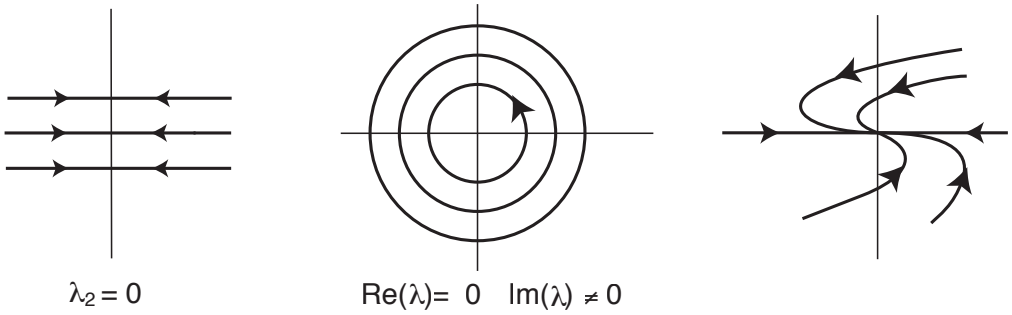
i) generic cases:



Note:

- For symmetric matrices $\mathbf{L}$ eigenvectors for different eigenvalues are orthogonal to each other. For non-symmetirc matrices this need not be the case.

ii) special cases:



At a degenerate node there is a repeated eigenvalue but only a single proper eigenvector,

$$\mathbf{L} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

The system is *almost* oscillating:

$$\mathbf{L} = \begin{pmatrix} \lambda & 1 \\ \epsilon & \lambda \end{pmatrix} \qquad (\lambda - \sigma)^2 = \epsilon \qquad \sigma = \lambda \pm \sqrt{\epsilon}$$

In 2 dimensions the dependence of the phase diagram on the parameters can be given easily:

Eigenvalues in 2d:

$$\det \mathbf{L} = \det\left(\mathbf{S}^{-1}\mathbf{L}\mathbf{S}\right) = \lambda_1\lambda_2 \qquad \text{tr } \mathbf{L} = \text{tr } \left(\mathbf{S}^{-1}\mathbf{L}\mathbf{S}\right) = \lambda_1 + \lambda_2$$

$$\lambda_{1,2} = \frac{+tr\mathbf{L} \pm \sqrt{tr\mathbf{L} - 4\det \mathbf{L}}}{2}$$

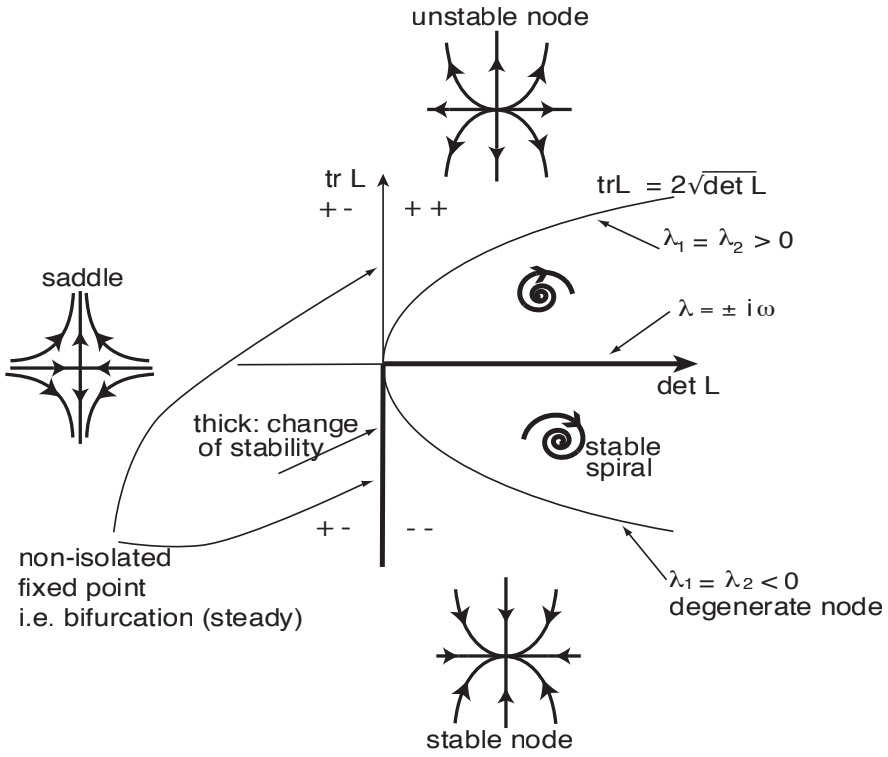
Change in stability: Re $(\lambda_i) = 0$

i) tr$\mathbf{L} = 0$ and $\det \mathbf{L} > 0$ $\Rightarrow$ $\lambda = \pm i\omega$ complex pair crossing imaginary axis

ii) tr$\mathbf{L} < 0$ and $\det \mathbf{L} = 0$ $\Rightarrow$ $\lambda_1 = 0 \;\; \lambda_2 < 0$ single zero eigenvalue

Change in the character of the phase diagram:

real $\leftrightarrow$ complex $(\text{tr}\mathbf{L})^2 = 4\det \mathbf{L}$



Notes:

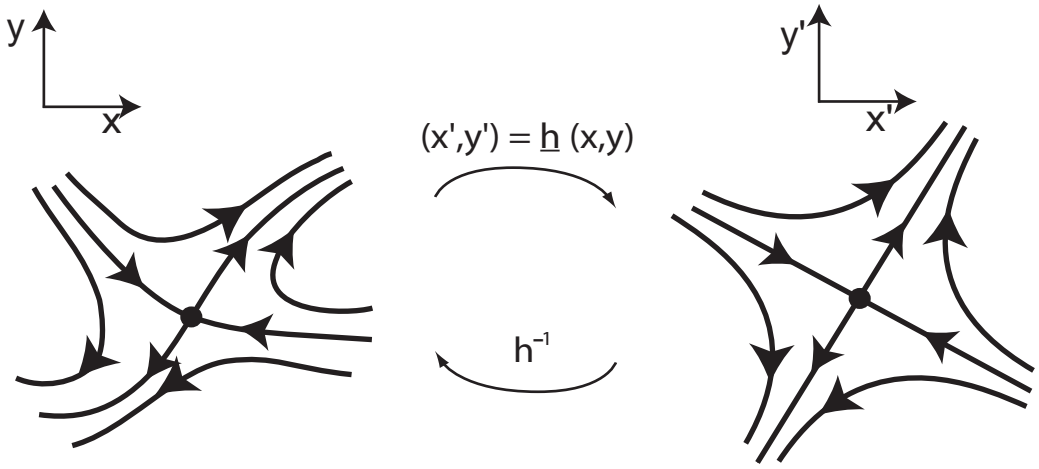
- degenerate node $\Rightarrow$ border between nodes and spirals, does not quite oscillate
- non-isolated fixed points: steady bifurcation, one or more fixed points are created/annihilated (details depend on nonlinearities)

2.1 Hartman-Grobman theorem

Linear systems can be completely understood without too much difficulty. How much of that can be transferred to nonlinear systems?

Definition: A fixed point $\mathbf{x}_0$ of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is called **hyperbolic** if all eigenvalues of the Jacobian $\frac{\partial f_i}{\partial x_j}$ have non-zero real parts.

Thus: in all directions a hyperbolic fixed point is either linearly attractive or repulsive. It has no marginal direction.



Hartman-Grobman Theorem:

Let $\mathbf{x} = 0$ be a hyperbolic fixed point of

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mu)$$

for some fixed μ and let ϕ_t be the corresponding nonlinear flow,

$$\mathbf{x}(t) = \phi_t(\mathbf{x}),$$

and $\tilde{\phi}_t$ the flow of the linearized problem

$$\dot{\mathbf{x}} = \mathbf{L}\,\mathbf{x} \qquad L_{ij} = \frac{\partial f_i}{\partial x_j}.$$

Then there exists a homeomorphism $\mathbf{h} : \mathbb{R} \to \mathbb{R}$ and a neighborhood $\mathcal{U}$ of $\mathbf{x} = 0$ such that

$$\phi_t(\mathbf{x}) = \mathbf{h}^{-1}\left(\tilde{\phi}_t(\mathbf{h}\,(\mathbf{x})\right)$$

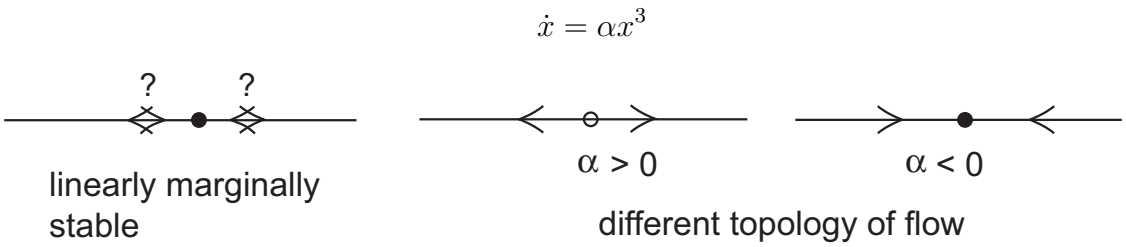
for $\mathbf{x} \in \mathcal{U}$. The homeomorphism $\mathbf{h}$ preserves the sense of orbits.

Note:

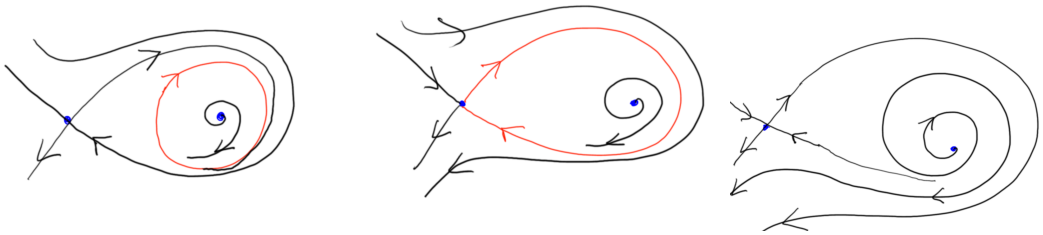
- A homeomorphism is a continuous mapping with a continuous inverse. It preserves the topology of the region.

Thus:

- For a hyperbolic fixed point $\mathbf{x}_0$ the linearization of the flow gives the **topology** of the nonlinear flow in a neighborhood of $\mathbf{x}_0$. The nonlinear and the linear flow are **qualitatively the same.**
- If a fixed point is not hyperbolic, the linearization does not give sufficient information to determine the topology of the flow in its vicinity:



- Topological changes in the nonlinear flow that are **local** to the fixed point x_0 must be reflected in the **linearization** around that fixed point.
- When the parameter μ is changed the dimensions of the stable and unstable manifolds $W^{(s,u)}$ can only change if the dimensions of the corresponding linear eigenspaces $E^{(s,u)}$ change, i.e. the real part of some eigenvalue must pass through 0.
- Only *local changes* in phase space are indicated by changes in the linearization; *global changes* are not indicated by changes in the linearization.



In this scenario the periodic orbit disappears in a global bifurcation involving a *homoclinic orbit*.

3 Bifurcations in 1 Dimension

3.1 Implicit Function Theorem

What can happen when a fixed point is *not* hyperbolic?

For simplicity consider first a one-dimensional system,

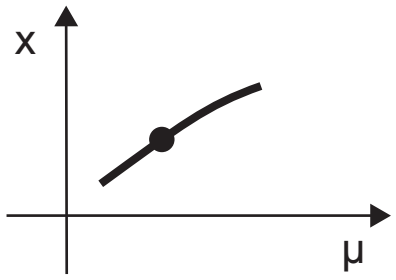
$$\dot{x} = f(x, \mu),$$

that has a fixed point x_0 for $\mu = \mu_0$,

$$f(x_0, \mu_0) = 0\,.$$

Under what conditions does that fixed point persist when the parameter μ is varied away from μ_0, i.e. under what conditions is there a **branch** of fixed points?

Can a small change in μ create or remove a fixed point?



Consider a *local* analysis near x_0 for small changes in μ away from μ_0:

Taylor expansion

$$f(x,\mu) \;=\; \underbrace{f(x_0,\mu_0)}_{=0} + \frac{\partial f}{\partial x}(x-x_0) + \frac{\partial f}{\partial \mu}(\mu-\mu_0) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x-x_0)^2 + \cdots \qquad (1)$$

(All derivatives evaluated at x_0, μ_0)

The fixed point condition implies $f(x_0,\mu_0)=0$.

If $\frac{\partial f}{\partial x}|_{x_0,\mu_0} \neq 0$ we can solve *uniquely* for x

$$x - x_0 = -(\mu-\mu_0)\frac{\frac{\partial f}{\partial u}}{\frac{\partial f}{\partial x}} + O\left((\mu-\mu_0)^2\right) .$$

Thus, in this case there is a *differentiable branch* of solutions. This is the statement of the *Implicit Function Theorem.*

It applies also more generally to systems in higher dimensions:

Consider solutions of

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},\mu) \qquad \mathbf{x}\in\mathbb{R}^n \qquad \mathbf{f} \text{ smooth in } \mathbf{x} \text{ and } \mu\,.$$

Expand again around a fixed point $\mathbf{x}_0$ at μ_0

$$\mathbf{f}(\mathbf{x},\mu) = \mathbf{f}(\mathbf{x}_0,\mu_0) + \mathbf{L}\,(\mathbf{x}-\mathbf{x}_0) + \left.\frac{\partial \mathbf{f}}{\partial \mu}\right|_{\mathbf{x}_0,\mu_0} (\mu-\mu_0) + \ldots$$

with the Jacobian $\mathbf{L}$ given by

$$L_{ij} = \left.\frac{\partial f_i}{\partial x_j}\right|_{\mathbf{x}_0,\mu_0} .$$

If $\mathbf{L}$ is invertible, we can solve for $\mathbf{x}$

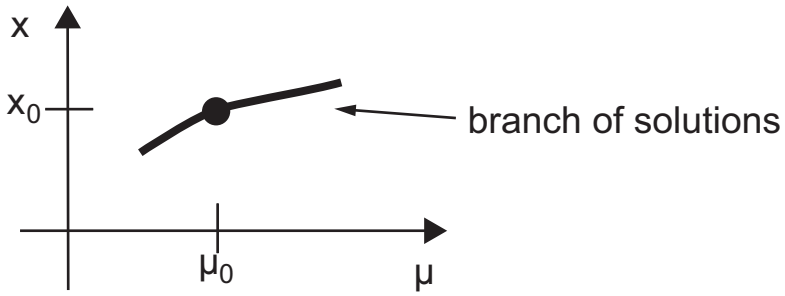
$$\mathbf{x}-\mathbf{x}_0 = -\mathbf{L}^{-1}\left.\frac{\partial \mathbf{f}}{\partial \mu}\right|_{\mathbf{x}_0,\mu_0} (\mu-\mu_0)\ .$$

Thus, if

$$\mathbf{f}(\mathbf{x}=\mathbf{x}_0,\mu=\mu_0)=0 \text{ and } \det\left(\frac{\partial f_i}{\partial x_j}\right)\neq 0 \text{ at } \mu=\mu_0 \text{ and } \mathbf{x}=\mathbf{x}_0,$$

then there is a **unique** differentiable $\mathbf{X}(\mu)$ that satisfies

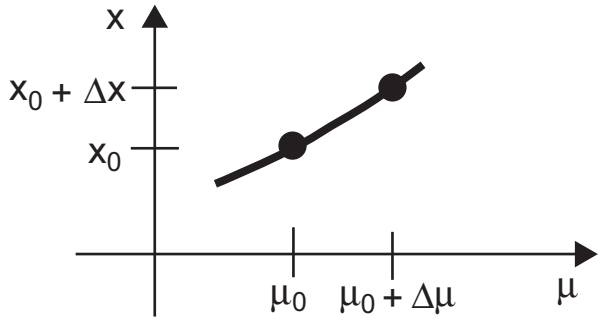
$$\mathbf{f}(\mathbf{X}(\mu),\mu)=0 \qquad \text{and} \qquad \mathbf{X}(\mu=\mu_0)=\mathbf{x}_0.$$



Notes:

- For $\det\mathbf{L}\neq 0$ the branch of fixed points *persists uniquely* $\Rightarrow$ the number of fixed points does not change
  - persistence: the fixed point does not disappear
  - uniqueness: no new fixed point appears
- The change of x is smooth in μ if $\frac{\partial f}{\partial x}\neq 0$

$$\Delta x\sim\Delta\mu$$



- **Generic properties** are those properties that do not require **any tuning** of the parameters
  When picking parameters randomly one expects $\det\mathbf{L}\neq 0$,
  i.e. in general one needs to *tune* μ to get $\det L=0$.
  $\Rightarrow$ **generically** there is a smooth branch.

The existence or non-existence of a smooth unique branch is directly connected with the linear stability of the fixed point:

For small perturbations around $\mathbf{x} = \mathbf{x}_0 + \Delta\mathbf{x}(t)$ the evolution can be approximated by the linearization

$$\frac{d}{dt}\Delta\mathbf{x} = \mathbf{L}\Delta\mathbf{x}\,.$$

Thus

- For the number of fixed points to change at μ_0 it is necessary that $\det\mathbf{L} = 0$, i.e. the fixed point needs to be non-hyperbolic, its stability has to change as μ is changed across μ_0.

3.2 Saddle-Node Bifurcation

Focus now on one-dimensional systems. What happens when $\frac{\partial f}{\partial x} = 0$?

We need to go to higher order in the Taylor expansion (choose $x_0 = 0, \mu_0 = 0$)

$$f(x,\mu) = \underbrace{f(0,0)}_{=0} + \underbrace{\frac{\partial f}{\partial x}}_{=0} x + \frac{\partial f}{\partial \mu}\mu + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}x^2 + \frac{\partial^2 f}{\partial x \partial \mu}x\mu + \frac{1}{2}\frac{\partial^2 f}{\partial \mu^2}\mu^2 + \ldots$$

Solve again for x,

$$x^2 = -\frac{2}{\frac{\partial^2 f}{\partial x^2}}\left\{\underbrace{\frac{\partial f}{\partial \mu}\mu}_{x=\mathcal{O}(\mu^{1/2})} + \frac{\partial^2 f}{\partial x \partial \mu}\underbrace{x\mu}_{\mathcal{O}(\mu^{3/2})} + \frac{1}{2}\frac{\partial^2 f}{\partial \mu^2}\mu^2 + \cdots\right\}.$$

Which terms are to be kept? By assumption we have $|x| \ll 1$ and $|\mu| \ll 1$. Even though we do not yet have a relationship between x and μ, we have $|x\mu| \ll |\mu|$ and $\mu^2 \ll |\mu|$. Therefore, to leading order only the first term on the right-hand side needs to be kept and we get

$$x_{1,2} = \pm\sqrt{-2\frac{\frac{\partial f}{\partial \mu}}{\frac{\partial^2 f}{\partial x^2}}\,\mu} + \mathcal{O}(\mu)\,.$$

Notes:

When the implicit function theorem fails

- one gets a nonlinear equation for x with *multiple* solutions
- the number of solutions changes with the parameter μ
- the change in x is ***not smooth*** in μ.

Dynamics:

To assess the stability of these multiple solutions we need to reintroduce the dynamics,

$$\dot{x} \;=\; f(x,\mu) = a\mu + \frac{1}{2}bx^2 + \text{h.o.t.}\,, \tag{2}$$

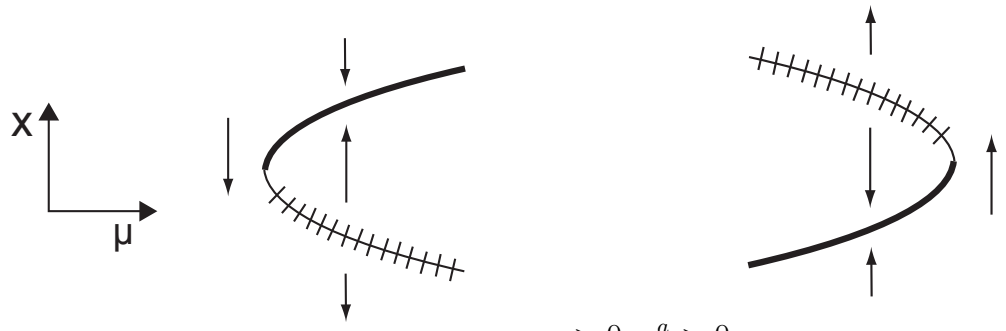
where the relevant parameters are given by

$$a = \frac{\partial f}{\partial \mu} \equiv \partial_\mu f \qquad b = \frac{\partial^2 f}{\partial x^2} \equiv \partial_x^2 f$$

Bifurcation diagrams:

To get an overview plot all solution branches as a function of μ,

$$x_{1,2} = \pm\sqrt{-2\frac{a}{b}\,\mu} + \mathcal{O}(\mu)\,.$$



$a>0 \quad \frac{a}{b}<0$: $a>0 \quad \frac{a}{b}>0$:

In total there are four qualitatively different cases: switching the sign of a with a/b fixed reverses the flow and flips the arrows in the bifurcation diagram.

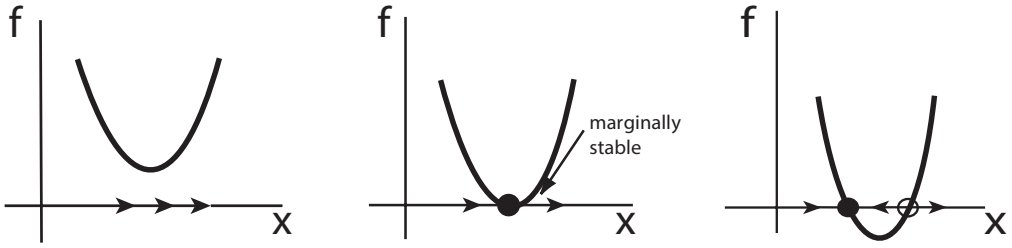
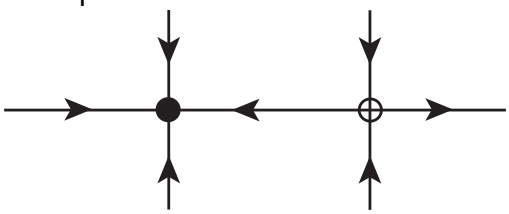


Figure 2: Phase line for increasing values of μ for $a<0$, $b>0$. The arrows indicate the flow on the phase line (x-axis).

Notes:

- Generically the minimum of f in x is quadratic $\Rightarrow b \neq 0$. Equation (2) is therefore the *normal form* for a saddle-node bifurcation. The same equation will be obtained in general systems - including higher-dimensional systems - in the vicinity of the bifurcation (cf. later).

- 2 fixed points are created/destroyed simultaneously: single solutions do not simply pop up or disappear.
  - The roots of a *real* polynomial can only become complex as complex pairs $\Rightarrow$ solutions disappear in pairs.
  - Single solutions can only disappear by diverging at infinity
- When the two fixed points coincide at $\mu = 0$ they are marginally stable ($det\mathbf{L} = 0$): going along the solution branch, $\partial_x f$ changes sign and the solution **changes stability**, consistent with the earlier statement that a change in the number of fixed points is associated witha a change in stability.
- The flow changes direction **only locally**:
  only when μ goes through 0 and only near the fixed point $x = 0$ does the flow change direction.
  Away from bifurcation point the flow is qualitatively unchanged when μ changes (the arrows far away remain the same).
- Why are these bifurcations called saddle-node bifurcations? In higher dimensions a saddle-node bifurcation occurs when a node collides with a saddle, eliminating both fixed points.



- The only condition for a saddle-node bifurcation to occur is $\partial_x f = 0$. This is the condition for **any** (steady) bifurcation to occur.
  Thus: If there is a bifurcation because a real eigenvalue goes through 0, one should 'expect' a saddle-node bifurcation.
- Saddle-node bifurcations are sometimes also called "blue-sky bifurcations", because solutions appear out of the 'blue sky'.

Examples:
A compressed, upward-curved beam under a transverse load can 'snap through'. At that point the stable solution collides with an unstable solution that is less buckled.

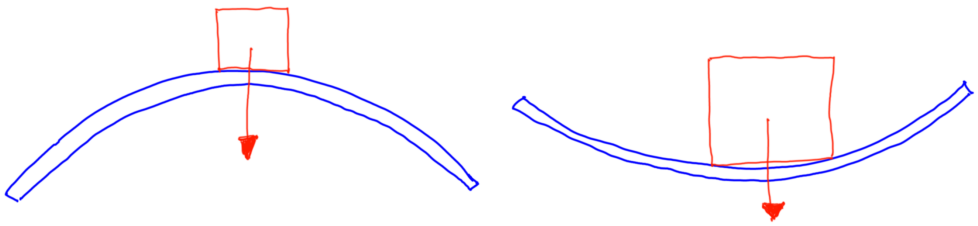


Figure 3: A buckled beam undergoes a saddle-node bifurcation when the load becomes too large.

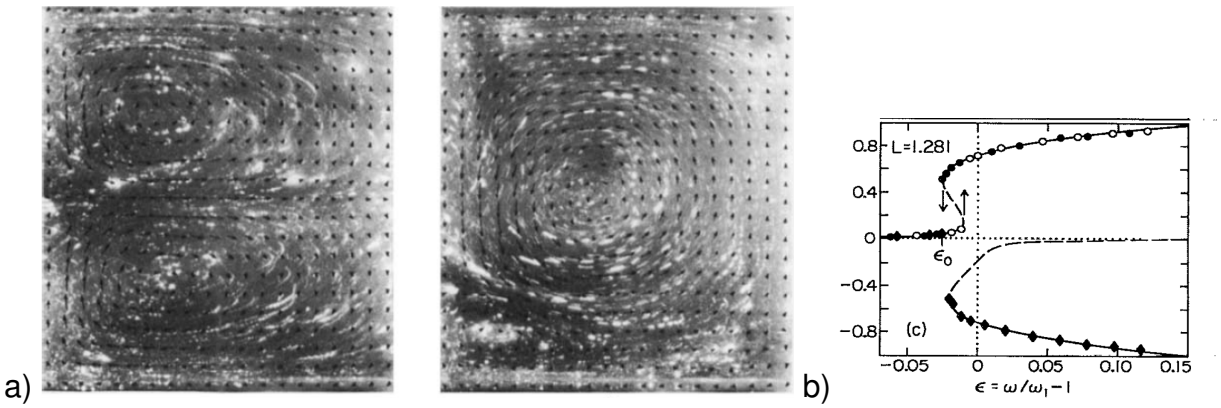


Figure 4: Three saddle-node bifurcations in Taylor vortex flow in a short cylinder. a) left: symmetric vortices below the bifurcation, right: asymmetric vortices (note the small vortex in the bottom left corner) above the transition (Lücke et al., 1984). b) Bifurcation diagram in terms of the degree of asymmetry as a function of the rotation rate ω of the cylinder (Aitta et al., 1985).

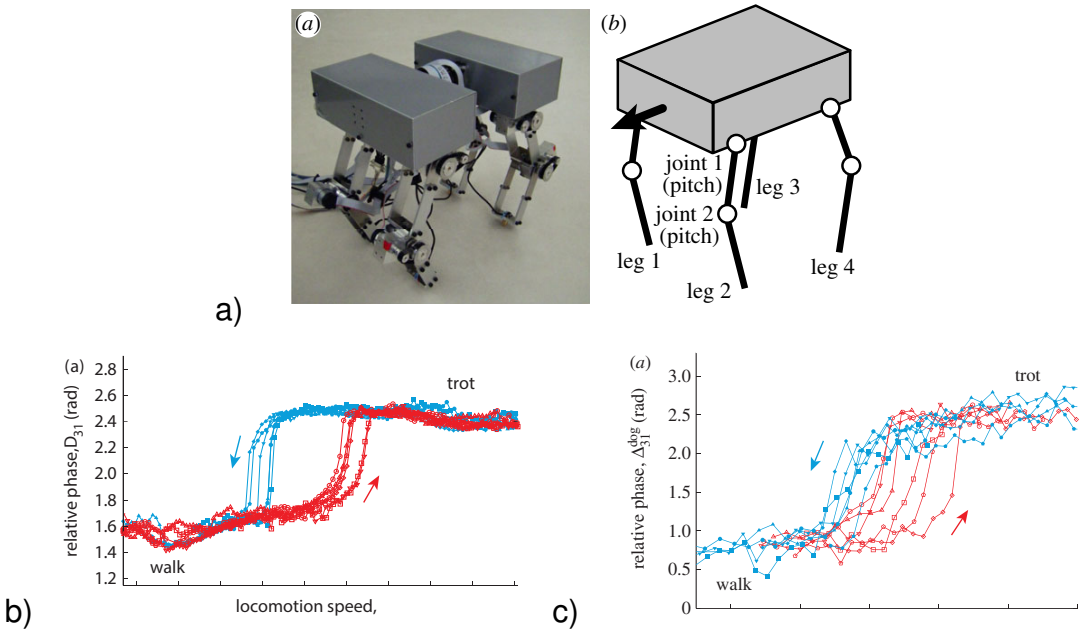


Figure 5: Hysteresis via saddle-node bifurcations in the transition between walk and trot gait as a function of locomotion speed. The gait is characterized by the phase difference between front and rear legs. a) 4-legged robot. b) hysteresis in the robot gait, c) hysteresis in dog gait (Aoi et al., 2013).

3.3 Transcritical Bifurcation

Consider a system that satisfies an additional condition beyond that of the occurrence of a bifurcation: assume a fixed-point solution exists for all μ. For simplicity assume that

solution is $x = 0$:

$$\dot{x} = f(x, \mu) \qquad \text{with} \qquad f(0, \mu) = 0 \qquad \text{for all } \mu\,.$$

Performing again a Taylor expansion around the bifurcationpoint $\mu = 0$,

$$f(x,\mu) = \underbrace{f(0,0)}_{=0} + \underbrace{\partial_x f|_{0,0}}_{=0}\, x + \underbrace{\partial_\mu f|_{0,0}}_{=0}\,\mu + \frac{1}{2}\,\partial_x^2 f\big|_{0,0}\, x^2 + \partial_{x\mu}^2 f\big|_{0,0}\, x\mu + \underbrace{\frac{1}{2}\,\partial_\mu^2 f\big|_{0,0}}_{=0}\,\mu^2 + \dots \quad (3)$$

Based on our assumption the following terms vanish:

- $x = 0$ fixed point: $f(0, 0) = 0\,.$
- a bifurcation occurs: $\partial_x f|_{0,0} = 0\,.$
- $x = 0$ is a fixed point for all μ: $\partial_\mu^n f\big|_{0,0} = 0$ for all n.

To leading order we obtain then

$$\dot{x} = x\,(a\,\mu + b\,x) + \cdots \quad (4)$$

with

$$a = \partial_{x\mu}^2 f \qquad b = \frac{1}{2}\partial_x^2 f$$

It has two fixed points:

$$x_1 = 0 \qquad x_2 = -\frac{a}{b}\mu \equiv -\frac{\partial_{x\mu}^2 f}{\frac{1}{2}\partial_x^2 f}\,\mu$$

Depending on the signs of a and b there are four cases (cf. Fig.6).

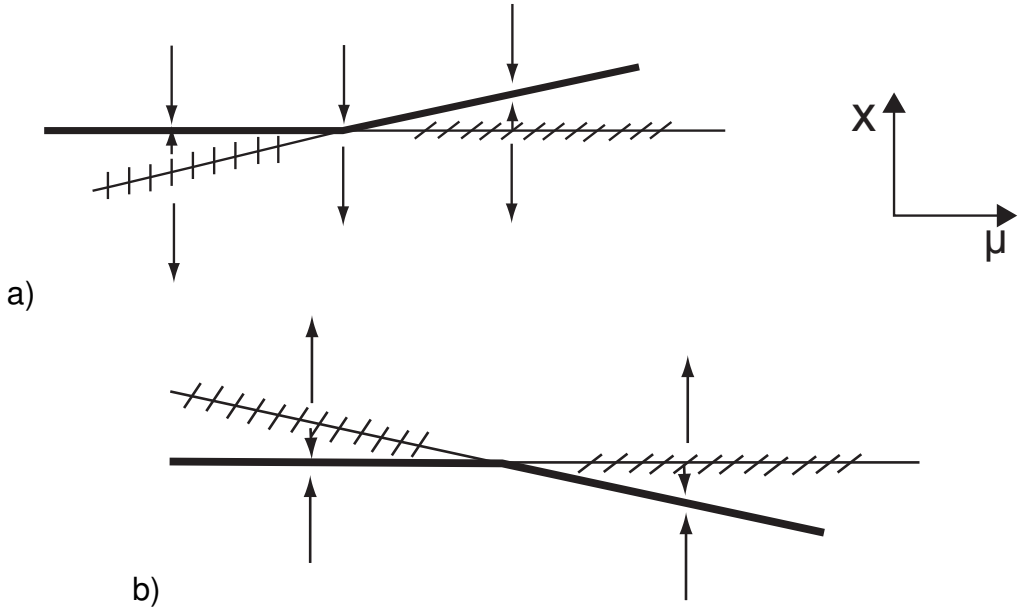


Figure 6: a) $\frac{a}{b} < 0$, $a > 0$. b) $\frac{a}{b} > 0$, $a > 0$. Switching the sign of a with a/b fixed reverses the flow, i.e. flips the arrows in bifurcation diagram, and switches the stability.

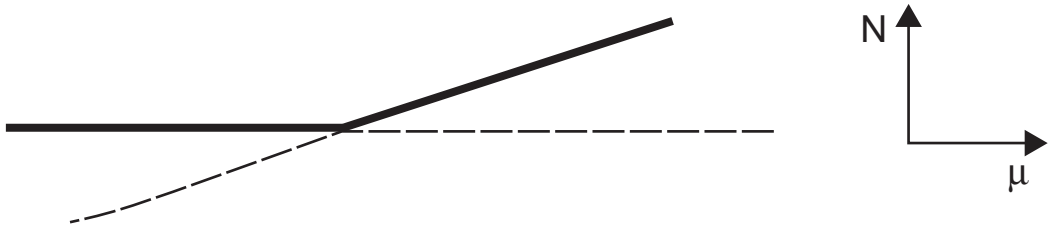
Notes:

- Both fixed points exist below and above the bifurcation (for $\mu < 0$ and $\mu > 0$). Consistent with the statement of the implicit function theorem, there is, however, no *unique* branch of solutions going through μ.
- Equation (4) is the *normal form* for a transcritical bifurcation.
- The transcritical bifurcation is characterized by an *exchange of stability* between the two branches.
- There is a subcritical branch:
  Sufficiently large perturbations can lead away from the (linearly) stable fixed point.

Examples:

a) **Logistic equation for population growth**

$$\dot{N} = \mu N - N^2$$



- The branch existing for all μ corresponds to a vanishing population size N.
- For $\mu < 0$ the lower branch is unphysical since the population N cannot be negative.

b) **Rayleigh-Benard convection in a fluid layer heated from below**

- The state without fluid flow (corresponding to $x = 0$) exists for all temperature differences.
- Hexagonal flow patterns arise in a transcritical bifurcation at $\epsilon = 0$ (corresponding to $\mu = 0$).
- The branch emerging from the transcritical bifurcation undergoes a saddle-node bifurcation.
- Large perturbations can kick the solution without fluid flow above the unstable branch of the transcritical bifurcation and trigger the formation of hexagonal convection patterns.
- There is bistability between the hexagonal pattern and the homogeneous state $x = 0$.

- For $\mu > 0$ the lower branch associated with the transcritical bifurcation is unstable in a different way; this instability is not contained in the single equation (4).

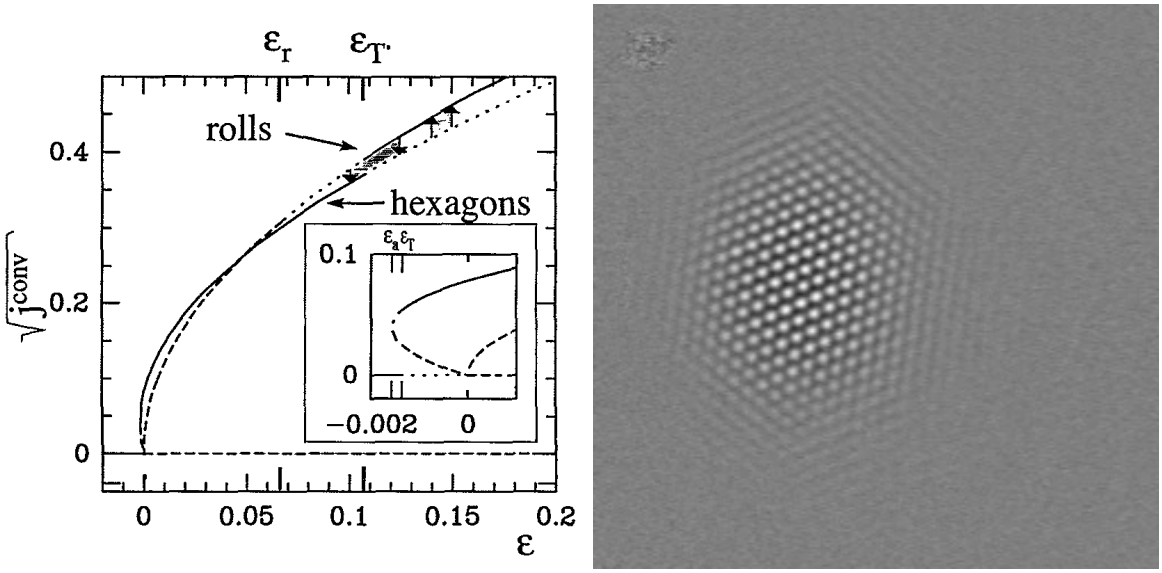


Figure 7: Convection in very thin fluid layers sets in via a transcritical bifurcation to hexagonal convection patterns. The root of the heat flux, $\sqrt{j^{(conv)}}$, plays the role of the magnitude $|x|$ of the amplitude x. The hexagons and the convection-less state are simultaneously linearly stable in a (very small) range of parameters. If the heating is increased the pattern expands into the whole system (Bodenschatz et al., 1991).

The transcritical bifurcation is part of a larger bifurcation scenario (see inset in Fig.7:

- A large perturbation can kick the solution above the unstable branch of the transcritical bifurcation leading to a stable branch of hexagonal convection
  - For $\mu > 0$ the lower branch is unstable in a different way (that instability not contained in the single equation).

3.4 Pitchfork Bifurcation

Consider systems that have a reflection symmetry $x \to -x$, which is defined by the requirement that if $x(t)$ is a solution, then $-x(t)$ is also a solution.

Thus, we have

$$\dot{x} = f(x, \mu) \qquad \text{and} \qquad -\dot{x} = f(-x, \mu)\,.$$

Multiplying one of the two equations by -1 implies that $f(x, \mu)$ is odd in x,

$$f(-x, \mu) = -f(x, \mu)\,.$$

As a consequence, $x = 0$ is a solution for all μ and all even terms in x of the Taylor expansion of $f(x, \mu)$ vanish.

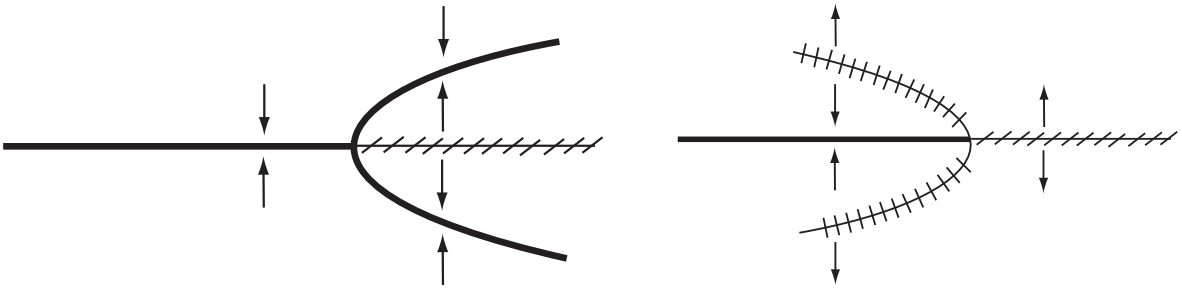
Taylor expansion:

$$f(x,\mu) = \underbrace{f(0,0)}_{=0} + \underbrace{\partial_x f}_{=0}\, x + \underbrace{\partial_\mu f}_{=0}\,\mu + \underbrace{\partial^2_{x\mu}}_{a}\, x\mu + \underbrace{\frac{1}{6}\partial^3_x}_{b}\, x^3 + \ldots$$

$$\dot{x} = a\mu x + bx^3 + h.o.t.$$

Fixed points:

$$\begin{aligned} x_0 &= 0 \\ x_{2,3} &= \pm\sqrt{-\frac{a}{b}\mu} \end{aligned}$$



$a > 0 \quad b < 0$ supercritial $\qquad$ $a > 0 \quad b > 0$ subcritical

Notes:

- At $\mu = 0$ the fixed point changes stability and two solutions appear/disappear.
- The bifurcation is called supercritical if the nonlinear branch arises for those values of μ for which the base state $x = 0$ is linearly unstable. In this case the instability saturates and leads to a stable new solution.
- In the subcritical case the nonlinear branch arises for μ-values for which $x = 0$ is stable. There is no saturation of the linear instability of the base state to cubic order. Higher-order terms determine whether a new stable solution arises when the base state becomes unstable.
- The system has reflection symmetry $x \to -x$
  - the solution $x_0 = 0$ has that symmetry as well.
  - the solutions $x_{2,3} = \pm\sqrt{\ldots}$ themselves are *not* reflection-symmetric. Instead they form a pair of symmetrically related solutions.

  $\Rightarrow$ the pitchform bifurcation is a *symmetry-breaking* bifurcation.

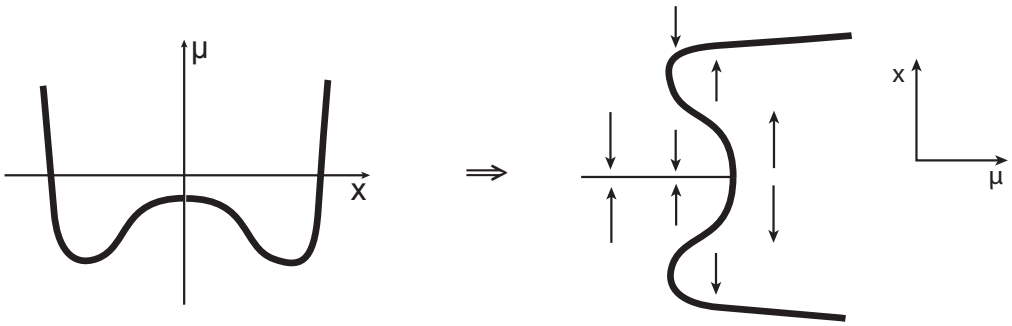
Subcritical Pitchfork Bifurcation:

For $b > 0$ we can include a quintic term with the aim to provide saturation of the instability,

$$\dot{x} = \mu x + \underbrace{bx^3}_{\text{destabilizing}} - \underbrace{cx^5}_{\text{stabilizing}} \tag{5}$$

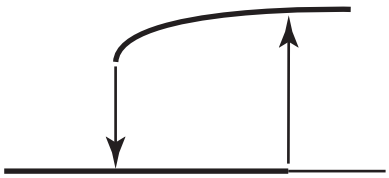
Assume $c > 0$. In general, the term at quintic order need not be saturating.

To get the bifurcation diagram, plot $\mu = \mu(x)$ and flip the plot about the diagonal



Notes:

- In this equation the subcritical pitchfork bifurcation is associated with 2 saddle-node bifurcations, which occur for $\mu < 0$.
- There is a range of bistability which is associated with hysteresis. I.e. the transition between the two states occurs at different values of μ depending on whether μ is increased or decreased.



Question: are the conclusions about the saddle-node bifurcations based on the analysis of (5) guaranteed to be valid? Since we used a Taylor expansion in μ and in x the results are only guaranteed to be valid in the limit $\mu \to 0$ and $x \to 0$. Concretely, in the expansion we ignored terms of $\mathcal{O}(x^7)$ and higher.

On the lower branch we have $x \to 0$ as $\mu \to 0$, as is the case for the other bifurcations discussed above.Therefore the results are correct in the vicinity of the bifurcation point, $|\mu| \ll 1$.

On the upper branch, however, we have

$$x^2_{upper} = \frac{b + \sqrt{b^2 - 4\mu c}}{2c} \,. \tag{6}$$

Therefore, even for $\mu = 0$ we have

$$x^2_{upper} = \frac{b}{c} = \mathcal{O}(1) \,.$$

Validity therefore requires that b be small *in addition* to μ. Thus, we need to expand around the *tricritical* point $b = 0$, which is the point at which the bifurcation changes from supercritical ($b < 0$) to subcritical ($b > 0$), i.e. the bifurcation has to be *weakly subcritical*.

This can also be seen by noting that the solution (6) involves all three terms of (5). These three terms should therefore be of the same order

$$\mu x \sim b x^3 \sim c x^5 \,.$$

For $c = \mathcal{O}(1)$, this implies the scaling

$$\mu = \mathcal{O}(x^4) \qquad b = \mathcal{O}(x^2)\,.$$

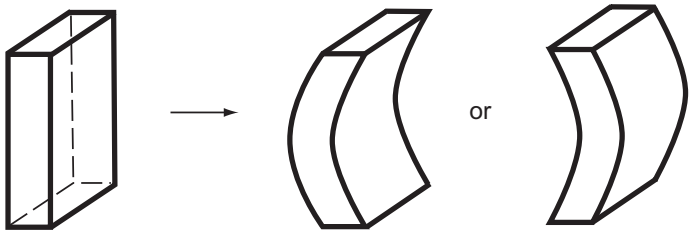
Then all three terms are $\mathcal{O}(x^5)$, compared to which the neglected terms $\mathcal{O}(x^7)$ are indeed small. Equation (5) is therefore valid in the *distinguished limit*

$$\mu \to 0 \qquad \text{and} \qquad b \to 0 \qquad \text{with} \quad b = \hat{b}\,\mu^{\frac{1}{2}}\,.$$

Conversely, if $b = \mathcal{O}(1)$, then $x_{upper} = \mathcal{O}(1)$ and there is no reason that any of the omitted terms ($\mathcal{O}(x^7)$ and higher) can be neglected and stopping the expansion at quintic order is not justfied. As a result, it could be that there is no saddle-node bifurcation at all, even if $c > 0$.

Examples:

a) buckling of a straight beam



reflection symmetric symmetry related

b) Rayleigh-Bénard roll convection:

up-flow down-flow

Note:

- up $\Rightarrow$ down corresponds to translations by half a wavelength
  intermediate positions also possible $\Rightarrow$ larger symmetry

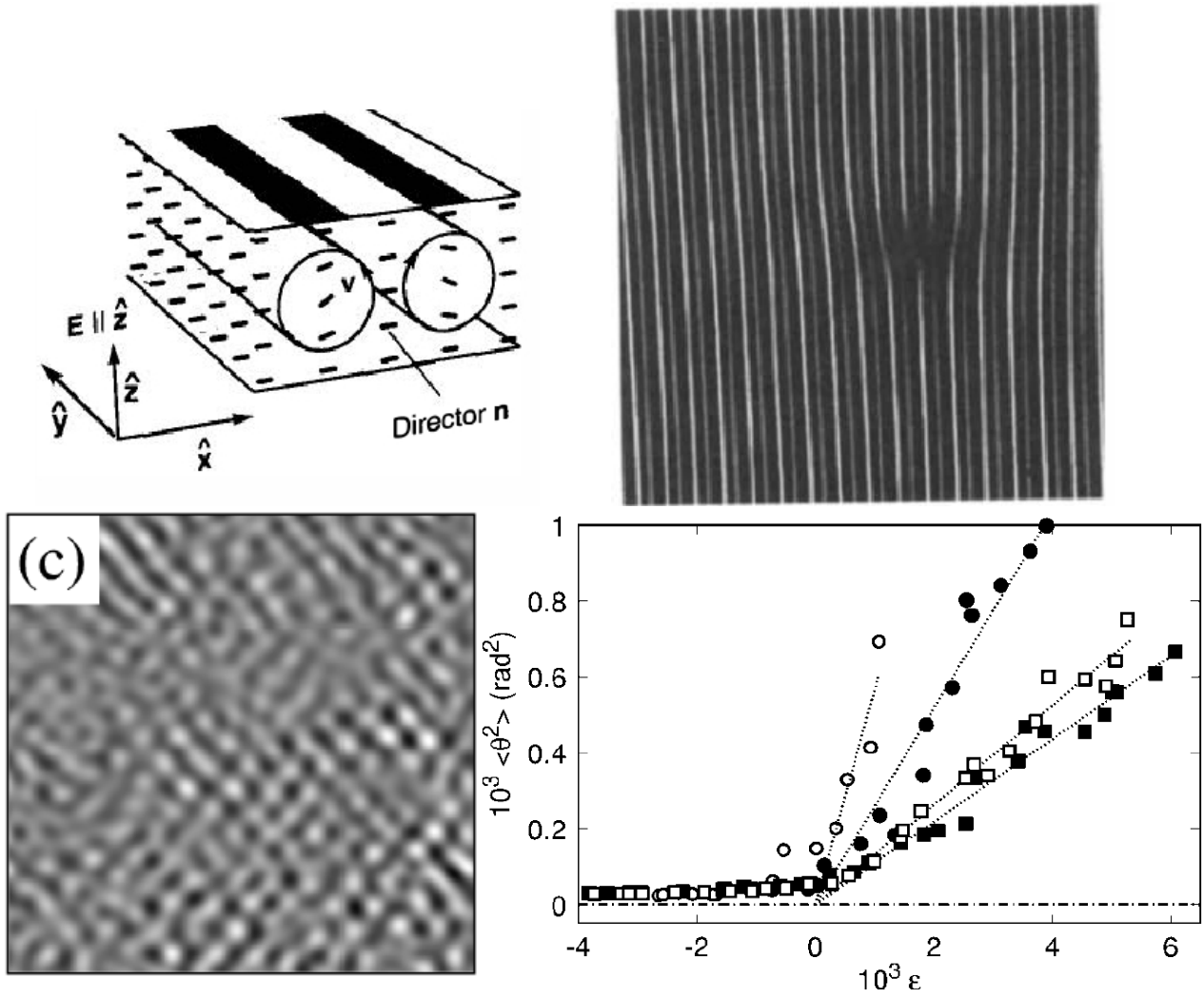


Figure 8: a) An AC electric field applied transversally to a thin layer of nematic liquid crystal can drive fluid flow in the form of rolls. b) Top view of a roll pattern with a dislocation defect. c) Disordered convection pattern slightly above the bifurcation point. d) The square of the pattern amplitude grows linearly at the bifurcation point reflecting the square-root law for the amplitude. As the electrical conductivity of the liquid crystal is changed (different symbols) a tricritical point is approached (the line becomes vertical): the pitch-fork bifurcation eventually becomes subcritical. (Scherer et al., 2000).

3.5 Structural Stability of Bifurcations

Since we are mostly interested in the qualitative behavior of a system, a relevant question is whether the bifurcations that we have identified are *robust* with respect to small changes in the parameters of the equations or small changes *in the equations themselves*.

Define:

- A bifurcation is called **structurally stable** if small perturbations δ of the *equations* do not change the bifurcations qualitatively.

Note:

- Even in a structurally stable bifurcation an individual solution can change qualitatively when a parameter is changed infinitesimally; it is the overall set of solutions that should remain qualitatively the same.
- This stability is to be contrasted with the dynamical stability of a solution to a given equation, which refers to the stability under perturbations of the solution with the equation and its parameters fixed.

3.5.1 Saddle-Node Bifurcation

Consider small perturbations of $\mathcal{O}(\delta)$ of the equation for the saddle-node bifurcation by allowing f to depend also on $\delta \ll 1$,

$$\begin{aligned}
\dot{x} &= f(x,\mu,\delta) = \mu + x^2 + \delta\left(\partial_\delta f + \partial_{\delta\mu} f\,\mu + \partial_{x\delta} f\,x + \frac{1}{2}\partial_{\delta xx} f\,x^2\right) + \mathcal{O}(\delta^2, \delta\mu^2, \delta x^3) \\
&= x^2\left(1 + \frac{1}{2}\delta\partial_{\delta xx} f\right) + x\left(\delta\partial_{\delta x} f\right) + \left(1 + \delta\partial_{\delta\mu} f\right)\mu + \delta\partial_\delta f \\
&= \left(1 + \frac{1}{2}\delta\partial_{\delta xx} f\right)(x - \Delta x_0)^2 + \left(1 + \delta\partial_{\delta\mu} f\right)(\mu - \Delta\mu_0) + \mathcal{O}(\delta^2, \delta\mu^2, \delta x^3)\,,
\end{aligned}$$

with the shifts Δx_0 and $\Delta\mu_0$ given by

$$-\left(1 + \frac{1}{2}\delta\partial_{\delta xx} f\right)2\,\Delta x_0 = \delta\partial_{\delta x} f \qquad \Rightarrow \qquad \Delta x_0 = -\frac{1}{2}\delta\partial_{\delta x} f + \mathcal{O}(\delta^2)$$

$$-\left(1 + \delta\partial_{\delta\mu} f\right)\Delta\mu_0 = \delta\partial_\delta f \qquad \Rightarrow \qquad \Delta\mu_0 = -\delta\partial_\delta f + \mathcal{O}(\delta^2)\,.$$

Thus

- Small perturbations only shift the position of the 'nose' of the saddle-node bifurcation in μ and x. But the nose as such, i.e. the structure of the solution set, persists.
- The saddle-node bifurcation is structurally stable.

3.5.2 Transcritical Bifurcation

Consider again small perturbations

$$\dot{x} = x\,(a\,\mu + b\,x) + \delta\,(\partial_\delta f + \partial_{\delta x} f\,x + ...) = bx^2 + (a\mu + \delta\partial_{\delta x} f)\,x + \delta\partial_\delta f + \ldots$$

We focus on the term $\delta\partial_\delta f$, since it introduces a term that is not present in the unperturbed equation: it destroys the solution $x = 0$.

The fixed points are given now by

$$x_{1,2} = \frac{1}{2b}\left(-a\mu - \delta\partial_{\delta x}f \pm \sqrt{(a\mu + \delta\partial_{\delta x}f)^2 - 4b\delta\,\partial_\delta f}\right) \tag{7}$$

For $b\delta\partial_\delta f > 0$ there is no fixed point for $(a\mu + \delta\partial_{\delta x}f)^2 < 4b\delta\partial_\delta f$: the transcritical bifurcation is transformed into two saddle-node bifurcations.

For $b\delta\partial_\delta f < 0$ there is no value of μ for which the square root vanishes, i.e. the two solution branches do not touch each other: the transcritical bifurcation is transformed into two smoothly changing solution branches without any bifurcation.

Thus: the transcritical bifurcation is **not structurally stable**.

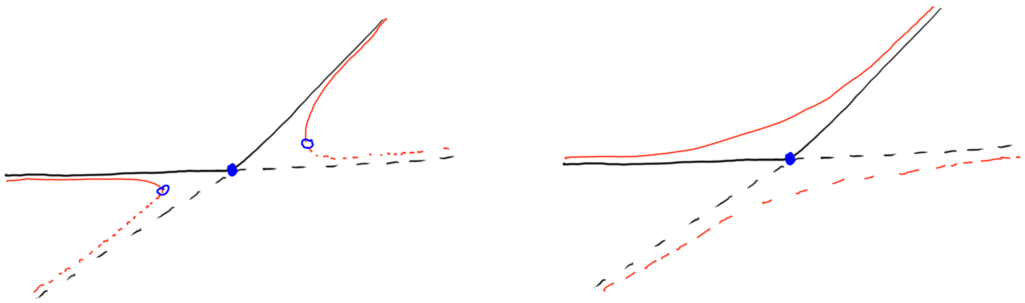


Figure 9: Generic perturbations of the transcritical bifurcation. a) $b\delta\partial_\delta f > 0$. b) $b\delta\partial_\delta f < 0$.

Notes:

- The transcritical bifurcation is structurally stable if the class of systems is restricted to those for which $x = 0$ is a fixed point for all values of $\mu = 0$. The perturbation that breaks the transcritical bifurcation is then not allowed.
- A bifurcation is called **degenerate** if additional conditions "happen" to be satisfied. In a general system a transcritical bifurcation would be considered to be degenerate since $\partial_\mu f$ would vanish at the bifurcation point only 'by chance'.
- To obtain a transcritical bifurcation the condition that $x = 0$ is a solution for all μ is not necessary. The condition $\partial_\mu f = 0$ is sufficient as long as $\partial^2_\mu f$ is not too large or has the correct sign. Solving (3) with $\partial^2_\mu f \mu^2$ retained, one obtains

  $$x_{1,2} = \frac{1}{2b}\left(-a\mu \pm \sqrt{\left(a^2 - 2b\partial^2_\mu f\right)\mu^2}\right),$$

  which yields two branches of fixed points as long as $2b\partial^2_\mu f < a^2$. Otherwise, there is only a single, isolated fixed point $(0, 0)$.

4 1d-Bifurcations in Higher Dimensions: Reduction of Dynamics

Higher-dimensional systems can undergo the same bifurcations as 1-dimensional systems.

$\Rightarrow$ can we reduce dynamics to 1 dimension near the bifurcation?

Local bifurcations change the behavior of phase space *only locally*, e.g. in the vicinity of a fixed point. We want to capture the long-term dynamics near the fixed point $\mathbf{x}_{FP}$.

We want:

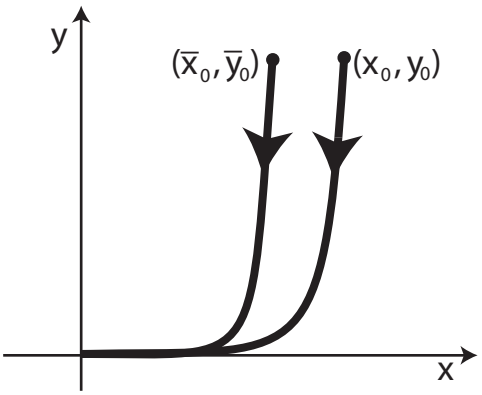
- a manifold of lower dimension that captures the *complete dynamics* of the system as long as they remain *local* to that fixed point.
- that manifold should be invariant under the flow

  $$\mathbf{x}(0) \in \mathcal{M} \qquad \Rightarrow \qquad \mathbf{x}(t) \in \mathcal{M} \quad \text{for all t i.e. for } -\infty < t < \infty$$

  i.e. forward and backward evolution must remain in $\mathcal{M}$.

How could this work? Consider first simple linear example: stable node

$$\left.\begin{aligned} \dot{x} &= \mu x \\ \dot{y} &= -y \end{aligned}\right\} \; y = y_0 \left(\frac{x}{x_0}\right)^{+\frac{1}{|\mu|}} \qquad \mu < 0$$



For small $|\mu|$ the approach $y \to 0$ is extremely rapid as $x \to 0$

$\Rightarrow$ after a short time **any initial condition** approaches the x-axis and stays in its vicinity

Nonlinear Example

How relevant are these linear eigenspaces for nonlinear systems? Explore this using the system

$$\dot{x} = \mu x + \alpha x y - \gamma x^3 \tag{8}$$
$$\dot{y} = -y + x^2 \tag{9}$$

If it was not for the term involving y, the equation for x would describe a supercritical pitch-fork bifurcation. What effect does the additioanl variable y have? Fig.10 shows the phase plane just below and just the bifurcation point. There appears to be a pitch-fork bifurcation, but the new fixed points are not on the x-axis but seem to lie on a smooth curve (manifold) that is tangent to the x-axis, which is the center eigenspace for $\mu = 0$.

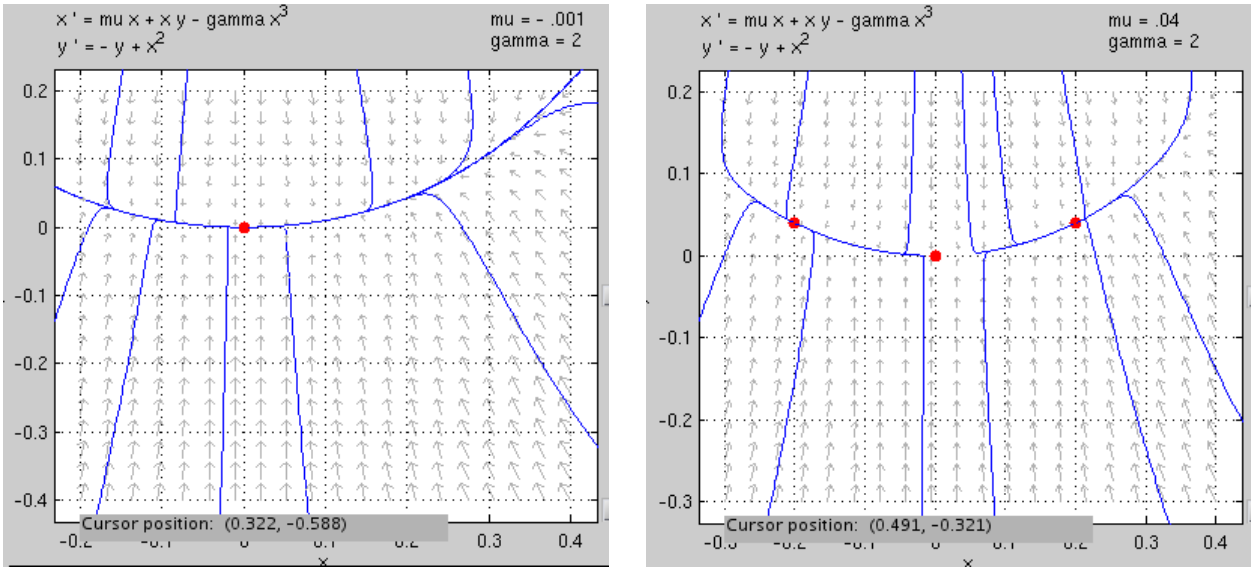


Figure 10: Phase plane for $\mu = -0.001$ (single fixed point) and $\mu = 0.04$ (3 fixed points) showing the slow manifold and the dynamics on it.

Thus:

- The dynamics become effectively one-dimensional

Goal:

- We want to obtain a description of the higher-dimensional system in terms of these one-dimensional dynamics.

Note:

- The description will be valid at most **after the decay** of transients: the approach will forget certain details of the initial conditions.
- From the perspective of perturbation theory there is an initial layer with fast dynamics that connects the slow one-dimensional dynamics to the initial conditions.

4.1 Center Manifold Theorem

To get a mathematically justified description we need the **separation of time scales**, i.e. the ratio of the time scales has to become infinite:

$$\mu \to 0 \qquad \text{we need to be at the bifurcation point}$$

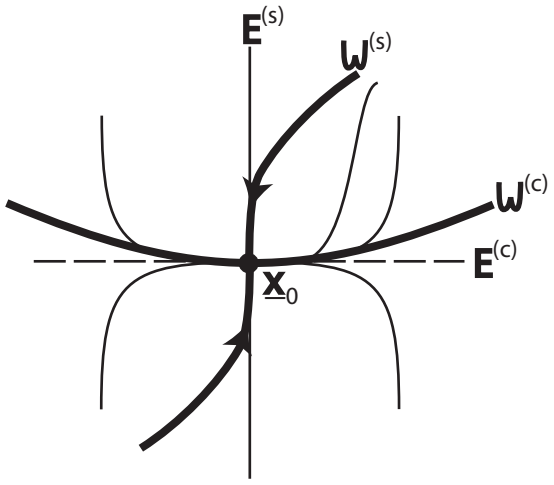
For $\mu = 0$, i.e. at the bifurcation, there are 3 types of eigenvectors/eigenspaces:

- stable eigenspace $E^{(s)} = \{\mathbf{x} \,|\, \mathbf{x} = \sum \alpha_i \mathbf{v}_i^{(s)}\}$
  where the $\underline{v}_i^{(s)}$ are the eigenvectors of the linear system with $Re(\lambda(_i^s)) < 0$.
- unstable eigenspace $E^{(u)} = \{\mathbf{x} \,|\, \mathbf{x} = \sum \alpha_i \mathbf{v}_i^{(u)}\}$ with $Re(\lambda_i^{(u)}) > 0$.
- *center eigenspace* $E^{(c)} = \{\mathbf{x} \,|\, \mathbf{x} = \sum \alpha_i \mathbf{v}_i^{(c)}\}$ with $Re(\lambda_i^{(c)}) = 0$.

Center Manifold Theorem:

For a fixed point $\underline{x}_0$ with eigenspaces $E^{(s,u,c)}$ there exist stable, unstable, and center manifolds $W^{(s,u,c)}$ such that $W^{(s)}$ and $W^{(u)}$ are tangent to $E^{(s)}$ and $E^{(u)}$ at $\underline{x}_0$ and $W^{(c)}$ is tangent to $E^{(c)}$ at $\mathbf{x}_0$.

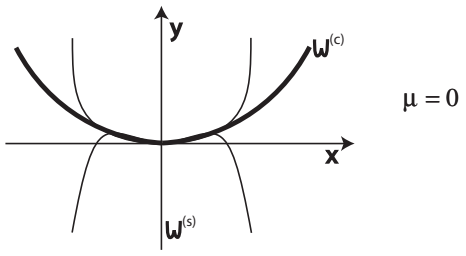
$W^{(s,u,c)}$ are invariant under the flow. $W^{(s)}$ and $W^{(u)}$ are unique. $W^{(c)}$ need not be unique.



For the example from above

$$\begin{aligned} \dot{x} &= \mu x + xy - \gamma x^3 \\ \dot{y} &= -y + x^2 - y^2 \end{aligned}$$

$\mu < 0$:	$E^{(s)} = R^2$	$E^{(c)}$ empty	$E^{(u)}$ empty
$\mu = 0$:	$E^{(s)}$ = y-axis	$E^{(c)}$ = x-axis	$E^{(u)}$ empty
$\mu > 0$:	$E^{(s)}$ = y-axis	$E^{(c)}$ empty	$E^{(u)}$ = x-axis



The center manifold $W^{(c)}$ captures the dynamics local to the fixed point, i.e. in a neighborhood of the fixed point.

- $W^{(c)}$ is *locally attracting*:
  if $\mathbf{x}(0)$ has a forward trajectory in a neighborhood $\mathcal{U}$ of $\mathbf{x}_0$, i.e. if $\mathbf{x}(t)$ remains in U for $t \to +\infty$, then $\mathbf{x}(t)$ converges to $W^{(c)}$ for $t \to \infty$.
  - all the points that do not leave $\mathcal{U}$ are captured by $W^{(c)}$ (after transients).
  - $W^{(c)}$ need not be truly attractive: if $W^{(u)}$ is not empty, the points that are outside $W^{(s)} \cup W^{(c)}$ leave $\mathcal{U}$ as $t \to +\infty$.
- $W^{(c)}$ contains *all local trajectories*:
  if $x(t)$ in $\mathcal{U}$ for all $-\infty < t < +\infty$ then $\mathbf{x}(t)$ in $W^{(c)}$.
  (if $\mathbf{x}(t)$ had a component in $W^{(s,u)}$ then it would diverge away from the fixed point and leave $\mathcal{U}$ for $t \to \pm\infty$.)
  - this does not imply that all points on $W^{(c)}$ stay in $\mathcal{U}$: the flow on the center manifold could diverge away from the fixed point
- $W^{(c)}$ contains *all locally recurrent points*:
  if the forward trajectory $\mathbf{x}(t)$ of a recurrent point $\mathbf{x}(0)$ is contained in a neighborhood $\mathcal{U}$ of $\mathbf{x}_0$ then $\mathbf{x}(0)$ is already on $W^{(c)}$.
  - $\mathbf{x}(t) \in U$ for $t \to \infty$ therefore $\mathbf{x}(t)$ converges to $W^{(c)}$, but $\mathbf{x}(0)$ recurrs $\Rightarrow \mathbf{x}(0) \in W^{(c)}$
  - all the points on a periodic orbit are recurrent. A periodic orbit that is confined to a neighborhood of $\mathbf{x}_0$ is therefore contained in $W^{(c)}$.

Note:

- These statements suggest that the dynamics that are local to $\mathbf{x}_0$ can be described completely within the center manifold $W^{(c)}$.

Expect:

- For $0 \neq |\mu| \ll 1$ one still has a fast contraction onto a manifold close to $W^{(c)}(\mu = 0)$.
- The dynamics on that manifold may depend strongly on μ since the linear growth rate within that manifold changes sign at $\mu = 0$.

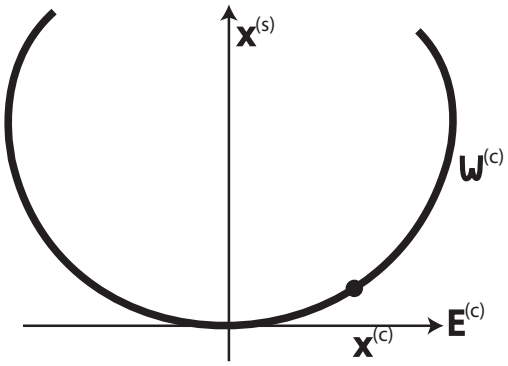
4.2 Center-Manifold Reduction

We want a description of the dynamics *on* $W^{(c)}$,

$$\mathbf{x} = (\mathbf{x}^{(c)}, \mathbf{x}^{(s)})$$

with $\mathbf{x}^{(s)} = \mathbf{h}(\mathbf{x}^{(c)})$ and $\mathbf{x}^{(c)} \in E^{(c)}$.

For simplicity we assume here that there is no unstable manifold $W^{(u)}$.



Note:

- Since $W^{(c)}$ is tangent to $E^{(c)}$ at the fixed point, this description is possible *locally* (near the fixed point).
- Further away the correspondence between $\mathbf{x}^{(s)}$ and $\mathbf{x}^{(c)}$ may become multivalued.

At the Bifurcation Point

Consider the example from before

$$\dot{x} = \mu x + xy - \gamma x^3 \tag{10}$$
$$\dot{y} = -y + x^2 - y^2 \tag{11}$$

For $W^{(c)}$ to exist we need to be at a bifurcation point: $\mu = 0$

$$E^{(c)} = \{(x, 0)\}, \qquad E^{(s)} = \{(0, y)\}$$

Since $E^{(c)}$ is the x-axis and therefore tangential to $W^{(c)}$, x is a good coordinate to parameterize $W^{(c)}$, i.e. we write $(x, y) = (x, h(x))$ with $h(x)$ yet to be determined.

The condition $y(t) = h(x(t))$ leads to two differential equations for $y(t)$,

$$\dot{y} = -y + x^2 - y^2 \qquad \text{and} \qquad \dot{y} = \frac{dh}{dx}\dot{x} = \frac{dh}{dx}\left(xy - \gamma x^3\right) .$$

This yields an equation for $h(x)$

$$\frac{dh}{dx}\left(x\, h(x) - \gamma x^3\right) = -h(x) + x^2 - (h(x))^2 \tag{12}$$

Thus:

- We obtain a differential equation for $h(x)$, which in general is nonlinear and may not be exactly solvable.
- We are interested in a local analysis in the vicinity of the fixed point, i.e. for small x $\Rightarrow$ expand $h(x)$ for small x.

Expand $h(x)$,

$$h(x) = h_0 + h_1 x + h_2 x^2 + h_3 x^3 + h_4 x^4 + \cdots \tag{13}$$

By the center-manifold theorem we have

- The fixed point $(0,0)$ is on $W^{(c)} \Rightarrow h_0 = 0$.
- $W^{(c)}$ is tangent to $E^{(c)}$, i.e. $\frac{dh}{dx} = 0$ at $x = 0 \Rightarrow h_1 = 0$, i.e. $h(x)$ is strictly nonlinear (cf. Fig.11).

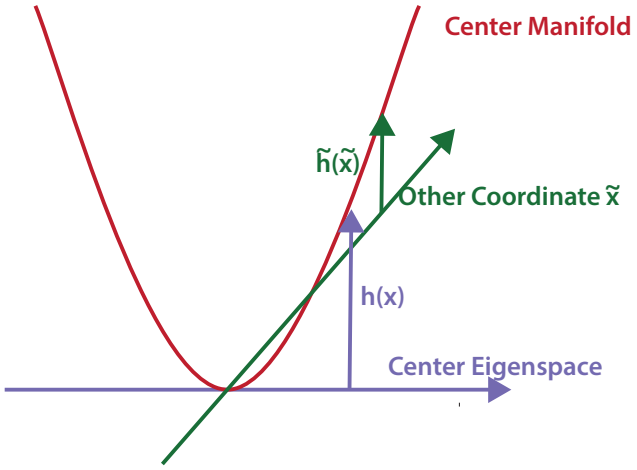


Figure 11: If the center manifold is parametrized by the coordinates on the center eigenspace, the function $h(x)$ is strictly nonlinear. For other coordinates $\tilde{x}$ there would also be a linear contribution to $\tilde{h}(\tilde{x})$. The linear contribution $\tilde{h}_1$ is then determined by a nonlinear equation with multiple solutions, corresponding to the additional invariant manifolds of the system (e.g. the stable manifolds).

Inserting the expansion into (12) yields then

$$\left(2h_2 x + 3h_3 x^2 + \cdots\right)\left\{x\left(h_2 x^2 + h_3 x^3\right) - \gamma x^3\right\} =$$
$$\overset{!}{=} -h_2 x^2 - h_3 x^3 - h_4 x^4 + x^2 - \left(h_2 x^2 + h_3 x^3 + \ldots\right)^2 \quad .$$

Collecting powers of x,

$$\begin{aligned}
\mathcal{O}(x^2) &: \quad 0 = -h_2 + 1 \quad \Rightarrow \quad h_2 = 1 \\
\mathcal{O}(x^3) &: \quad 0 = h_3 \quad \Rightarrow \quad h_3 = 0 \\
\mathcal{O}(x^4) &: \quad 2h_2(h_2 - \gamma) = -h_4 - h_2^2 \\
&\quad \Rightarrow \quad h_4 = 2(\gamma - 1) - 1\,,
\end{aligned}$$

we obtain

$$y = h(x) = x^2 + (2\gamma - 3)\, x^4 + \mathcal{O}(x^5)\,.$$

Inserting $h(x)$ into the equation for $\dot{x}$ results in

$$\dot{x} = x\left(x^2 + (2\gamma - 3)\, x^4 + \cdots\right) - \gamma x^3\,.$$

Thus:

The evolution equation on the center manifold is given by

$$\dot{x} = (1-\gamma)\, x^3 + (2\gamma - 3)\, x^5 + \cdots .$$

Note:

- In the absence of y the origin would be attractive for all values of γ. However, the coupling between x and y modifies these dynamics.
  - For $\gamma > 1$ the fixed point is locally attracting at the bifurcation point.
  - For $\gamma < 1$ the cubic term in (10) is not sufficient to compensate for the growth induced by the quadratic term in the equation for y.
- Thus, although y is linearly damped, we cannot simply set $y = 0$ and read off the equation for x. We need to include that y is driven by x and then feeds back onto x.

In the Vicinity of the Bifurcation Point

We want also a description that is valid for $0 \neq |\mu| \ll 1$. However, to use the center-manifold theorem, there must be a center manifold.

Consider the ***suspended* system** in which the control parameter is taken to be another dynamical variable

$$\dot{\mu} = 0 \tag{14}$$
$$\dot{x} = \mu x + xy - \gamma x^3 \tag{15}$$
$$\dot{y} = -y + x^2 - y^2 \tag{16}$$

Thus:

- The dynamics in the μ-direction are trivial: the value of μ is simply given by the initial condition. There is a 0 eigenvalue associated with the μ-direction.
- μx plays now the role of a *nonlinear* term and the linearization of the suspended system has 0 eigenvalue associated with the x-direction as it is the case at the bifurcation point.
- The center eigenspace is now two-dimensional,

  $$E^{(c)} = \{(\mu, x, 0)\} \qquad E^{(s)} = \{(0, 0, y)\}\,,$$

  resulting in a two-dimensional center manifold, which can be parameterized by the coordinates of the center eigenspace,

  $$y = h(\mu, x) \qquad \text{for} \qquad (\mu, x, y) \in W^{(c)}\,.$$

Local analysis: expand $h(\mu, x)$ in μ and x:

$$h(\mu, x) = \sum_{k,l=0} h_{kl} \mu^k x^l \,.$$

Since $W^{(c)}$ includes $(0, 0, 0)$ and is tangential to $E^{(c)}$ at that point we have

$$h_{00} = 0 \qquad h_{10} = 0 \qquad h_{01} = 0 \,,$$

yielding

$$h(\mu, x) = \sum_{k+l\geq 2} h_{kl} \mu^k x^l \,.$$

To go to 4^{th}- order as before would still require a large number of terms. It is therefore useful if we can guess a relationship among the variables in the expected equation on $W^{(c)}$.

Symmetries:

Eqs.(14,15,16) have a reflection symmetry. They are *equivariant* under the operation

$$(\mu, x, y) \to (\mu, -x, y) \,,$$

i.e. (14) and (16) do not change under the reflection, whereas *both sides* of (15) do switch signs.

Expect that the equation determining the center manifold respects the same symmetry,

$$y = h(\mu, x) \qquad \text{even in} x \,.$$

The expansion of $h(\mu, x)$ is then

$$h(\mu, x) = x^0 \left(h_{20}\mu^2 + h_{30}\mu^3 + \ldots\right) + x^2 \left(h_{02} + h_{12}\mu + \ldots\right) + x^4 \left(h_{04} + h_{14}\mu + \ldots\right) + \ldots \,.$$

Insert into (15,16),

$$\begin{aligned} \dot{y} = \frac{dh}{dx}\dot{x} + \frac{dh}{d\mu}\underbrace{\dot{\mu}}_{0} \quad &= \quad \left[2x \left(h_{02} + h_{12}\mu + \ldots\right) + 4x^3 \left(h_{04} + \ldots\right) + \ldots\right] \times \\ &\quad \times \left[\mu x + x \left(\left(h_{20}\mu^2 + \ldots\right) + x^2 \left(h_{02} + \ldots\right) + \ldots\right) - \gamma x^3\right] \\ &\overset{!}{=} \quad - \left(h_{20}\mu^2 + x^2 \left(h_{02} + h_{12}\mu\right) + x^4 \left(h_{04} + \ldots\right) + \cdots\right) + x^2 - \\ &\quad - \left(h_{20}\mu^2 + x^2 \left(h_{02} + h_{12}\mu\right) + \cdots\right)^2 \end{aligned}$$

$\mathcal{O}(\mu^2 x^0)$:

$$0 = -h_{20} \qquad \Rightarrow \qquad h_{20} = 0$$

$\mathcal{O}(\mu^1 x^1)$:

$$0 = 0$$

$\mathcal{O}(\mu^0 x^2)$:

$$0 = -h_{02} + 1 \qquad \Rightarrow \qquad h_{02} = 1$$

$\mathcal{O}(\mu^1 x^2)$:

$$2h_{02} = -h_{12} \qquad \Rightarrow \qquad h_{12} = -2$$

$\mathcal{O}(\mu^0 x^4)$:

$$-2\gamma h_{02} + 2h_{02}^2 = -h_{04} - h_{02}^2 \qquad \Rightarrow \qquad h_{04} = 2\gamma - 3$$

Thus, we have for the center manifold

$$y = x^2 - 2\mu x^2 + (2\gamma - 3)x^4 + \ldots$$

and for the evolution on the center manifold we get from (15)

$$\dot{x} \;=\; \mu x - (\gamma - 1 + 2\mu)x^3 + \left[(2\gamma - 3)x^5 + \ldots\right] \tag{17}$$

Thus:

- For $\gamma > 1$ we have a supercritical pitchfork bifurcation.
- For $\gamma < 1$ we have a subcritical pitchfork bifurcation.
- For $\gamma \approx 1$ the μ-dependence of the cubic coefficient has to be taken into account. This makes the bifurcation scenario more subtle. It is worth investigating.

4.3 Non-Uniqueness of the Center Manifold

The center-manifold theorem states that the center manifold $W^{(c)}$ may be non-unique. This is, because the center manifold $W^{(c)}$ is defined via the tangency condition $W^{(c)} \parallel E^{(c)}$ at the fixed point.

Consider as illustration a simplification of (10,11),

$$\begin{aligned} \dot{x} &= -x^3 \\ \dot{y} &= -y + x^2 \end{aligned}$$

The center manifold $y = h(x)$ is determined by the condition

$$\frac{dh}{dx}\left(-x^3\right) = -h + x^2 \,.$$

This equation is linear in h and can be solved exactly

$$\frac{dh}{dx} - \frac{1}{x^3}h = -\frac{1}{x}$$

$$\frac{d}{dx}\left(e^{\frac{1}{2x^2}}h\right) = -e^{\frac{1}{2x^2}}\frac{1}{x}$$

$$
\begin{aligned}
h(x,C) &= -e^{-\frac{1}{2x^2}} \int^x e^{\frac{1}{2x'^2}} \frac{1}{x'} dx' + Ce^{-\frac{1}{2x^2}} \\
&= \frac{1}{2} e^{-\frac{1}{2x^2}} \,\mathrm{Ei}\left(\frac{1}{2x^2}\right) + Ce^{-\frac{1}{2x^2}} \qquad (18) \\
&= x^2 + 2x^4 + 8x^6 + \mathcal{O}(x^8) + Ce^{-\frac{1}{2x^2}} \,. \qquad (19)
\end{aligned}
$$

with C an arbitrary integration constant. Here the exponential integral is given by $\mathrm{Ei}(x) = \int_{-x}^{\infty} \frac{e^{-t}}{t} dt$.

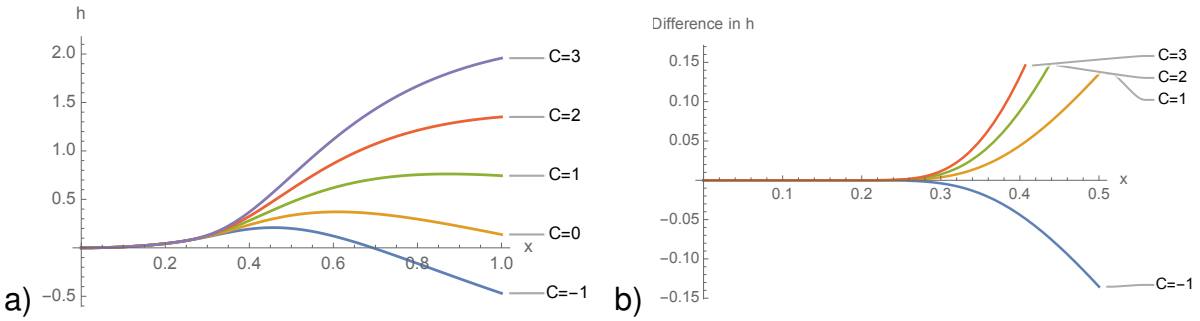


Figure 12: a) Different center manifolds (18). b) Difference between different center manifolds $h(x,C) - h(x,0)$.

Note:

- C allows to satisfy arbitrary initial conditions $h(x_0) = y_0$.
- For any C the manifold $\{(x,y)|y = h(x,C)\}$ is tangent to $E^{(c)}$,

$$\frac{d}{dx} e^{-\frac{1}{2x^2}} = \frac{1}{x^3} e^{-\frac{1}{2x^2}} \to 0 \qquad \text{for} \quad x \to 0$$

  i.e. all these manifolds are center manifolds.
- For small x the difference between the different $W^{(c)}$ becomes extremely small (cf. Fig.12). In fact, near the fixed point they differ only by *exponentially small terms*. In terms of their Taylor expansions they are identical to all orders, since all derivatives of $e^{-\frac{1}{2x^2}}$ vanish at $x = 0$ (cf. (19)).
- In Roberts (1985) the emergence of the non-uniqueness as the bifurcation point is approached is illustrated in a simple, solvable example.

4.4 Comparison with a Multiple-Scale Analysis

Reconsider example (10,11) from Sec.4.2,

$$
\begin{aligned}
\dot{x} &= \mu x + xy - \gamma x^3 \\
\dot{y} &= -y + x^2 - y^2 \,.
\end{aligned}
$$

The result from the center-manifold reduction shows

- in the neighborhood of the fixed point $(x, y) = (0, 0)$, i.e. for x small, x and with it y evolves very slowly on $W^{(c)}$. The time scale is set by $1/\mu \to \infty$ at the bifurcation point
- $y = h(x, \mu)$ is strictly nonlinear in x, i.e. much smaller than x.

Since the evolution becomes slow at the bifurcation point, we introduce a **slow time**

$$T = \Phi(\epsilon)\, t\,.$$

How should we choose $\Phi(\epsilon)$?

In this case: considering the center-manifold result (17) to cubic order,

$$\dot{x} = \mu x - (\gamma - 1)x^3\,,$$

suggests

$$\mu = \epsilon^2 \mu_2 \qquad T = \epsilon^2 t \qquad \Rightarrow \qquad \frac{d}{dt}x = \epsilon^2 \frac{d}{dT}x$$

and the expansion of (x, y) as

$$\begin{aligned} x &= \epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3 + \ldots \\ y &= \epsilon y_1 + \epsilon^2 y_2 + \epsilon^3 y_3 + \ldots \end{aligned}$$

Then the three terms in the evolution equation are all of the same order.

Insert the expansions into the basic equations (10,11) and solve order by order in ϵ:

$\mathcal{O}(\epsilon^1)$:

$$\begin{aligned} 0 &= 0 \\ 0 &= -y_1 \end{aligned}$$

Thus,

$$y_1 = 0 \qquad x_1 \text{ is still undetermined}$$

$\mathcal{O}(\epsilon^2)$:

$$\begin{aligned} 0 &= x_1 y_1 \\ 0 &= -y_2 + x_1^2 \end{aligned}$$

Thus:

$$y_2 = x_1^2 \qquad x_2 \text{ is still undetermined}$$

$\mathcal{O}(\epsilon^3)$:

$$\begin{aligned} \frac{d}{dT}x_1 &= \mu_2 x_1 + x_1 y_2 - \gamma x_1^3 \\ \frac{d}{dT}y_1 &= -y_3 + 2x_1 x_2 - 2y_1 y_2 \end{aligned}$$

Using the results from lower orders we get

$$\frac{d}{dT}x_1 = \mu_2 x_1 - (\gamma - 1)\, x_1^3$$

and

$$y_3 = 2x_1 x_2\,.$$

Notes:

- This result agrees to leading order with that of the center-manifold reduction (17).
- To obtain the equation (17) all the way to fifth order in x we need to $\mathcal{O}(\epsilon^5)$ in this expansion and determine also $x_{2,3}$. Combining $x_{1,2,3}$ into x yields then the center-manifold result.

For more complicated problems it is important to understand the mathematical structure of the problem.

Consider more generally

$$\dot{\mathbf{u}} \;=\; \mathbf{L}\,\mathbf{u} + \mathbf{N}_2(\mathbf{u},\mathbf{u}) + \mathbf{N}_3(\mathbf{u},\mathbf{u},\mathbf{u})\,, \tag{20}$$

where $\mathbf{L}$ is the Jacobian of the linearization around the fixed point $\mathbf{u} = 0$. In our example, writing $\mathbf{u} = (x, y)$,

$$\mathbf{L}(\mu) = \begin{pmatrix} \mu & 0 \\ 0 & -1 \end{pmatrix}$$

and

$$\begin{aligned} \mathbf{N}_2(\mathbf{u},\mathbf{u}) &= \begin{pmatrix} xy \\ x^2 - y^2 \end{pmatrix} \\ \mathbf{N}_3(\mathbf{u},\mathbf{u},\mathbf{u}) &= \begin{pmatrix} -\gamma x^3 \\ 0 \end{pmatrix}. \end{aligned}$$

The essential feature of the linear operator $\mathbf{L}$ is that at the bifurcation point it has at least one zero eigenvalue and is therefore singular:

$$\mathbf{L}(\mu = 0)\mathbf{v}_1 = 0\,.$$

The eigenvector $\mathbf{v}_1$ spans the *center* eigenspace $E^{(c)}$, which is tangent to the center manifold $W^{(c)}$. The dominant component of $\mathbf{u}$ is therefore proportional to $\mathbf{v}_1$. In the expansion of $\mathbf{u}$ it is therefore useful to introduce an amplitude A for the component along the center eigenspace

$$\mathbf{u} = \epsilon^\beta A(T)\,\mathbf{v}_1 + \epsilon^{2\beta}\mathbf{u}_2(T) + \ldots$$

Here we have also introduced a slow time T,

$$T = \epsilon^\alpha t \qquad \Rightarrow \qquad \frac{d}{dt} = \epsilon^\alpha \frac{d}{dT}\,.$$

How to choose the scalings, i.e. α and β and that of the control parameter $\mu = \epsilon^\delta \mu_\delta$?

The evolution equation on the center manifold depends on the type of bifurcation at hand $\Rightarrow$ **symmetries** of the original system are important.

The choice of the scaling is dictated by the fact that we need to balance simultaneously 3 types of terms in the **equation for the evolution on the center manifold**

- The slow time derivative $\frac{dA}{dT}$.
- The linear term, which reflects the linearization of the original equations and captures eigenvalue that goes through 0 at the bifurcation point, i.e. at $\mu = 0$,

$$\begin{aligned} \text{saddle-node bifurcation:} &\sim \mu \\ \text{transcritical or pitch-fork bifurcation:} &\sim \mu A\,. \end{aligned}$$

- Nonlinear terms

$$\begin{aligned} \text{saddle-node or transcritical bifurcation:} &\sim A^2 \\ \text{pitch-fork bifurcation:} &\sim A^3\,. \end{aligned}$$

Note:

- The nonlinearities in the *original* equations do *not* give directly the information about the scaling. The scaling is also affected by the eigenvectors spanning the center eigenspace. Together with the nonlinearities they determine the type of bifurcation and the associated scaling. For example, for $\gamma = 0$ there is no cubic nonlinearity in the original equation of our example, but we still get a cubic nonlinearity in the equation for the amplitude A.

It is best to discuss the approach in terms of a concrete bifurcation type, e.g. the pitchfork bifurcation of example (10,11),

$$\underbrace{\frac{d}{dT}A}_{\epsilon^\alpha\epsilon^\beta} \sim \underbrace{\mu_\delta A}_{\epsilon^\delta\epsilon^\beta} \sim \underbrace{A^3}_{\epsilon^{3\beta}}$$

Thus, we need

$$\alpha = \delta = 2\beta\,.$$

For simplicity choose $\beta = 1$, $\alpha = \delta = 2$.

The linear operator $\mathbf{L}$ is then also expanded in ϵ,

$$\mathbf{L} = \mathbf{L}_0 + \epsilon\mathbf{L}_1 + \epsilon^2\mathbf{L}_2 + \ldots$$

with

$$\mathbf{L}_0 = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \qquad \mathbf{L}_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad \mathbf{L}_2 = \begin{pmatrix} \mu_2 & 0 \\ 0 & 0 \end{pmatrix}.$$

The expansion then yields:

$\mathcal{O}(\epsilon^1)$:

$$\mathbf{L}_0\, A\, \mathbf{v}_1 = 0\,. \tag{21}$$

This reproduces the linear stability analysis. It does not give any condition for the amplitude A.

$\mathcal{O}(\epsilon^2)$:

$$\mathbf{L}_0\, \mathbf{u}_2 = -\mathbf{N}_2(A\mathbf{v}_1, A\mathbf{v}_1) \tag{22}$$

To solve for $\mathbf{u}_2$ we would need to invert $\mathbf{L}_0$: $\mathbf{u}_2 = -\mathbf{L}_0^{-1}\mathbf{N}_2$.

But:

- $\mathbf{L}_0$ is singular due to (21). Therefore $\mathbf{L}_0^{-1}$ **does not exist** $\Rightarrow$ (22) does not always have a solution.

This situation is captured by the **Fredholm Alternative Theorem**, which states the condition under which such an equation does have a solution:

If the matrix $\mathbf{M}$ is singular with left 0-eigenvector $\mathbf{v}^+$,

$$\mathbf{v}^+\mathbf{M} = 0\,,$$

then the inhomogeneous equation

$$\mathbf{M}\mathbf{x} = \mathbf{b}$$

has either

- infinitely many solutions (if $\mathbf{v}^+\mathbf{b} = 0$)

  or

- no solution at all (if $\mathbf{v}^+\mathbf{b} \neq 0$).

This statement is easily understood, if one multiplies the equation with the left 0-eigenvector

$$\underbrace{\mathbf{v}^+\mathbf{M}}_{=0}\mathbf{x} = \mathbf{v}^+\mathbf{b}\,.$$

Thus, the Fredholm Alternative Theorem states a **solvability condition**, that needs to be satisfied in order for the equation at order $\mathcal{O}(\epsilon^2)$ to be solved.

Thus, we need the left 0-eigenvector $\mathbf{v}_1^+$, which satisfies

$$\mathbf{v}_1^+\mathbf{L}_0 = 0\,. \tag{23}$$

Multiplying (22) by $\mathbf{v}_1^+$ from the left,

$$\mathbf{v}_1^+\,\mathbf{L}_0\,\mathbf{u}_2 = -\mathbf{v}_1^+\mathbf{N}_2(A\mathbf{v}_1, A\mathbf{v}_1)\,,$$

one obtains the solvability condition

$$\mathbf{v}_1^+\mathbf{N}_2(A\mathbf{v}_1, A\mathbf{v}_1) = 0\,. \tag{24}$$

In our example

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \mathbf{v}_1^+ = (1,0) \qquad \mathbf{N}_2(A\mathbf{v}_1, A\mathbf{v}_1) = \begin{pmatrix} 0 \\ A^2 \end{pmatrix}$$

Thus,

$$\mathbf{v}_1^+\mathbf{N}_2(A\mathbf{v}_1, A\mathbf{v}_1) = 0 \qquad \text{for any } A$$

and (22) can be solved for $\mathbf{u}_2$,

$$\mathbf{u}_2 = \begin{pmatrix} 0 \\ A^2 \end{pmatrix}\,.$$

$\mathcal{O}(\epsilon^3)$:

$$\mathbf{L}_0\,\mathbf{u}_3 = \frac{d}{dT}A\,\mathbf{v}_1 - \mathbf{L}_2\,\mathbf{u}_1 - \mathbf{N}_2(\mathbf{u}_2, A\mathbf{v}_1) - \mathbf{N}_2(A\mathbf{v}_1, \mathbf{u}_2) - \mathbf{N}_3(A\mathbf{v}_1, A\mathbf{v}_1, A\mathbf{v}_1)\,.$$

Again, to solve for $\mathbf{u}_3$ we would have to invert $\mathbf{L}_0$. We therefore get again a solvability condition,

$$\mathbf{v}_1^+\left(\frac{d}{dT}A\,\mathbf{v}_1 - \mathbf{L}_2\,A\mathbf{v}_1 - \mathbf{N}_2(\mathbf{u}_2, A\mathbf{v}_1) - \mathbf{N}_2(A\mathbf{v}_1, \mathbf{u}_2) - \mathbf{N}_3(A\mathbf{v}_1, A\mathbf{v}_1, A\mathbf{v}_1)\right) = 0$$

In our example

$$\frac{d}{dT}A - A\,(1,0)\begin{pmatrix}\mu_2 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} - (1,0)\left(\begin{pmatrix}0\\0\end{pmatrix} - \begin{pmatrix}A\cdot A^2\\0\end{pmatrix} - \begin{pmatrix}-\gamma A^3\\0\end{pmatrix}\right) = 0$$

Thus

$$\frac{d}{dT}A = \mu_2 A - (\gamma - 1)A^3\,.$$

Notes:

- The amplitude A at $\mathcal{O}(\epsilon)$ is not determined until we take the expansion to $\mathcal{O}(\epsilon^3)$. There the solvability condition leads to an evolution equation for A.
- The quadratic nonlinearity contributes to the solvability condition at $\mathcal{O}(\epsilon^3)$ by being 'cycled through twice'.
- If the solvability at $\mathcal{O}(\epsilon^2)$ had not been satisfied, a faster slow time $T_1 = \epsilon t$ would have to be introduced and one would have obtained at $\mathcal{O}(\epsilon^2)$ an equation of the form

$$\frac{d}{dT_1}A = \mathbf{v}_1^+\mathbf{N}_2(A\mathbf{v}_1, A\mathbf{v}_1)\,.$$

- Since $\underline{v}_1^+\underline{v}_1 \neq 0$, the term with the time-derivative guarantees that the solvability condition can be satisfied.

Note:

- the center-manifold reduction can also be described as the *adiabatic elimination* of the damped modes since the damped modes follow the active critical mode $A\underline{v}_1$ *adiabatically* (compare 'adiabatic' changes in thermodynamics: they are so slow that no entropy is generated, the system is in equilibrium at all times).

5 Numerical Approaches to Bifurcations I

A good introduction can be found in (Spence and Graham, 1999). A more comprehensive, approachable source is (Seydel, 2009), useful is also (Doedel, 2007). These papers are on Canvas under Files/Resources.

5.1 Introduction

So far

- we always assumed a (base) solution is available
- we focused on the bifurcations off that solution
- we performed a *weakly* nonlinear *local* analysis around that solution
- connections between multiple bifurcations can be captured only via higher singularities, i.e. bifurcations with higher codimension (e.g. subcritical pitchfork and saddle-node bifurcations in the quintic subcritical pitchfork scenario; hexagons/rolls for *weakly* subcritical bifurcation)

In general:

- We need to obtain the base solution.
- We are interested also in connections between bifurcations that are not close to each other, i.e. separated by a finite distance, and that cannot be connected via a higher singularity.
- We want to follow branches away from and between bifurcations.

We want to employ also numerical methods . They should allows us to follow solution branches and to navigate bifurcations.

- Time-stepping
  - Follow branch by slightly changing bifurcation parameter and using previous solution as initial condition.
  - The temporal evolution becomes *very slow* near local bifurcation points ('critical slowing down'), since the real part of at least one eigenvalue vanishes at the bifurcation point. .
  - Captures fixed points, oscillations, and complex dynamics, but only for attractors, i.e. only for stable solutions. Bifurcation structures typically involve also unstable solutions that are necessary to understand the structure, e.g. saddle-node bifurcation.
- Root Finding for Fixed Points
  - As in time-stepping, follow the branch by slightly changing the bifurcation parameter and using the previous solution as initial condition.
  - No slow-down near bifurcations and no restriction to dynamically stable solutions.

– Newton iteration to solve $\mathbf{F}(\mathbf{x},\lambda)=0$
Starting from an approximation $\mathbf{x}^{(k)}$ for the root, determine an approximation of the change $\Delta\mathbf{x}=\mathbf{x}^{(\mathbf{k+1})}-\mathbf{x}^{(k)}$ by extrapolation

$$\mathbf{F}(\mathbf{x}^{(k+1)},\lambda)=\mathbf{F}(\mathbf{x}^{(k)},\lambda)+\mathbf{F}_{\mathbf{x}}(\mathbf{x}^{(k)},\lambda)\,\left(\mathbf{x}^{(k+1)}-\mathbf{x}^{(k)}\right)\overbrace{=}^{!}0$$

and iterate

$$\mathbf{x}^{(k+1)}=\mathbf{x}^{(k)}-\left(\mathbf{F}_{\mathbf{x}}\left(\mathbf{x}^{(k)},\lambda\right)\right)^{-1}\mathbf{F}(\mathbf{x}^{(k)},\lambda)\,.$$

Numerically, it is better to solve

$$\mathbf{F}_{\mathbf{x}}\left(\mathbf{x}^{(k)},\lambda\right)\mathbf{x}^{(k+1)}=\mathbf{F}_{\mathbf{x}}\left(\mathbf{x}^{(k)},\lambda\right)\mathbf{x}^{(k)}-\mathbf{F}(\mathbf{x}^{(k)},\lambda)\,,$$

to avoid computing the inverse.

The Newton iteration converges very fast if the initial guess is sufficiently close to the solution u^∞, i.e. if the initial guess is in the basin of attraction of the fixed point.

To see this, consider the scalar case

$$u^{(l+1)}=u^{(l)}-\frac{F(u^{(l)})}{F'(u^{(l)})}$$

and expand around the fixed point u^∞,

$$u^{(l)}=u^\infty+\epsilon^{(l)}\qquad\left|\epsilon^{(l)}\right|\ll 1\,.$$

Inserting this into the Newton iteration yields

$$\begin{aligned}
u^\infty+\epsilon^{(l+1)} &= u^\infty+\epsilon^{(l)}-\frac{F(u^\infty+\epsilon^{(l)})}{F'(u^\infty+\epsilon^{(l)})}\\
\epsilon^{(l+1)} &= \epsilon^{(l)}-\frac{\overbrace{F(u^\infty)}^{=0}+F'(u^\infty)\,\epsilon^{(l)}+\frac{1}{2}F''(u^\infty)\left(\epsilon^{(l)}\right)^2}{F'(u^\infty)+\epsilon^{(l)}F''(u^\infty)}\\
&= \epsilon^{(l)}-\epsilon^{(l)}\frac{1}{1+\epsilon^{(l)}\frac{F''(u^\infty)}{F'(u^\infty)}}-\left(\epsilon^{(l)}\right)^2\frac{1}{2}\frac{F''(u^\infty)}{F'(u^\infty)}\frac{1}{1+\epsilon^{(l)}\frac{F''(u^\infty)}{F'(u^\infty)}}\\
\epsilon^{(l+1)} &= +\frac{1}{2}\frac{F''(u^\infty)}{F'(u^\infty)}\,\left(\epsilon^{(l)}\right)^2+\mathcal{O}\left(\left(\epsilon^{(l)}\right)^3\right)\,.
\end{aligned}$$

Definition

The order of the convergence of an iteration method is p if the limit

$$\lim_{l\to\infty}\frac{\left|\epsilon^{(l+1)}\right|}{\left|(\epsilon^{(l)})^p\right|}=r$$

exists and is non-zero.

Notes:

- For linear convergence, $p = 1$, the approach to the fixed point is exponential

$$\epsilon^{(n)} \propto \epsilon^{(0)}\, r^n$$

- For quadratic convergence, $p = 2$, the approach to the fixed point is much faster than exponential

$$\epsilon^{(l+1)} = r\, \left(\epsilon^{(l)}\right)^2 = r\, \left(r\, \left(\epsilon^{(l-1)}\right)^2\, r\, \left(\epsilon^{(l-1)}\right)^2\right)$$

Thus, in each iteration the number of *factors* $\epsilon^{(0)}$ is doubled and the number $n^{(l)}$ of factors r satisfies

$$n^{(l+1)} = 2n^{(l)} + 1\,,$$

i.e.

$$\begin{aligned} n^{(1)} &= 1 \\ n^{(2)} &= 2n^{(1)} + 1 = 3 \\ n^{(3)} &= 2\,(2 \cdot 1 + 1) + 1 = 7 \\ n^{(4)} &= 2\,(2\,(2 \cdot 1 + 1) + 1) + 1 = 15 \end{aligned}$$

Unrolling $n^{(4)}$ from the last term backwards we get

$$n^{(4)} = 1 + 1 \cdot 2 + 1 \cdot 2^2 + 1 \cdot 2^3$$

or in general

$$n^{(n)} = \sum_{l=0}^{n-1} 2^l = 2^n - 1\,.$$

Combined we have then

$$\epsilon^{(n)} = r^{-1}\, \left(r\epsilon^{(0)}\right)^{2^n}\,.$$

- For example for $r\epsilon^{(0)} = 0.1$:
while for linear convergence the number of correct digits increases by one in each step, it *doubles* in each step for quadratic convergence

exponential	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}
super-exponential	10^{-1}	10^{-2}	10^{-4}	10^{-8}	10^{-16}	10^{-32}

- **Issue**: Newton converges only if a good initial guess is available.

• Issues:

- These methods do not allow to follow a branch around a saddle-node bifurcation (fold).
  - One would have to happen upon an initial guess for the other branch that emerges from the saddle-node bifurcation.
  - We need a method that can follow a branch smoothly around a fold.
- How to choose the desired branch at a bifurcation involving more than 1 branch?
  - We need a method that can switch from one branch to another branch at a bifurcation point.

5.2 Pseudo-Arclength Continuation

At a saddle-node bifurcation (fold point) the number of solutions goes from 0 to 2 as the bifurcation parameter λ is changed $\Rightarrow$ λ is not suitable for a smooth parametrization of the solution branch. But when walking *along* the branch the path is unique; for a saddle-node bifurcation there is no intersection with other branches as there is for transcritical, pitch-fork, or Hopf bifurcations.

Consider again the Taylor expansion of the fixed-point equation for $\mathbf{x} \in \mathbb{R}^n$,

$$0 = \mathbf{F}(\mathbf{x}, \lambda) = \underbrace{\mathbf{F}(\mathbf{x}^{(0)}, \lambda_0)}_{=0} + \underbrace{\mathbf{F}_x(\mathbf{x}^{(0)}, \lambda_0)}_{\mathbf{F}^{(0)}_{\mathbf{x}}} \left(\mathbf{x} - \mathbf{x}^{(0)}\right) + \underbrace{\mathbf{F}_\lambda(\mathbf{x}^{(0)}, \lambda_0)}_{\mathbf{F}^{(0)}_{\lambda}} (\lambda - \lambda_0) + h.o.t.$$

where

$$(\mathbf{F}_\mathbf{x})_{ij} = \frac{\partial F_i}{\partial x_j} \qquad \text{and} \qquad (\mathbf{F}_\lambda)_i = \frac{\partial F_i}{\partial \lambda}\,.$$

Definitions

- The point $(\mathbf{x}^{(0)}, \lambda_0)$ is a singular point if $det\mathbf{F}_\mathbf{x} = 0$.
- The point $(\mathbf{x}^{(0)}, \lambda_0)$ is a fold point if $rank(\mathbf{F}_\mathbf{x}) = n - 1$ and $\mathbf{F}^{(0)}_\lambda \notin range(\mathbf{F}_\mathbf{x})$.
- The point $(\mathbf{x}^{(0)}, \lambda_0)$ is a simple stationary bifurcation point if $rank(\mathbf{F}_\mathbf{x}) = n - 1$ and $\mathbf{F}^{(0)}_\lambda \in range(\mathbf{F}_\mathbf{x})$.

Notes:

- If $\mathbf{F}^{(0)}_\lambda \in range(\mathbf{F}_\mathbf{x})$ one can still solve the Taylor-expanded equation for $\mathbf{x}$ for arbitrary λ near λ_0, i.e. the branch continues across λ_0.
- If $\mathbf{F}^{(0)}_\lambda \notin range(\mathbf{F}_\mathbf{x})$ one cannot compute any solution at this lowest order: the solutions are determined at higher order and a bifurcation occurs.

Illustration of the different singular points in the scalar case

$$0 = \underbrace{F(0,0)}_{=0} + \underbrace{\partial_x F|_{0,0}}_{=0}\, x + \partial_\lambda F|_{0,0}\, \lambda + \frac{1}{2}\,\partial_x^2 F\big|_{0,0}\, x^2 + \partial^2_{x\lambda} F\big|_{0,0}\, x\lambda + \frac{1}{2}\,\partial_\lambda^2 F\big|_{0,0}\, \lambda^2 + \ldots$$

- Saddle-node bifurcation (fold):

  $0 = \lambda + x^2 \quad \Rightarrow \quad \partial_x F^{(0)} = 0, \quad$ i.e. $range(\partial_x F^{(0)}) = 0 \qquad \partial_\lambda F^{(0)} \neq 0 \Rightarrow \partial_\lambda F^{(0)} \notin range(\partial_x F)$.

- Transcritical bifurcation ($F(0, \lambda) = 0$ for all λ):

  $0 = \lambda x + x^2 \quad \Rightarrow \quad \partial_x F^{(0)} = 0 \quad$ i.e. $range(\partial_x F^{(0)}) = 0 \qquad \partial_\lambda F^{(0)} = 0 \in range(\partial_x F^{(0)})$.

Two-dimensional example:

$$\mathbf{F}(\mathbf{x};\lambda) = \begin{pmatrix} x_1^2 + x_2^2 - \lambda \\ x_2^2 - 2x_1 + 1 \end{pmatrix}$$

For given parameter λ, the fixed point(s) correspond to the intersections of the circle $x_1^2 + x_2^2 = \lambda$ and the parabola $x_1 = \frac{1}{2}(x_2^2 + 1)$.

At $\lambda = \lambda_0 = \frac{1}{4}$ and $\mathbf{x} = (\frac{1}{2}, 0)$ the system is singular

$$\mathbf{F}_{\mathbf{x}}^{(0)} = \begin{pmatrix} 2x_1 & 2x_2 \\ -2 & 2x_2 \end{pmatrix}\Big|_{(\mathbf{x}^{(0)},\lambda_0)} = \begin{pmatrix} 1 & 0 \\ -2 & 0 \end{pmatrix} \qquad det\mathbf{F}_{\mathbf{x}}^{(0)} = 0 \qquad rank\mathbf{F}_{\mathbf{x}}^{(0)} = 1\,,$$

reflecting that the circle touches the parabola, annihilating the pair of intersection points as λ is decreased.

Thus, the parametrization of $\mathbf{x}$ in terms of λ ceases to exist as λ is decreased.

Rewrite the fixed-point equations, considering x_1 and λ as the solution and x_2 the parameter,

$$\mathbf{G}(\mathbf{y};x_2) = \begin{pmatrix} x_1^2 - \lambda + x_2^2 \\ -2x_1 + 1 + x_2^2 \end{pmatrix} \qquad \text{with} \qquad \mathbf{y} = (x_1, \lambda)\,.$$

The solution set of $\mathbf{G}(\mathbf{y};x_2) = 0$ is identical to that of $\mathbf{F}(\mathbf{x};\lambda) = 0$. The Jacobian $\mathbf{G}_{\mathbf{y}}$ is non-singular at $x_1 = \frac{1}{2}$ and $\lambda = \frac{1}{4}$,

$$\mathbf{G}_{\mathbf{y}}^{(0)} = \begin{pmatrix} 2x_1 & -1 \\ -2 & 0 \end{pmatrix}\Big|_{(\mathbf{y}^0),x_2^{(0)})} \qquad det\mathbf{G}_{\mathbf{y}}^{(0)} = -2 \neq 0\,,$$

implying that there is a smooth branch of solutions passing through $(\frac{1}{2}, 0, \frac{1}{4})$ as x_2 is varied across $x_2 = 0$.

Note:

- By changing the parametrization one can follow smoothly along the solution branch passing through the fold point.

Our goal is to characterize the fixed-point branch that is traced out in the $(\mathbf{x},\lambda)$-space when λ is scanned. This curve does not have to be parametrized by any of the coordinates x_i or λ. Instead, we can introduce an additional parameter s that parametrizes the solution along the branch, $\mathbf{x} = \mathbf{x}(s)$, $\lambda = \lambda(s)$, somewhat similar to an arclength, and determine $\mathbf{y}(s) \equiv (\mathbf{x}(s), \lambda(s))$.

The main idea is to extrapolate a distance $s - s_0$ from the known fixed point $(\mathbf{x}(s_0), \lambda(s_0))$ along the tangent $\boldsymbol{\tau}$ in the $(\mathbf{x},\lambda)$-space ,

$$\boldsymbol{\tau} = \left(\dot{\mathbf{x}}(s), \dot{\lambda}(s)\right) \equiv \left(\frac{d\mathbf{x}(s)}{ds}, \frac{d\lambda(s)}{ds}\right)\,,$$

as a prediction and then perform Newton iterations to correct that prediction in order to get back onto the branch, but staying on the plane perpendicular to $\boldsymbol{\tau}$ at a distance $s - s_0$.

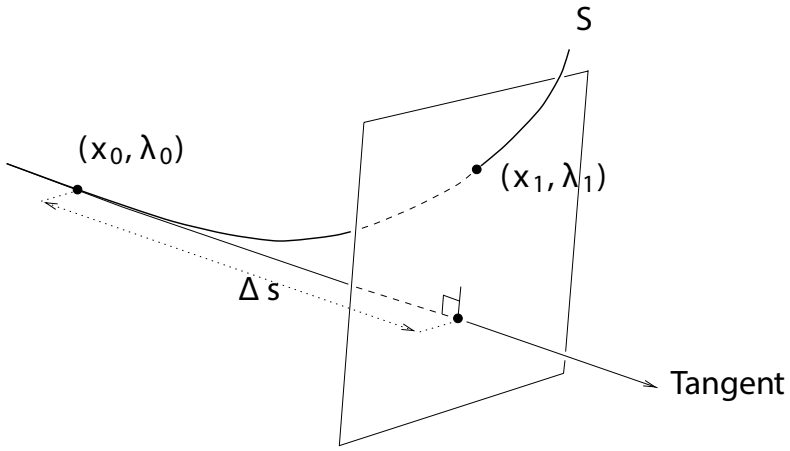


Figure 13: Pseudo-arclength continuation extrapolates along the tangent and then corrects within the plane perpendicular to the tangent (from Spence and Graham (1999), they use t rather than s as the parameter).

Since $(\mathbf{x}(s), \lambda(s))$ satisfies $\mathbf{F}(\mathbf{x}(s), \lambda(s)) = 0$ for all s, the tangent vector $\boldsymbol{\tau}$ satisfies

$$0 = \frac{d}{ds}\mathbf{F}\left(\mathbf{x}(s), \lambda(s)\right) = \mathbf{F}_{\mathbf{x}}\dot{\mathbf{x}}^t + \mathbf{F}_\lambda\dot{\lambda} = (\mathbf{F}_{\mathbf{x}}|\mathbf{F}_\lambda)\begin{pmatrix}\dot{\mathbf{x}}^t\\ \dot{\lambda}\end{pmatrix}. \tag{25}$$

Here the extended Jacobian $(\mathbf{F}_{\mathbf{x}}|\mathbf{F}_\lambda)$ was introduced, which has dimensions $n \times (n+1)$, i.e. the column $\mathbf{F}_\lambda$ is attached to the Jacobian $\mathbf{F}_{\mathbf{x}}$.

We introduce the normalized tangent vector[3]

$$\boldsymbol{\tau}^{(0)} = (\boldsymbol{\tau}_{\mathbf{x}}^{(0)}, \tau_\lambda^{(0)}) = \frac{(\dot{\mathbf{x}}, \dot{\lambda})}{\left\|(\dot{\mathbf{x}}, \dot{\lambda})\right\|}, \qquad \boldsymbol{\tau}^{(0)}\boldsymbol{\tau}^{(0)t} = 1,$$

at $\mathbf{y}^{(0)} = (\mathbf{x}(s_0), \lambda(s_0)) \equiv (\mathbf{x}^{(0)}, \lambda_0)$. Points $(\mathbf{x}, \lambda)$ on the plane perpendicular to the tangent $\boldsymbol{\tau}^{(0)}$ have the same projection onto $\boldsymbol{\tau}^{(0)}$ and satisfy (cf. Fig.13)

$$\boldsymbol{\tau}^{(0)}\left(\mathbf{x} - \mathbf{x}^{(0)}, \lambda - \lambda_0\right)^t = (s - s_0). \tag{26}$$

Among those points we are looking for those that also satisfy $\mathbf{F}(\mathbf{x}, \lambda) = 0$. For given s, we therefore look for solutions of

$$\mathbf{H}(\mathbf{y}; s) = 0, \tag{27}$$

where

$$\mathbf{H}(\mathbf{y}; s) = \begin{pmatrix}\mathbf{F}(\mathbf{x}, \lambda)\\ \boldsymbol{\tau}^{(0)}\left(\mathbf{x} - \mathbf{x}^{(0)}, \lambda - \lambda_0\right)^t - (s - s_0)\end{pmatrix} = \begin{pmatrix}\mathbf{F}(\mathbf{x}, \lambda)\\ \boldsymbol{\tau}_{\mathbf{x}}^{(0)}\left(\mathbf{x} - \mathbf{x}^{(0)}\right)^t + \tau_\lambda^{(0)}(\lambda - \lambda_0) - (s - s_0)\end{pmatrix}. \tag{28}$$

[3]Here $\mathbf{x}$, $\boldsymbol{\tau}^{(0)}$, and $\dot{\mathbf{x}}$ are written as row vectors.

Thus $\mathbf{H}(\mathbf{y}; s) = 0$ defines the point on the solution branch that lies at the intersection of the solution branch and that plane.

Note:

- The approach to add a tangent condition (bottom line in (28)) to the equations defining the point of interest (here the equations $\mathbf{F}(\mathbf{x}, \lambda) = 0$) is suitable more generally, e.g. when one wants to follow the locus of the folds themselves rather than the solution branch.

For the implicit function theorem to guarantee a unique, smooth branch $\mathbf{H} = 0$ through $\mathbf{y}^{(0)}$ at s_0, the Jacobian of $\mathbf{H}$ must be non-singular at $(\mathbf{y}^{(0)}; s_0)$. We have

$$\mathbf{H}_\mathbf{y}(\mathbf{y}^{(0)}; s) = \begin{pmatrix} \mathbf{F}_\mathbf{x}^{(0)} & \mathbf{F}_\lambda^{(0)} \\ \boldsymbol{\tau}_\mathbf{x}^{(0)} & \tau_\lambda^{(0)} \end{pmatrix} . \tag{29}$$

We are assuming that the only singularity on this part of the branch is a fold, i.e. $rank(\mathbf{F}_\mathbf{x}^{(0)}) = n - 1$ and $\mathbf{F}_\lambda^{(0)} \notin range(\mathbf{F}_\mathbf{x}^{(0)})$. The $n \times (n+1)$-dimensional matrix $(\mathbf{F}_\mathbf{x}^{(0)} | \mathbf{F}_\lambda^{(0)})$ has therefore n linearly independent columns and therefore also n linearly independent rows.

In addition, (25) implies that $(\boldsymbol{\tau}_\mathbf{x}^{(0)}, \tau_\lambda^{(0)})$ is linearly independent of all rows of $(\mathbf{F}_\mathbf{x}^{(0)} | \mathbf{F}_\lambda^{(0)})$, which constitute all other rows in $\mathbf{H}_\mathbf{y}$. Thus, the rank of the matrix $\mathbf{H}_\mathbf{y}(\mathbf{y}^{(0)}; s)$ is $n + 1$ and it is not singular.

For illustration, perform the first step of the Newton iteration. We start with the previously determined fixed point $\mathbf{y}^{(0)} \equiv (\mathbf{x}^{(0)}, \lambda_0)$ at $s = s_0$, which satisfies $\mathbf{F}(\mathbf{x}^{(0)}, \lambda_0) = 0$, and look for a solution of $\mathbf{H}(\mathbf{y}; s) = 0$ with $s > s_0$,

$$\mathbf{H}_\mathbf{y}(\mathbf{y}^{(0)}; s) \left(\mathbf{y}^{(1)} - \mathbf{y}^{(0)}\right) = -\mathbf{H}(\mathbf{y}^{(0)}; s) .$$

This results in

$$\mathbf{F}_\mathbf{x}^{(0)} \left(\mathbf{x}^{(1)} - \mathbf{x}^{(0)}\right)^t + \mathbf{F}_\lambda^{(0)} (\lambda_1 - \lambda_0) = 0 \tag{30}$$

$$\boldsymbol{\tau}_\mathbf{x}^{(0)} \left(\mathbf{x}^{(1)} - \mathbf{x}^{(0)}\right)^t + \tau_\lambda^{(0)} (\lambda_1 - \lambda_0) = s - s_0 . \tag{31}$$

Since $(\mathbf{F}_\mathbf{x}^{(0)} | \mathbf{F}_\lambda^{(0)})$ has rank n its kernel has dimension 1 and is spanned by $\boldsymbol{\tau}$. Therefore (30) implies $\left(\mathbf{x}^{(1)} - \mathbf{x}^{(0)}, \lambda_1 - \lambda_0\right) = \alpha \boldsymbol{\tau}^{(0)}$ for some $\alpha \in \mathbb{R}$. (31) then yields $\boldsymbol{\tau}^{(0)} \alpha \boldsymbol{\tau}^{(0)t} = s - s_0$. Due to the normalization of $\boldsymbol{\tau}$ this yields $\alpha = s - s_0$. Thus,

$$\left(\mathbf{x}^{(1)} - \mathbf{x}^{(0)}, \lambda_1 - \lambda_0\right) = (s - s_0) \boldsymbol{\tau}^{(0)}$$

and the first Newton step extrapolates along the tangent $\boldsymbol{\tau}^{(0)}$ a distance $s - s_0$. Subsequent iterations converge to the solution branch while staying on the plane perpendicular to $\boldsymbol{\tau}^{(0)}$.

The continuation algorithm can therefore be implemented as

1. At the current value s_0 of s determine $\boldsymbol{\tau}$ via (25) by solving

$$\mathbf{F}_\mathbf{x}^{(0)} \mathbf{z} = -\mathbf{F}_\lambda^{(0)} \tag{32}$$

for $\mathbf{z}$ and setting[4]

$$\boldsymbol{\tau} \equiv (\boldsymbol{\tau}_{\mathbf{x}}{}^{(0)}, \tau_\lambda^{(0)}) = \pm \frac{1}{\sqrt{\mathbf{z}^t\mathbf{z}+1}} \left(\mathbf{z}^t, 1\right) . \tag{33}$$

The sign in (33) is chosen such that the orientation of $\boldsymbol{\tau}$ is close to that of the tangent vector $\boldsymbol{\tau}_{previous}$ at the previous value of s, i.e. require $\boldsymbol{\tau}\boldsymbol{\tau}_{previous}^t > 0$. This allows λ to pass through an extremum as the branch is followed around a fold point.
Solving (32) assumes $\mathbf{F}_\mathbf{x}^{(0)}$ is non-singular, which is the case except right at the fold point.

2. Increment the arclength parameter

$$s = s_0 + \Delta s$$

and determine $\mathbf{x}(s)$ and $\lambda(s)$ by Newton iteration. For $k \geq 1$ iterate

$$\left(\mathbf{x}^{(k+1)}, \lambda_{k+1}\right) = \left(\mathbf{x}^{(k)}, \lambda_k\right) + (\Delta\mathbf{x}, \Delta\lambda)$$

until convergence, where

$$\begin{pmatrix} \mathbf{F}_\mathbf{x}(\mathbf{x}^{(k)}, \lambda_k) & \mathbf{F}_\lambda(\mathbf{x}^{(k)}, \lambda_k) \\ \boldsymbol{\tau}_\mathbf{x}^{(0)} & \tau_\lambda^{(0)} \end{pmatrix} \begin{pmatrix} \Delta\mathbf{x} \\ \Delta\lambda \end{pmatrix} = - \begin{pmatrix} \mathbf{F}(\mathbf{x}^{(k)}, \lambda_k) \\ \boldsymbol{\tau}_\mathbf{x}{}^{(0)} \left(\mathbf{x}^{(k)} - \mathbf{x}^{(0)}\right)^t + \tau_\lambda^{(0)}(\lambda_k - \lambda_0) - \Delta s \end{pmatrix}$$

and the values $\mathbf{x}^{(0)} = \mathbf{x}(s_0)$ and $\lambda_0 = \lambda(s_0)$ serve as starting points for the iteration.

Notes:

- At the fold itself $\mathbf{F}_\mathbf{x}$ is singular and cannot be inverted. Since $\mathbf{F}_\mathbf{x}^{(0)}\dot{\mathbf{x}} + \mathbf{F}_\lambda^{(0)}\dot{\lambda} = 0$, at the fold we have $\dot{\lambda} = 0$ and the tangent vector is given by $(\mathbf{z}, 0)$ with $\mathbf{F}_\mathbf{x}^{(0)}\mathbf{z} = 0$. The required 0-eigenvector $\mathbf{z}$ can be determined by the inverse power method. Define $\mathbf{A} = \left(\mathbf{F}_\mathbf{x}^{(0)} - \rho\mathbf{I}\right)^{-1}$ with $|\rho|$ sufficiently small and iterate

$$\mathbf{z}^{(n+1)} = \mathbf{A}\mathbf{z}^{(n)} .$$

Then $\mathbf{z}^{(n)}$ converges to the eigenvector corresponding to the eigenvalue of $\mathbf{A}$ with largest magnitude, i.e. $\max(\frac{1}{|\rho|}, \frac{1}{|\mu_j - \rho|})$, where μ_j are the eigenvalues of $\mathbf{F}_\mathbf{x}^{(0)}$. This procedure is usually not needed, since the algorithm rarely ends up close enough to the fold point itself.

- By including further conditions in the equation (28) for $\mathbf{H}$ one can use the same technique to follow the locus of bifurcations, e.g. by including the bottom row of (28) in (35) below.

[4]Here $\mathbf{z}$ is a column vector.

5.3 Branch Switching

Along the branch the solution can undergo other bifurcations than saddle-node bifurcations. They need to be identified and then the appropriate branch needs to be followed.

Identifying Bifurcations

- The continuation method itself will not be affected by it, since it always follows the tangent along the branch 'before' the bifurcation.
- Monitor the occurrence of bifurcations using suitable test functions $\mathcal{T}(\mathbf{x}, \lambda)$
  - Steady bifurcations (pitchfork, transcritical): an eigenvalue μ_j goes through 0, i.e. $\det \mathbf{F_x} = 0 \Rightarrow$ one could choose $\mathcal{T}(\mathbf{x}, \lambda) = \det \mathbf{F_x}$.
    This test does not distinguish between fold points (saddle-node bifurcations) and other steady bifurcations. It is therefore not really suitable to determine whether a new branch arises or not. A new branch arises when the branch cannot be followed smoothly and uniquely, i.e. when the implicit function theorem fails for the branch-following algorithm based on $\mathbf{H}(\mathbf{y}) = 0$ (cf. (27)). This is the case when $\det \mathbf{H_y} = 0$, which therefore identifies bifurcations involving more than 1 branch.
  - At Hopf bifurcations a complex pair of eigenvalues crosses the imaginary axis, i.e. the sum of one pair of eigenvalues goes through 0 $\Rightarrow$ one could choose $\mathcal{T}(\mathbf{x}, \lambda) = \prod_{1 \le j < \le k \le n} (\mu_j + \mu_k)$.
  - There are other test functions that are computationally more efficient (by not requiring to compute all eigenvalues) (cf. Seydel (2009, p.220)
- The parameter value λ_c of the bifurcation can be estimated by linear interpolation between the values $\lambda_n \equiv \lambda(s_n)$ and $\lambda_{n+1} \equiv \lambda(s_{n+1})$ between which $\mathcal{T}(\mathbf{x}, \lambda)$ changes sign,
$$\lambda_c \approx \lambda_n + \xi \, (\lambda_{n+1} - \lambda_n) \tag{34}$$
where
$$\xi = \frac{\mathcal{T}(\mathbf{x}^{(n)}, \lambda_n)}{\mathcal{T}(\mathbf{x}^{(n)}, \lambda_n) - \mathcal{T}(\mathbf{x}^{(n+1)}, \lambda_{n+1})} \qquad \text{and} \qquad \mathbf{x}^{(n)} \equiv \mathbf{x}(\lambda_n) \, .$$
Higher-order interpolations can improve that estimate further.
- Extrapolation of $\mathcal{T}(\mathbf{x}, \lambda)$ to an expected bifurcation point can be used to adjust the step size Δs of the continuation method to localize the bifurcation more precisely.
- Instead of solving $\mathbf{F}(\mathbf{x}, \lambda) = 0$ at the estimated λ_c to obtain $\mathbf{x}(\lambda_c)$ one can also use ξ to interpolate between $\mathbf{x}(s_n)$ and $\mathbf{x}(s_{n+1})$. This is much faster and may be accurate enough if $s_{n+1} - s_n$ is small enough.

To locate a steady bifurcation precisely one can use Newton's method to solve the system of equations
$$\mathbf{T}(\mathbf{y}) \equiv \begin{pmatrix} \mathbf{F}(\mathbf{x}, \lambda) \\ \mathbf{F_x}(\mathbf{x}, \lambda) \, \phi \\ \phi^t \phi - 1 \end{pmatrix} = 0 \qquad \text{for} \qquad \mathbf{y} = \begin{pmatrix} \mathbf{x}^t \\ \phi \\ \lambda \end{pmatrix} \in \mathbb{R}^{2n+1} \, . \tag{35}$$

The Newton iteration converges only if a good guess of $(\mathbf{x}_c, \phi_c^t, \lambda_c)$ is available. Good initial guesses $(\bar{\mathbf{x}}, \bar{\lambda})$ for $(\mathbf{x}_c, \lambda_c)$ are available from the interpolation. We still need a good guess $\bar{\phi}$ for ϕ_c, i.e. the solution of

$$\mathbf{F}_\mathbf{x}(\bar{\mathbf{x}}, \bar{\lambda})\bar{\phi} = 0\,.$$

Away from the bifurcation point $\mathbf{F}_\mathbf{x}$ is not singular and therefore no such $\bar{\phi}$ exists. Removing one of the equations (e.g. the l^{th} equation) makes the equation solvable. Replace the equation by a normalization condition for one of the components of $\bar{\phi}$, e.g. $\bar{\phi}_k = 1$. This yields the system

$$\left(\left[\mathbf{I} - \mathbf{e}^{(l)}\mathbf{e}^{(l)t}\right]\mathbf{F}_\mathbf{x}(\bar{\mathbf{x}}, \bar{\lambda}) + \mathbf{e}^{(l)}\mathbf{e}^{(k)t}\right)\bar{\phi} = \mathbf{e}^{(l)}\,, \tag{36}$$

where $\mathbf{e}_i^{(l)} = \delta_{il}$ is the unit vector in the l-direction (Seydel, 2009, p.214).

The top part of $\mathbf{T}(\mathbf{y})$ corresponds to the top part of $\mathbf{H}(\mathbf{y})$ (cf. (28)). The convergence of the Newton iteration depends on the Jacobian $\mathbf{T}_\mathbf{y}$; like $\mathbf{H}_\mathbf{y}$ it contains the block $(\mathbf{F}_\mathbf{x}|\mathbf{F}_\lambda)$. In the fold case $\mathbf{F}_\lambda \notin range\mathbf{F}_\mathbf{x}$ and $\mathbf{H}_\mathbf{y}$ is not singular. In the transcritical and the pitchfork case, however, $\mathbf{F}_\lambda \in range\mathbf{F}_\mathbf{x}$, rendering $\mathbf{T}_\mathbf{y}$ singular. Apparently, this slows down the convergence only very close to the bifurcation point (i.e. when high accuracy is required) and is practically not a real issue (Seydel, 2009, p.216).

Note:

- The direct method (35) of computing the bifurcation point is precise, but expensive since it requires the solution of a system that is twice as large as the original system ($2n+1$ instead of n). This is particularly relevant when applying these techniques to PDEs for which the spatial discretization determines n.

Branch Switching

How to switch to the other branch emanating from the bifurcation? One needs to obtain a sufficiently good guess for the solution on the other branch for the Newton iteration to converge to that branch rather than converging back to the already known branch.

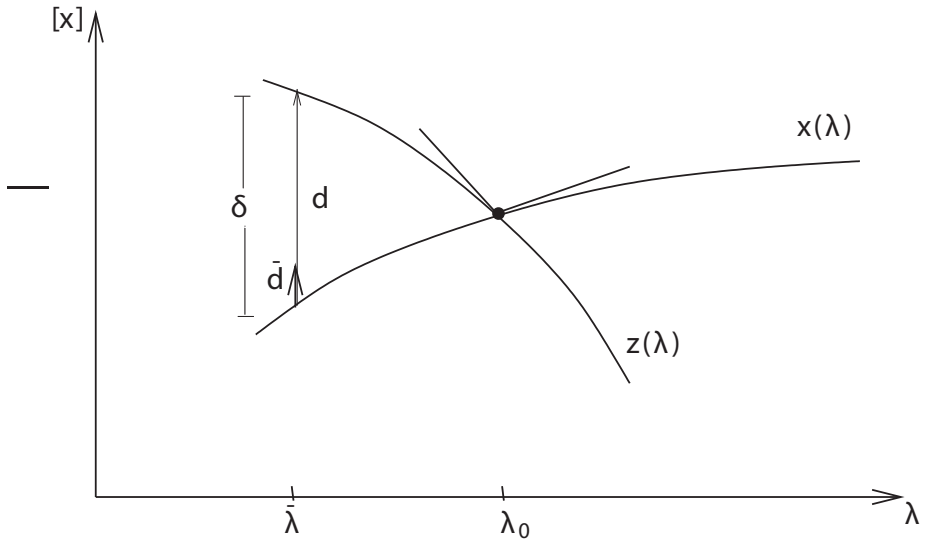


Figure 14: Tangents along the two branches intersecting at the bifurcation point. Also shown is a distance vector $\mathbf{d}$ used for switching (modified from (Seydel, 2009, p.233)).

One could compute both tangents that emanate at the bifurcation point. But that involves computing second derivatives of $\mathbf{F}$ with respect to $\mathbf{x}$ at the bifurcation point

- the bifurcation point needs to be computed accurately
- computational costs of the second derivatives is typically high.

Instead, exploit the fact that the eigenvector ϕ associated with the instability generating the bifurcation is a linear approximation to the direction $\mathbf{d}$ of the difference vector between the two branches (cf. Fig.14) in the vicinity of the bifurcation ($\lambda \neq \lambda_c$). ϕ lies in the plane spanned by the two tangent vectors. We need

1. a good approximation $\bar{\phi}$ of ϕ,
2. the distance δ between the branches at some value $\lambda \neq \lambda_c$.

Predictor:

A good initial predictor for a point on the other branch $\boldsymbol{z}(\lambda)$ is then

$$\bar{\mathbf{z}} = \mathbf{x}(\lambda) + \delta(\lambda)\bar{\phi}\,, \tag{37}$$

where λ is a point sufficiently close to the bifurcation point at which $\mathbf{x}$ has been calculated.

The approximation $\bar{\phi}$ is obtained via (36). It is normalized via its k^{th}-component, $\bar{\phi}_k = 1$.

To determine an approximation for the distance δ as a function of λ is challenging. Consider the simple one-dimensional case of a bifurcation off the state $x = 0$,

$$\begin{aligned} 0 \;=\; & \underbrace{F(0,\lambda_c)}_{=0} + \underbrace{\partial_x F|_{0,\lambda_c}}_{=0} x + \partial_\lambda F|_{0,\lambda_c}\,(\lambda-\lambda_c) + \frac{1}{2}\,\partial_x^2 F\big|_{0,\lambda_c}\,x^2 + \\ & + \partial_{x\lambda}^2 F\big|_{0,\lambda_c}\,x\,(\lambda-\lambda_c) + \frac{1}{2}\,\partial_\lambda^2 F\big|_{0,\lambda_c}\,(\lambda-\lambda_c)^2 + \ldots \end{aligned}$$

To get an estimate for x, which amounts to the distance between the two branches, one needs to determine again an accurate bifurcation point λ_c and second derivatives $\partial_x^2 F|_{(0,\lambda_c)}$.

Instead: fix δ and determine - via a corrector step - the value of λ for which this δ is attained.

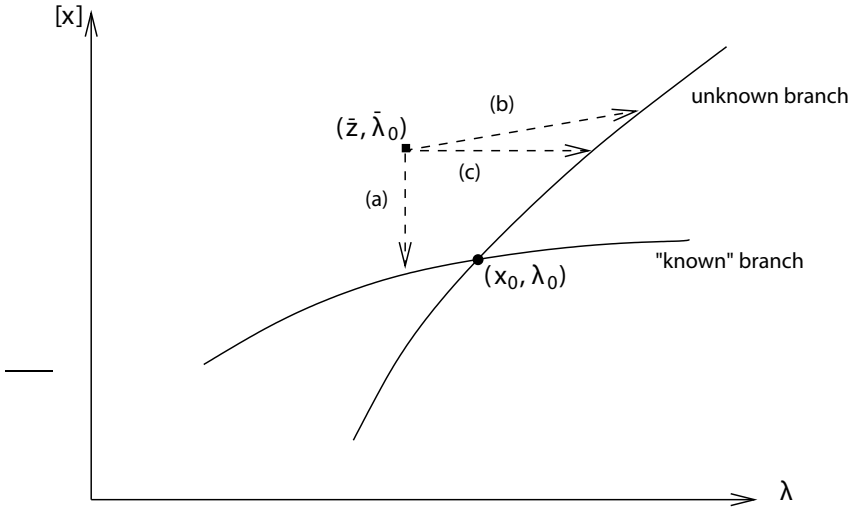


Figure 15: Comparison of the two corrector methods, fixing either $z_k - x_k$ or $z_k - \bar{z}_k$ (modified from (Seydel, 2009, p.235))

Corrector:

It is important that the correction step does not lead back to the known branch. This can be achieved by setting the distance $|\delta|$ between $\bar{\mathbf{z}}$ and $\mathbf{x}(\lambda)$ and fixing it to that value in the iterations on $\mathbf{z}$. Since in the predictor the k^{th}-component is normalized, $\bar{\phi}_k = 1$, fix the difference in that component $z_k = x_k \pm |\delta|$. Allow both signs since the orientation of the difference is not known. This yields the equations

$$\mathbf{K}(\mathbf{y}) \equiv \begin{pmatrix} \mathbf{F}(\mathbf{x}, \lambda) \\ \mathbf{F}(\mathbf{z}, \lambda) \\ z_k - x_k \mp |\delta| \end{pmatrix} = 0\,, \tag{38}$$

to be solved again via Newton. The initial guess is from the predictor $(\bar{\mathbf{x}} = \mathbf{x}(\lambda), \bar{\lambda} = \lambda, \bar{\mathbf{z}})^t$.

In this approach the solutions on both branches are computed in parallel. It guarantees that the iteration does not simply fall back to the known branch. But it requires recomputing solutions on the known branch $\mathbf{x}$. Instead, one could seek the value of λ for which the solution $\boldsymbol{z}$ has the same value in the k^{th} component as the predictor,

$$\begin{pmatrix} \mathbf{F}(\mathbf{z}, \lambda) \\ z_k - \bar{z}_k \end{pmatrix} = 0$$

with initial guess $(\bar{\mathbf{z}}, \bar{\lambda})$ with $\bar{z}_k = x_k \pm |\delta|$. This does not guarantee that the solution is on the new branch, but it is significantly faster.

Notes:

- For each transcritical or pitchfork bifurcation point 4 branch-following computations are triggered corresponding to the two values of z_k and to the two directions in which the branch is followed starting from each $\mathbf{z}$. Two of the four computations will recover the previously identified bifurcation and need not be continued further

- Since multiple solutions on the known branch are available, a better approximation for the predictor can be obtained by interpolating between them to get predictor values that are closer to the values at the bifurcation point λ_c, analogously to (34),

$$\begin{aligned}\bar{\mathbf{x}} &= \mathbf{x}^{(n)} + \xi\left(\mathbf{x}^{(n+1)} - \mathbf{x}^{(n)}\right)\\ \bar{\phi} &= \bar{\phi}^{(n)} + \xi\left(\bar{\phi}^{(n+1)} - \bar{\phi}^{(n)}\right)\\ \bar{z} &= \bar{\mathbf{x}} \pm |\delta|\,\bar{\phi}\,.\end{aligned}$$

6 Higher-Dimensional Center Manifolds: Hopf Bifurcation

So far the center manifold $W^{(c)}$ was always one-dimensional, because only a single eigenvalue passed through 0.

Higher-dimensional center manifolds arise for

- bifurcations involving interaction of multiple modes (multiple vanishing eigenvalues)
- bifurcations to oscillatory solutions (complex pair of eigenvalues).

So far the linearization only had real eigenvalues going through 0.

Consider a pair of complex eigenvalues crossing the imaginary axis.

Investigate using an example:

$$\begin{aligned}\dot{x} &= -x - y + 2y^2\\ \dot{y} &= 2x + (1+\mu)\,y - 4xy - 4y^2\end{aligned}$$

The linearization about the fixed point $(0,0)$,

$$\mathbf{L} = \begin{pmatrix} -1 & -1 \\ 2 & 1+\mu \end{pmatrix},$$

has eigenvalues

$$\lambda_{1,2} = \frac{\mu \pm \sqrt{\mu^2 - 4(1-\mu)}}{2} = \frac{1}{2}\mu \pm i\sqrt{1 - \mu - \frac{1}{4}\mu^2} \equiv \sigma \pm i\omega\,. \tag{39}$$

At $\mu = 0$ we have $\lambda_{1,2} = \pm i$ and the eigenvectors

$$\tilde{\mathbf{v}}_{1,2} = \begin{pmatrix} -\frac{1}{2} \pm i\frac{1}{2} \\ 1 \end{pmatrix} \qquad \text{and} \qquad \tilde{\mathbf{v}}_2 = \tilde{\mathbf{v}}_1^*\,.$$

6.1 Center Manifold Approach

The center eigenspace is two-dimensional: $E^{(c)} = \mathbb{R}^2$.

The vectors in the center eigenspace must be real: consider therefore suitable combinations of $\tilde{\mathbf{v}}_{1,2}$

$$\mathbf{v}_1 = \frac{1}{2}\left(\tilde{\mathbf{v}}_1 + \tilde{\mathbf{v}}_1^*\right) = \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix} \qquad \mathbf{v}_2 = \frac{1}{2i}\left(\tilde{\mathbf{v}}_1 - \tilde{\mathbf{v}}_1^*\right) = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix},$$

which represent the real and the imaginar part of $\tilde{\mathbf{v}}_1$. We then have

$$\begin{aligned} \mathbf{L}\mathbf{v}_1 &= \frac{1}{2}\left(i\tilde{\mathbf{v}}_1 + (-i)\tilde{\mathbf{v}}_2\right) = \frac{i}{2}\left(\tilde{\mathbf{v}}_1 - \tilde{\mathbf{v}}_1^*\right) = \frac{i}{2} 2i \mathbf{v}_2 = -\mathbf{v}_2 \\ \mathbf{L}\mathbf{v}_2 &= \mathbf{v}_1 \,. \end{aligned}$$

Thus, the space spanned by $\mathbf{v}_{1,.2}$ is invariant under the linear dynamics, as has to be the case for the center eigenspace.
Rewrite (x, y) in terms of the components of these vectors

$$\begin{pmatrix} x \\ y \end{pmatrix} = u\mathbf{v}_1 + v\mathbf{v}_2 = \begin{pmatrix} -\frac{1}{2}u + \frac{1}{2}v \\ u \end{pmatrix} = \mathbf{S}\begin{pmatrix} u \\ v \end{pmatrix} \qquad \mathbf{S} = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix} \quad \mathbf{S}^{-1} = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix},$$

which yields after a little algebra

$$\begin{aligned} \dot{u} &= \mu u + v - 2u^2 - 2uv \\ \dot{v} &= -u + \mu u + 2u^2 - 2uv \,. \end{aligned}$$

We expect oscillations: it is therefore useful to rewrite the equations in terms of a complex amplitude $A = u + iv$, i.e. $u = \frac{1}{2}(A + A^*)$ and $v = \frac{1}{2i}(A - A^*)$,

$$\dot{A} = -iA + \frac{1}{2}\mu(1+i)\,A + \frac{1}{2}\mu(1+i)\,A^* - (1-i)\,A^2 - (1-i)\,A^*A \,. \tag{40}$$

Notes:

- Since this system is only two-dimensional, this process has only led to a rewriting of the equations in terms of a new variable A,
  no approximation has been made: μ and A have not been assumed to be small so far.
  No real simplification has been achieved yet.
- We will see that near the bifurcation this equation can be substantially simplified by a *near-identity transformation.*

In higher-dimensional systems a true reduction in terms of the dimension is achieved just as it was for steady bifurcations. For concreteness, assume a three-dimensional system with the center eigenspace spanned by $\mathbf{v}_{1,2}$ and the stable eigenspace by $\mathbf{v}_3$,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = u\mathbf{v}_1 + v\mathbf{v}_2 + w\mathbf{v}_3 \qquad . \begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} f_u(u,v,w) \\ f_v(u,v,w) \\ f_w(u,v,w) \end{pmatrix}.$$

Parametrize the points (x, y, z) on the two-dimensional center manifold in terms of the coordinates on the center eigenspace

$$w = h(u, v)\,.$$

The function h defining the center manifold is then determined by the condition

$$\dot{w} = \frac{\partial h}{\partial u}\dot{u} + \frac{\partial h}{\partial v}\dot{v} = \frac{\partial h}{\partial u} f_u\left(u, v, h(u, v)\right) + \frac{\partial h}{\partial v} f_v\left(u, v, h(u, v)\right) \overbrace{=}^{!} f_w\left(u, v, h(u, v)\right)\,,$$

which is solved in terms of an expansion in u and v, making again use of the fact that $W^{(c)}$ is tangential to $E^{(c)}$.

Notes:

- The dynamics and the center manifold equation can also be given in terms of a complex amplitude A.

6.2 Multiple-Scale Analysis[5]

Consider the same example as above,

$$\begin{aligned} \dot{x} &= -x - y + 2y^2 \\ \dot{y} &= 2x + (1+\mu)\, y - 4xy - 4y^2\,. \end{aligned}$$

Near the bifurcation point there are now two time scales in the problem (cf. (39))

- oscillations occur on the fast time scale t: $\omega = i$
- growth and decay occurs on the slow time scale T: $\sigma = \mathcal{O}(\mu) \ll 1$

How to choose the scaling of the oscillation amplitude?
We will see that because of time translation symmetry the oscillation amplitude is $\mathcal{O}(\mu^{1/2})$.

Choose therefore

$$\underline{u} = \mathcal{O}(\epsilon), \qquad T = \epsilon^2 t \qquad \mu = \epsilon^2 \mu_2\,.$$

Note:

- $\underline{u}$ depends on the fast and the slow time independently:

$$\frac{d}{dt}\underline{u} \to \left(\frac{\partial}{\partial t} + \epsilon^2 \frac{\partial}{\partial T}\right)\underline{u}$$

The equations can now be written as

[5] For a simpler start-up example see Notes for 322.

$$\underbrace{\begin{pmatrix} 1+\frac{\partial}{\partial t} & 1 \\ -2 & -1+\frac{\partial}{\partial t} \end{pmatrix}}_{\underline{\underline{\mathcal{L}}}_0} \begin{pmatrix} x(t,T) \\ y(t,T) \end{pmatrix} = -\begin{pmatrix} \frac{\partial}{\partial T}x(t,T) \\ \frac{\partial}{\partial T}y(t,T) \end{pmatrix} + \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & \mu_2 \end{pmatrix}}_{\underline{\underline{\mathcal{L}}}_2} \begin{pmatrix} x(t,T) \\ y(t,T) \end{pmatrix} + + \underbrace{\begin{pmatrix} 2y(t,T)^2 \\ -4x(t,T)\,y(t,T) - 4y(t,T)^2 \end{pmatrix}}_{\mathbf{N}_2(\mathbf{u},\mathbf{u})}$$

Expand

$$\begin{aligned} \begin{pmatrix} x(t,T) \\ y(t,T) \end{pmatrix} &= \epsilon\mathbf{u}_1(t,T) + \epsilon\mathbf{u}_2(t,T) + \epsilon\mathbf{u}_3(t,T) + \ldots \qquad (41) \\ &= \epsilon A(T) \underbrace{\begin{pmatrix} -\frac{1}{2}+\frac{i}{2} \\ 1 \end{pmatrix}}_{\tilde{\underline{v}}_1} e^{+it} + \epsilon A^*(T) \underbrace{\begin{pmatrix} -\frac{1}{2}-\frac{i}{2} \\ 1 \end{pmatrix}}_{\tilde{\underline{v}}_1^* \equiv \tilde{\underline{v}}_2} e^{-it} + \\ &\quad +\epsilon^2 \left(\underline{u}_{20}(T) + \underline{u}_{22}(T)e^{2it} + \underline{u}_{22}^*(T)e^{-2it}\right) + \\ &\quad +\epsilon^3 \left(\underline{u}_{31}(T)e^{it} + \underline{u}_{33}(T)e^{3it} + c.c.\right) + h.o.t. \end{aligned}$$

Note:

- quadratic nonlinearity
  - introduces frequencies $\pm 2\omega$ at quadratic order
  - at cubic order the term from $\mathcal{O}(\epsilon^2)$ combines with $\mathcal{O}(\epsilon)$ to give frequencies $\pm 3\omega$ and $\pm\omega$

At each order in ϵ we will get an inhomogeneous linear equation involving the linear operator $\mathbf{L}_0$. Because the Fourier modes for different frequencies are linearly independent it is useful to consider different frequencies separately (somewhat similar to Fourier analysis),

$$\begin{pmatrix} 1+\frac{\partial}{\partial t} & 1 \\ -2 & -1+\frac{\partial}{\partial t} \end{pmatrix} \underline{u}e^{int} = \underbrace{\begin{pmatrix} 1+in & 1 \\ -2 & -1+in \end{pmatrix}}_{\underline{\underline{L}}_0(n)} \underline{u}e^{int}\,.$$

For different frequencies we therefore get different matrices $\mathbf{L}_0(n)$. Which of them are singular? We have

$$det\mathbf{L}_0(n) = -(1+n^2) + 2\,.$$

Thus, $\mathbf{L}_0(n)$ is only singular for $n = \pm 1$, and from the linear stability analysis we have the corresponding 0-eigenvectors $\tilde{\mathbf{v}}_{1,2}$,

$$\mathbf{L}_0(\pm 1) \begin{pmatrix} -\frac{1}{2}\pm\frac{i}{2} \\ 1 \end{pmatrix} = \begin{pmatrix} 1+(\pm i) & 1 \\ -2 & -1+(\pm i) \end{pmatrix} \begin{pmatrix} -\frac{1}{2}\pm\frac{i}{2} \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

Therefore, only $\mathbf{L}_0(\pm 1)$ will lead to a solvability condition.

Expansion

$\mathcal{O}(\epsilon)$:

$$\underline{\underline{L}}_0(+1)\underline{v}_1 A e^{+it} + \underline{\underline{L}}_0(-1)\underline{v}_1^* A^* e^{-it} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

confirming the results of the linear stability analysis.

$\mathcal{O}(\epsilon^2)$:

$$\underline{\underline{L}}_0(0)\underline{u}_{20}(T) + \underline{\underline{L}}_0(+2)\underline{u}_{22}(T)e^{2it} + \underline{\underline{L}}_0(-2)\underline{u}_{22}^*(T)e^{-2it} = \underline{\mathcal{N}}_2\left(\mathbf{u}_1, \mathbf{u}_1\right)$$

Note:

- None of these linear operators are singular: *no solvability condition* arises *at quadratic order.*
  that is the reason for the scaling of T, μ, and A.

Solve separately for different Fourier modes:

e^{0it} :

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} x_{20} \\ y_{20} \end{pmatrix}}_{\mathbf{u}_{20}} + \begin{pmatrix} 2\left(1\cdot 1 + 1\cdot 1\right) \\ -4\left(\left(-\frac{1}{2}+\frac{i}{2}\right)\cdot 1 + \left(-\frac{1}{2}-\frac{i}{2}\right)\cdot 1\right) - 4\cdot 2 \end{pmatrix} AA^*$$

$$\begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_{20} \\ y_{20} \end{pmatrix} = \begin{pmatrix} -4 \\ 4 \end{pmatrix} AA^*$$

$$\begin{pmatrix} x_{20} \\ y_{20} \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} AA^*$$

e^{2it} :

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1-2i & -1 \\ 2 & 1-2i \end{pmatrix} \begin{pmatrix} x_{22} \\ y_{22} \end{pmatrix} + \begin{pmatrix} 2\cdot 1 \\ -4\left(-\frac{1}{2}+\frac{i}{2}\right)\cdot 1 - 4\cdot 1 \end{pmatrix} A^2$$

$$\begin{pmatrix} x_{22} \\ y_{22} \end{pmatrix} = \begin{pmatrix} -2i \\ 2(-1+i) \end{pmatrix} A^2$$

$\mathcal{O}(\epsilon^3)$:

$$\begin{aligned} \underline{\underline{L}}_0(1)\,\underline{u}_{31}(T)e^{it} + \underline{\underline{L}}_0(3)\,\underline{u}_{33}(T)e^{3it} \;&=\; -\underline{v}_1\frac{d}{dT}Ae^{it} - \underline{v^*}_1\frac{d}{dT}A^*e^{-it} + \\ &\quad + \underline{\underline{\mathcal{L}}}_2\left(\underline{v}_1 e^{it}A + \underline{v}_1^* e^{-it}A^*\right) + \underline{\mathcal{N}}_2(\underline{u}_1, \underline{u}_2) + \underline{\mathcal{N}}_2(\underline{u}_2, \underline{u}_1) \end{aligned}$$

$\underline{\underline{L}}_0(\pm 1)$ is singular: solvability conditions arise only for the equations involving e^{it} and e^{-it}. We therefore need to consider only the e^{it}-component of the $\mathcal{O}(\epsilon^3)$-equation (we could obtain that term by multiplying equation by e^{-it} and integrating over one period $0 \le t \le 2\pi/\omega = 2\pi$).

We still need the left-eigenvector $\underline{v}_1^+$ for the zero eigenvalue:

$$(0,0) = \underline{v}_1^+ \underline{\underline{L}}_0(\pm 1) = (x_\pm^+, y_\pm^+) \begin{pmatrix} 1+(\pm i) & 1 \\ -2 & -1+(\pm i) \end{pmatrix}$$

$$x_\pm^+(1\pm i)-2y_\pm^+=0 \qquad \Rightarrow y_\pm^+=\frac{1}{2}(1+i)x_\pm^+ .$$

The first nonlinear term $\mathbf{N}_2(\mathbf{u}_1,\mathbf{u}_2)$contributes

$$\underline{\mathcal{N}}_2(\underline{u}_1,\underline{u}_2) = \begin{pmatrix} 2y_1y_2 \\ -4x_1y_2-4y_1y_2 \end{pmatrix} = e^{it}\underbrace{\begin{pmatrix} 2\left(Av_{1y}y_{20}+A^*v_{1y}^*y_{22}\right) \\ -4\left(Av_{1x}y_{20}+A^*v_{1x}^*y_{22}\right)-4\left(Av_{1y}y_{20}+A^*v_{1y}^*y_{22}\right) \end{pmatrix}}_{\underline{\mathcal{N}}_{21}(\underline{u}_1,\underline{u}_2)} +$$
$$+e^{-it}(\ldots)+e^{3it}(\ldots)+e^{-3it}(\ldots)$$

Insert $y_{20}=-2AA^*$, $y_{22}=(-1+\frac{1}{3}i)A^2$ etc. $\Rightarrow$ all nonlinear terms have the form AAA^*. Analogously for $\underline{\mathcal{N}}_2(\underline{u}_2,\underline{u}_1)$.

The solvability condition is then given by

$$\frac{d}{dT}A\underline{v}_1^+\cdot\underline{v}_1=\underline{v}_1^+\left(\underline{\underline{\mathcal{L}}}_2\underline{v}_1A+\underline{\mathcal{N}}_{21}(\underline{u}_1,\underline{u}_2)+\underline{\mathcal{N}}_{21}(\underline{u}_2,\underline{u}_1)\right),$$

which after some more algebra yields

$$\frac{d}{dT}A=\frac{1}{2}\mu_2(1-i)A-8(1+i)|A|^2A . \tag{42}$$

Note:

- The result of the multiple-scale analysis looks very different than the equation for the dynamics on the center manifold (40)
- In the center-manifold approach (40) no second time scale was introduced, i.e. there was no slow time that evolves independent of the fast time. The time-scale separation is an approximation; therefore, certain phenomena that arise from the coupling of the two time scales are lost in the multiple-scales analysis. These tend to be more subtle, e.g. associated with homoclinic or heteroclinic orbits.

6.3 Normal Form Transformations

The center-manifold reduction and the multiple-scale analysis gave very different looking results for the Hopf bifurcation. *Is there a disagreement?*

Compare the two equations:

Multiple scales:

$$\frac{d}{dT}A=\frac{1}{2}\mu_2(1-i)A-8(1+i)|A|^2A$$

- There is only a single, cubic nonlinearity, which has a special form that allows very simple exact solutions:

  $$A=\mathcal{A}e^{i\Omega T} \qquad \text{with} \quad \mathcal{A}=\frac{1}{4}\sqrt{\mu_2} \quad \Omega=-\frac{5}{2}\mu_2 .$$

  This periodic orbit traces out a perfect circle in the complex A-plane.

Center manifold:

$$\frac{d}{dt}\tilde{A} = -i\tilde{A} + \frac{1}{2}\mu(1+i)\,\tilde{A} + \frac{1}{2}\mu(1+i)\,\tilde{A}^* + \underbrace{(i-1)}_{a_{20}}\,\tilde{A}^2 + \underbrace{(i-1)}_{a_{11}}\,\tilde{A}^*\tilde{A} + a_{02}A^{*2} \qquad (43)$$

- This equation has only quadratic nonlinearities.
- The equation has no solution of the form $e^{i\Omega t}$: the orbit is not simply a circle in the $\tilde{A}$-plane, which is identical to the (u,v)-plane and the (x,y)-plane.
- In the example we found $a_{02} = 0$. We keep it here for generality.

In the two approaches the amplitudes A and $\tilde{A}$ represent different quantities.

Center Manifold:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{S}\begin{pmatrix} u \\ v \end{pmatrix} = \mathbf{S}\begin{pmatrix} \tilde{A}_r \\ \tilde{A}_i \end{pmatrix}$$

Multiple Scales:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \epsilon\left(A\mathbf{v}e^{+it} + \epsilon A^*\mathbf{v}^*e^{-it}\right) + \epsilon^2\left(\mathbf{u}_{20}(T) + \mathbf{u}_{22}(T)e^{2it} + \mathbf{u}_{22}^*(T)e^{-2it}\right) + h.o.t. \qquad (44)$$

Thus:

- Even if the multiple-scale solution is given by the single Fourier mode $\propto e^{i\Omega T+it}$, the overall solution (x,y) has more Fourier components. These higher harmonics arise from the terms that are higher order in ϵ.
- The center-manifold solution may include those Fourier modes already in $\tilde{A}$.

One may expect that the two solutions agree in the basic Fourier mode - up to the factor e^{it} that has been pulled out explicitly in (44) - and differ only in the higher-order Fourier modes. The amplitudes A and $\tilde{A}$ may therefore be related to each other via a *near-identity transformation*

$$\tilde{A} = A + \alpha A^2 + \beta|A|^2 + \gamma A^{*2} + \delta A^3 + \sigma A^2A^* + \rho AA^{*2} + \tau A^{*3} + \mathcal{O}(A^4) \qquad (45)$$

Note:

- For general A this transformation does not have a unique inverse.
- However, for small $\tilde{A}$ there is always a unique small inverse A,

$$\begin{aligned} A &= \tilde{A} - \alpha A^2 - \beta|A|^2 - \gamma A^{*2} + \ldots \\ &= \tilde{A} - \alpha\left(\tilde{A} - \alpha A^2 - \beta|A|^2 - \gamma A^*A^2 + \ldots\right)^2 - \beta\left|\tilde{A} - \alpha A^2 - \ldots\right|^2 \\ &\quad -\gamma\left(\tilde{A}^* - \alpha A^{*2} - \ldots\right)^2 \ldots \\ &= \tilde{A} - \alpha\tilde{A}^2 - \beta\left|\tilde{A}\right|^2 - \gamma\tilde{A}^{*2} + \mathcal{O}(\tilde{A}A^2) \\ &= \ldots . \end{aligned}$$

  Determine the inverse iteratively by inserting the expression for A again into the h.o.t.
  .

The goal of the near-identity transformation is to choose the coefficients α, β, γ etc. to eliminate as many terms as possible for the evolution equation for A. Since the aim is to describe the behavior in the vicinity of the bifurcation, the transformation must be valid, in particular, for $\mu = 0$. We set therefore $\mu = 0$ at this point.

Insert (45) into (43)

$$\frac{d}{dt}A + 2\alpha A\frac{d}{dt}A + \beta\left(A\frac{d}{dt}A^* + \frac{d}{dt}A\,A^*\right) + 2\gamma A^*\frac{d}{dt}A^* \quad + \mathcal{O}\left(A^2\tfrac{d}{dt}A\right) =$$
$$-i\left(A + \alpha A^2 + \beta|A|^2 + \gamma A^{*2} + \mathcal{O}\left(A^3\right)\right) + \tag{46}$$
$$+a_{20}\left(A + \alpha A^2 + \ldots\right)^2 + a_{11}\left(A + \alpha A^2 + \ldots\right)\left(A^* + \alpha^* A^{*2} + \ldots\right) +$$
$$+a_{02}\left(A^* + \alpha A^{*2} + \ldots\right)^2 \tag{47}$$

Note:

- It turns out that the requirement for the transformation to be valid at $\mu = 0$ poses significant restrictions on the terms that actually can be eliminated (see below).

Since the transformation is done in an expansion in the small amplitude A we solve this equation order by order in A

$\mathcal{O}(A)$:

$$\frac{d}{dt}A = -iA\,.$$

$\mathcal{O}(A^2)$:

- the time derivative $\frac{d}{dt}A$ in the nonlinear terms can be replaced by $-iA$

$$-2i\alpha A^2 + \beta\left(i|A|^2 - i|A|^2\right) + 2\gamma i A^{*2} \;=\; -i\left(\alpha A^2 + \beta|A|^2 + \gamma A^{*2}\right) +$$
$$+a_{20}\,A^2 + a_{11}\,|A|^2 + a_{02}A^{*2}\,.$$

Choose

$$\begin{aligned} -2i\alpha &= -i\alpha + a_{20} \qquad & \alpha &= ia_{20} \\ 0 &= -i\beta + a_{11} \qquad & \beta &= -ia_{11} \\ 2i\gamma &= -i\gamma + a_{02} \qquad & \gamma &= -\frac{i}{3}a_{02} \end{aligned}$$

Thus: the transformation

$$\tilde{A} = A + ia_{20}\,A^2 - ia_{11}\,|A|^2 - \frac{i}{3}a_{02}A^{*2}$$

removes all quadratic terms in the equation for A and leads to the evolution equation

$$\frac{d}{dt}A = -iA + \mathcal{O}(A^3) \tag{48}$$

$\mathcal{O}(A^3)$: The calculation becomes somewhat unwieldy, but proceeds in the same way.

Question:

Can *all* nonlinear terms be removed with such a normal-form transformation? **No.**

Allowing $\mu \neq 0$ and keeping only the leading-order term in an expansion in μ one obtains using Mathematica

$$\begin{aligned}
\delta &= \frac{1}{12}(1-i)\left(2|a_{02}|^2 - (1+i)a_{02}^* a_{11} + 3|a_{11}|^2 - 3a_{11}a_{20} - 6(1+i)a_{20}^2\right) + \mathcal{O}(\mu) \\
\rho &= \frac{1}{12}\left(-6a_{11}^2 + a_{02}\left(12a_{11}^* - 4a_{20} - 6(1+i)a_{02}^*\right)\right) \\
&\quad + \frac{1}{4}(1+i)\left(a_{11}\left(-3a_{11}^* + 2a_{20} - (1-i)a_{20}^*\right) + a_{11}^* a_{20}^*\right) + \mathcal{O}(\mu) \\
\sigma &= \frac{i}{3}\frac{2|a_{02}|^2 + 3a_{11}(a_{11}^* - a_{20})}{\mu} + \mathcal{O}(\mu^0) \\
\tau &= -\frac{1}{12}a_{02}\left(a_{11} + 6a_{20}^*\right) + \mathcal{O}(\mu)
\end{aligned}$$

Away from the bifurcation point one can eliminate all cubic terms with this transformation. However, as the bifurcation point $\mu = 0$ is approached σ diverges $\Rightarrow$ the transformation is therefore not uniformly valid near the bifurcation since the expansion coefficients of the near-identity transformation need to remain of $\mathcal{O}(1)$.

Thus, the corresponding equation cannot be solved as the bifurcation is approached and the corresponding cubic term $|A|^2 A$ cannot be removed by this near-identity transformation

Setting $\mu = 0$ one obtains for the transformation

$$\begin{aligned}
\delta &= -\frac{1}{6}\left(a_{02}^* a_{11} + 6a_{20}^2\right) \\
\rho &= \frac{1}{6}\left(-3a_{11}^2 + 6a_{02}a_{11}^* - 2a_{02}a_{20} - 3a_{11}a_{20}^*\right) \\
\tau &= -\frac{1}{12}\left(a_{02}\left(a_{11} + 6a_{20}^*\right)\right)
\end{aligned}$$

which yields the evolution equation

$$\frac{d}{dt}A = -iA + \frac{i}{3}\left(2|a_{02}|^2 + 3|a_{11}|^2 - 3a_{11}a_{20}\right)|A|^2 A + \mathcal{O}(A^4) \tag{49}$$

which for our example becomes

$$\frac{d}{dt}A = -iA - 2(1+i)|A|^2 A + \mathcal{O}(A^4) \tag{50}$$

Note:

- The term $|A|^2 A$ cannot be removed: the term $\sigma A^2 A^*$ in the transformation drops out of the computation and does not achieve any reduction of the equation.
- (49) is the *normal form* for the Hopf bifurcation; for all systems the center-manifold equations for the Hopf bifurcation can be transformed to this form.

- The normal form agrees with the equation obtained my multiple scales:
  - the term $-iA$ corresponds to the factor e^{it} that has been factored out in the multiple-scale ansatz (41).
  - the difference in the prefactor of the cubic term is related to a different scaling of the amplitudes in the two cases (norm of $\mathbf{v}_1$).
  - the ratio of real to imaginary part of the cubic coefficient cannot be changed by scaling: that ratio agrees.
- This transformation can be done order-by-order to all orders in the expansion in A. At each order, terms from lower orders are pushed to higher orders. Terms that are 'beyond all orders in A', e.g. $e^{-\frac{1}{A}}$, cannot be removed.
- In general, the series defining the normal-form transformation does *not converge.*
- For an accessible discussion of the general structure of the normal-form transformation see Chapter VIII in Crawford (1991).
- The result obtained for non-zero μ and that for $\mu = 0$ do not agree with each other in the limit $\mu \to 0$, since for $\mu = 0$ only 6 of the 8 equations that arise at $\mathcal{O}(A^3)$ are solved (the equation corresponding to eliminating $|A|^2 A$ and its complex conjugate are omitted).

7 Numerical Approaches to Bifurcations II

7.1 Hopf Bifurcations and Continuing Periodic Orbits

To study the branches of solutions emerging from a Hopf bifurcation we need to compute periodic solutions, i.e. we compute the temporal evolution of the system with the condition that the solution is actually periodic. This turns the usual initial-value problem into a boundary value problem.

Consider

$$\partial_{\hat{t}} \mathbf{u}(\hat{t}) = \mathbf{F}(\mathbf{u}(\hat{t}), \lambda) \qquad \text{with} \qquad \mathbf{u}(0) = \mathbf{u}(T)\,.$$

The period T is actually an unknown in this case. It is therefore useful to rescale the time to set the period to a fixed value, i.e. use $t = \hat{t}/T$ which yields

$$\partial_t \mathbf{u}(t) = T\,\mathbf{F}(\mathbf{u}(t), \lambda) \qquad \text{with} \qquad \mathbf{u}(0) = \mathbf{u}(1)\,.$$

This does not define a unique solution, since the solution can be shifted arbitrarily in time, with $\mathbf{u}(t)$ the shifted function $\mathbf{u}(t+\theta)$ is also a solution. When continuing the solution along the branch, it is advantageous to obtain solutions that are close to each other, e.g., by minimizing the difference

$$D(\theta) = \int_0^1 \left\|\mathbf{u}^{(n+1)}(t+\theta) - \mathbf{u}^{(n)}(t)\right\|^2 dt = \int_0^1 \left(\mathbf{u}^{(n+1)}(t+\theta) - \mathbf{u}^{(n)}(t)\right)^t \left(\mathbf{u}^{(n+1)}(t+\theta) - \mathbf{u}^{(n)}(t)\right) dt$$

between steps n and $n+1$ in the continuation. Differentiation gives

$$\begin{aligned}\frac{d\,D(\theta)}{d\theta} &= 2\int_0^1 \frac{d}{d\theta}\mathbf{u}^{(n+1)}(t+\theta)^t\,\left\{\mathbf{u}^{(n+1)}(t+\theta)-\mathbf{u}^{(n)}(t)\right\}dt \\ &= 2\int_0^1 \frac{1}{2}\frac{d}{d\theta}\left\{\mathbf{u}^{(n+1)}(t+\theta)^t\mathbf{u}^{(n+1)}(t+\theta)\right\} - \frac{d}{d\theta}\left\{\mathbf{u}^{(n+1)}(t+\theta)^t\right\}\,\mathbf{u}^{(n)}(t)\,dt\end{aligned}$$

Using periodicity in the first term and integrating by parts with respect to t, using $\frac{d}{d\theta}\left\{\mathbf{u}^{(n+1)}(t+\theta)^t\right\} = \frac{d}{dt}\left\{\mathbf{u}^{(n+1)}(t+\theta)^t\right\}$, in the second term gives then the condition

$$\int_0^1 \mathbf{u}^{(n+1)}(t+\theta)^t\,\frac{d}{dt}\mathbf{u}^{(n)}(t)dt = 0\,.$$

Since this is a condition on the solution $\mathbf{u}^{(n+1)}$ that is still to be determined, we require that that solution satisfy this condition for $\theta = 0$, i.e. we get the *integral phase condition*

$$\int_0^1 \mathbf{u}^{(n+1)}(t)^t\,\frac{d}{d\theta}\mathbf{u}^{(n)}(t)dt = 0\,.$$

That is the condition used in Doedel's AUTO package (cf. (Doedel, 2007)).

To follow the branch of periodic orbits, use again pseudo-arclength continuation. Again the continuation amounts to a fixed distance Δs (cf. Fig.13) along the tangent vector combined with a correction in the plane perpendicular to the tangent vector. While in (28) the solution vector has only a finite number of components, the vector $(\mathbf{u}(t), T, \lambda)$ has infinitely many, i.e. $\mathbf{u}$ at all times $t \in [0,1]$. The scalar product in (28), which requires the projection of the difference between the iterations onto the tangent to be given by Δs, turns therefore into an integral[6],

$$\int_0^1 \left(\mathbf{u}^{(n+1)}(t)-\mathbf{u}^{(n)}\right)^t \dot{\mathbf{u}}^{(n)}(t)\,dt + \left(T^{(n+1)}-T^{(n)}\right)\,\dot{T}^{(n)} + (\lambda_{n+1}-\lambda_n)\,\dot{\lambda}_n = \Delta s\,,$$

where the dot denotes a derivative with respect to the continuation parameter s, i.e. $(\dot{\mathbf{u}}^{(n)}, \dot{T}^{(n)}, \dot{\lambda}_n)$ corresponds to the tangent vector $\boldsymbol{\tau}$ to the branch for $s = s_n$.

In each continuation step $(\mathbf{u}^{(n+1)}(t), T^{(n+1)}, \lambda_{n+1})$ is determined by solving the equations

$$\partial_t\mathbf{u}^{(n+1)}(t) = T^{(n+1)}\,\mathbf{F}(\mathbf{u}^{(n+1)}(t), \lambda_{n+1}) \qquad \text{with} \qquad \mathbf{u}^{(n+1)}(0) = \mathbf{u}^{(n+1)}(1)\,, \tag{51}$$

$$\int_0^1 \mathbf{u}^{(n+1)}(t)^t\,\frac{d}{dt}\mathbf{u}^{(n)}(t)dt = 0\,, \tag{52}$$

$$\int_0^1 \left(\mathbf{u}^{(n+1)}(t)-\mathbf{u}^{(n)}\right)^t \dot{\mathbf{u}}^{(n)}(t)\,dt + \left(T^{(n+1)}-T^{(n)}\right)\,\dot{T}^{(n)} + (\lambda_{n+1}-\lambda_n)\,\dot{\lambda}_n = \Delta s\,. \tag{53}$$

[6]Note, $\mathbf{u}^{(n)}(t)$ is the currently known solution at the continuation parameter s and corresponds to $\mathbf{x}^{(0)}$ in (28).

Discretizing the time turns the integrals into sums and equations (52,53) become algebraic equations. One can also use finite differences to turn (51) into algebraic equations and then use a standard Newton method.

In analogy to (25), the tangent vector $(\dot{\mathbf{u}}^{(n)}, \dot{T}^{(n)}, \dot{\lambda}_n)$ is obtained from the derivative of (51) with respect to the continuation parameter s,

$$\partial_t \dot{\mathbf{u}}^{(n)}(t) = \dot{T}^{(n)}\, \mathbf{F}(\mathbf{u}^{(n)}(t), \lambda_n) + T^{(n)}\, \mathbf{F}_{\mathbf{u}}(\mathbf{u}^{(n)}(t), \lambda_n)\dot{\mathbf{u}}^{(n)} + T^{(n)}\, \mathbf{F}_{\lambda}(\mathbf{u}^{(n)}(t), \lambda_n)\dot{\lambda}_n\,. \qquad (54)$$

with

$$\dot{\mathbf{u}}^{(n+1)}(0) = \dot{\mathbf{u}}^{(n+1)}(1)\,.$$

To get the continuation started an initial guess needs to be computed. Use again the linear eigenvector associated with the bifurcation. Linearization of (51) around the fixed point $\mathbf{u}_{FP}$ near the Hopf bifurcation point λ_H amounts to

$$\partial_t \phi = T_H\, \mathbf{F}_{\mathbf{u}}(\mathbf{u}_{FP}, \lambda_H)\, \phi \qquad \text{with} \qquad \phi(0) = \phi(1)\,, \qquad (55)$$

where T_H is the period determined from the eigenvalues crossing the imaginary axis at the bifurcation point λ_H. The eigenvector $\phi(t)$ is not unique: its amplitude and its phase are arbitrary. To identify conditions to select a unique solution among them write

$$\phi(t) = \frac{1}{2} e^{i2\pi t}\, (\mathbf{w}_c - i\mathbf{w}_s) + c.c. = \cos(2\pi t)\mathbf{w}_c + \sin(2\pi t)\mathbf{w}_s\,,$$

which yields the algebraic condition

$$2\pi i\, (\mathbf{w}_c - i\mathbf{w}_s) = T_H\, \mathbf{F}_{\mathbf{u}}(\mathbf{u}_{FP}, \lambda_0)\; (\mathbf{w}_c - i\mathbf{w}_s)\,,$$

i.e. separated in real and imaginary parts one gets

$$\begin{pmatrix} -2\pi & T_H\, \mathbf{F}_{\mathbf{u}}(\mathbf{u}_{FP}, \lambda_0) \\ T_H \mathbf{F}_{\mathbf{u}}(\mathbf{u}_{FP}, \lambda_0) & 2\pi \end{pmatrix} \begin{pmatrix} \mathbf{w}_s \\ \mathbf{w}_c \end{pmatrix} = 0\,. \qquad (56)$$

The vectors $\mathbf{w}_{c,s}$ are not unique since their magnitude is arbitrary and any phase-shifted $\phi - \!\!\rightarrow e^{i\theta}\phi$ is also an eigenvector. To make the vectors unique impose therefore a normalization condition,

$$\mathbf{c}^t \mathbf{w}_c = 1\,,$$

with some, essentially arbitrary, vector $\mathbf{c}$. To fix the phase require

$$\mathbf{c}^t \mathbf{w}_s = 0\,.$$

To see that this condition can be satisfied by a suitable phase shift θ start with $\phi = \frac{1}{2} e^{i2\pi t}\, (\mathbf{w}_c - i\mathbf{w}_s) + c.c$ satisfying (56). Then $\hat{\phi} = re^{i\theta}\, \frac{1}{2} e^{i2\pi t}\, (\mathbf{w}_c - i\mathbf{w}_s) + c.c$ also satisfies (56) because of linearity. We can write $\hat{\phi}$ as

$$\hat{\phi} = \frac{1}{2} re^{i2\pi t}\, (\mathbf{w}_c(\theta) - i\mathbf{w}_s(\theta)) + c.c$$

with

$$\mathbf{w}_c(\theta) = \cos(\theta)\mathbf{w}_c + \sin(\theta)\mathbf{w}_s\,, \qquad \mathbf{w}_s(\theta) = -\sin(\theta)\mathbf{w}_c + \cos(\theta)\mathbf{w}_s\,.$$

The normalization and phase conditions require

$$r\mathbf{c}^t\left(\cos(\theta)\mathbf{w}_c + \sin(\theta)\mathbf{w}_s\right) = 1 \qquad r\mathbf{c}^t\left(-\sin(\theta)\mathbf{w}_c + \cos(\theta)\mathbf{w}_s\right) = 0$$

$$r\begin{pmatrix} \mathbf{c}^t\mathbf{w}_c & \mathbf{c}^t\mathbf{w}_s \\ \mathbf{c}^t\mathbf{w}_s & -\mathbf{c}^t\mathbf{w}_c \end{pmatrix}\begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

which always has a solution if

$$\left(\mathbf{c}^t\mathbf{w}_c\right)^2 + \left(\mathbf{c}^t\mathbf{w}_s\right)^2 \neq 0\,,$$

which is the case for any generic choice of $\mathbf{c}$.

Thus, the Hopf bifurcation is obtained by solving the equations

$$\mathbf{H}(\mathbf{y}) = \begin{pmatrix} \mathbf{F}(\mathbf{u},\lambda) \\ -2\pi\mathbf{w}_s + T\mathbf{F}_\mathbf{u}(\mathbf{u},\lambda)\mathbf{w}_c \\ T\mathbf{F}_\mathbf{u}(\mathbf{u},\lambda)\mathbf{w}_s + 2\pi\mathbf{w}_c \\ \mathbf{c}^t\mathbf{w}_c - 1 \\ \mathbf{c}^t\mathbf{w}_s \end{pmatrix} = 0 \qquad \text{with} \qquad \mathbf{y} = \begin{pmatrix} \mathbf{u} \\ \mathbf{w_c} \\ \mathbf{w}_s \\ T \\ \lambda \end{pmatrix}\,,$$

using Newton iteration, yielding λ_H, T_H, and $\phi(t)$.

This solution $(\lambda_H, T_H, \phi(t))$ then serves as the initial guess for Newton's method to get the initial solution on the Hopf branch,

$$\mathbf{u}^{(0)} = \mathbf{u}_{FP} + \Delta s\,\phi(t), \qquad T^{(0)} = T_H \qquad \lambda^{(0)} = \lambda_H\,.$$

The phase condition on the first iteration is

$$\int_0^1 \mathbf{u}^{(1)}(t)\frac{d}{dt}\phi(t)\,dt = 0$$

and the pseudo-arclength equation is given by

$$\int_0^1 \left(\mathbf{u}^{(1)}(t) - \mathbf{u}^{(0)}(t)\right)^t \phi(t)\,dt = \Delta s\,.$$

In the latter the terms involving $\dot\lambda$ and $\dot T$ drop out since the ϕ used in this step satisfies (55), which corresponds to (54) for $\dot\lambda = 0 = \dot T$.

Notes:

- There are a number of packages available for numerical continuation of fixed points, periodic orbits, as well as for continuation of bifurcations. All of them are freely available.
  - `AUTO` by E. Doedel
    `http://indy.cs.concordia.ca/auto/`

- XPP by B. Ermentrout, stand-alone, also a general dynamical systems toolbox, continuation based on AUTO
  http://www.math.pitt.edu/~bard/xpp/xpp.html
- MATCONT by Y. Kuznetsov, written in MATLAB
  https://sourceforge.net/projects/matcont/
  There are detailed manuals and examples on Kuznetsov's web site. A few of those are also on Canvas.

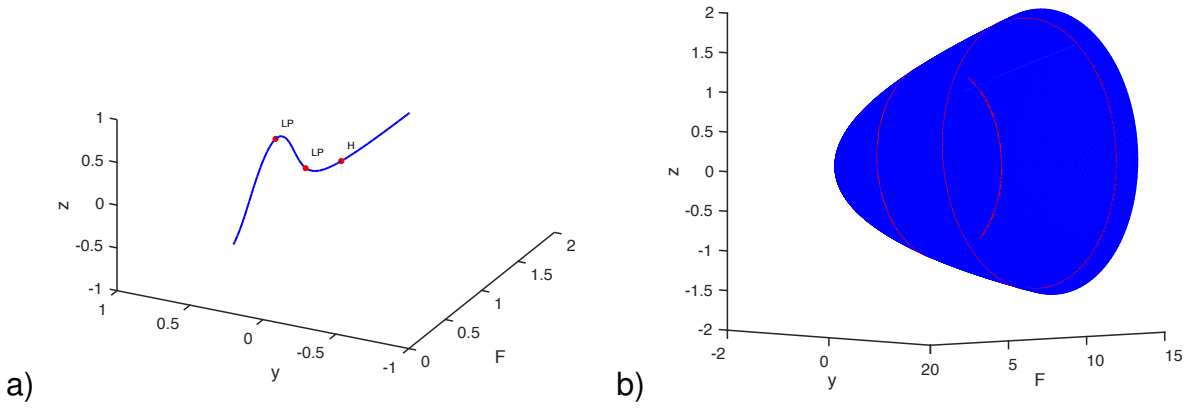


Figure 16: Lorenz84 model analyzed with MATCONT. a) Continuation of fixed point encountering two limit points and one Hopf bifurcation. b) Continuation of the periodic orbit emerging from the Hopf bifurcation. Red lines indicate period-doubling bifurcation. (following tutorial by H. Meijer).
Equations: $x' = -y*y - z*z - a*x + a*F; y' = x*y - b*x*z - y + G; z' = b*x*y + x*z - z$

8 Use of Symmetries: Forced Oscillators

The evolution equation for the oscillation amplitude that is obtained by multi-timing,

$$\frac{dA}{dT} = \mu A + c|A|^2 A\,,$$

has a symmetry that is not directly expected from the full equations from which the amplitude equation has been derived: it is equivariant under the transformation

$$A \to Ae^{i\phi} \quad \text{for any } \phi\,. \tag{57}$$

Where does this symmetry come from?

The phase-shift symmetry (57) is related to the time-translation symmetry of the original system. The original system is invariant under shifts in the origin of time,

$$t \to t + \Delta t\,.$$

In the expansion used in the multi-timing approach the evolution on the fast time is periodic. Therefore shifts in the fast time can be represented by shifts in the phase of the amplitude A of the critical mode,

$$x(t+\Delta t) = \epsilon A e^{i\omega(t+\Delta t)} + c.c. + h.o.t. = \epsilon A e^{i\omega\Delta t} e^{i\omega t} + c.c. + h.o.t.\,.$$

Thus, a shift Δt in the fast time is equivalent to multiplying the amplitude by $e^{i\omega\Delta t}$.
$\Rightarrow$ invariance under time translations implies equivariance under phase shift

$$t \to t + \Delta t \quad \Leftrightarrow \quad A \to A e^{i\omega\Delta t} . \tag{58}$$

Because of this symmetry no quadratic nonlinearities are allowed in the evolution equation for the amplitude $\Rightarrow$ the choice of the scaling of in the weakly nonlinear analysis is determined by this symmetry.

In the following we will use this time-translation symmetry argument to derive equations for the weakly nonlinear analysis of forced oscillators.

8.1 Resonant Forcing

Resonances in forced oscillations are important in many areas

- Dangerous resonances: stability of structures
  - Tacoma Narrows bridge collapse
  - Millenium Bridge swaying, cf. paper by D.M. Abrams.
  - Tae Bo class doing their exercises to this music made skyscraper sway

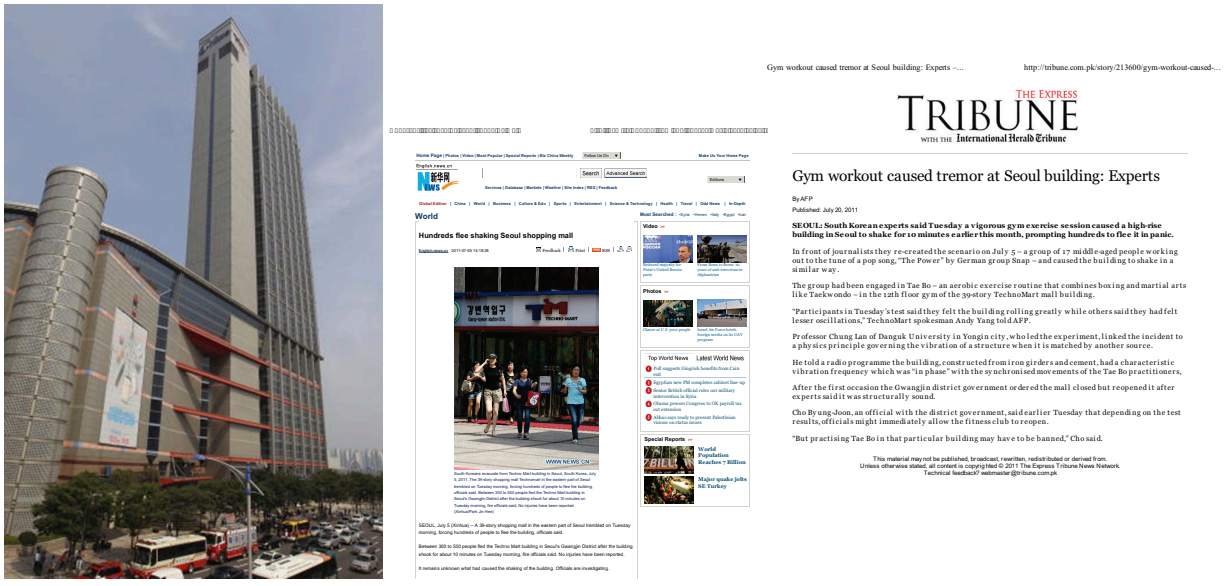


Figure 17: Seoul skyscraper resonance.

- Useful resonances: amplification of signals, e.g.

  - electronic circuits
  - double amplification in the ear: two staged oscillators.
    Otoacoustic emissions
    * evoked by sound indicate processing in the inner ear (used as non-invasive test for hearing defects)
    * spontaneous emissions.

Understanding how an individual externally forced oscillator can phase-lock to the forcing and essentially synchronize with the forcing provides also insight into resonantly coupled multiple oscillators, e.g.,

- Laser arrays: to enhance the laser power, the waves emitted by coupled lasers have to be in phase with each other.
- Heart cells: efficient pumping requires the near-synchronous activity of all muscle cells. During fibrillations the cells contract incoherently and do not achieve much pumping.
- Neurons
  - Synchronous spikes of different neurons have a stronger impact on down-stream neurons.
  - Too much synchrony may amount to epileptic seizures or Parkinsonian tremor.

Types of forcing

- non-parametric forcing: the forcing introduces an additional term in the equation
  Simple example: Pushing on a swing

$$ml\ddot{\theta} + mg\sin\theta = F(t) \qquad \Rightarrow \ddot{u} + \frac{g}{l}\sin\theta = \frac{1}{m}F(t)$$

- parametric forcing: a parameter of the system is modified in time
  Simple example: Pumping on a swing

$$ml(t)\ddot{\theta} + mg\sin\theta = 0 \qquad \Rightarrow \ddot{u} + \frac{g}{l(t)}\sin\theta = 0$$

  with l being the distance of the center of mass to the pivot.
  By shifting his/her center of mass the person changes the effective length l of the pendulum

Useful asymptotic expansions can be obtained for weak forcing near and away from resonances. The expansions and results depend on the type of resonance and the type of forcing, which often reflect the symmetries of the overall system.

This system provides a good example to illustrate

- the important role symmetries can play in the reduction of complex systems
- how important qualitative features of a complex nonlinear system can be extracted by expanding around special (singular) points, i.e.considering distinguished limits.

8.2 Symmetries, Selection Rule, and Scaling

In all generality, our weakly nonlinear analysis involves essentially a Taylor expansion. It will therefore lead to an equation of the form

$$\frac{dA}{dT} = F(A, A^*) \tag{59}$$

with $F(A, A^*)$ being a general polynomial in its arguments. But apparently the solvability condition does not lead to the most general polynomial: certain terms do not appear. Why not?

Very often, terms in an equation do not appear because of special *symmetries*. For instance, the Taylor expansion of $\sin x$ does not include any even terms because $\sin x$ is odd in x. The situation here is analogous: in the weakly nonlinear regime all oscillators have a specific symmetry that eliminates certain terms in the amplitude expansion.

We are considering systems that - in the absence of forcing - are equivariant under arbitrary time translations,

$$t \to t + \Delta t\,,$$

i.e. if $\mathbf{y}(t)$ is a solution of the original equations so is $\mathbf{y}(t + \Delta t)$.
This equivariance must be reflected in the resulting amplitude equation (59). However, A does not depend on the fast time t. How do the translations in the fast time t show up in the equation for A?

Consider a solution $y(t)$ and the time-shifted solution $y(t + \Delta t)$ and their expansions in terms of the complex amplitude A,

$$\begin{aligned} \mathbf{y}(t) &= A(T)\, e^{i\omega t}\mathbf{v} + c.c. + h.o.t. \\ \mathbf{y}(t+\Delta t) &= A(T) e^{i\omega(t+\Delta t)}\mathbf{v} + c.c. + h.o.t. \\ &= \left(A(T) e^{i\omega\Delta t}\right)\, e^{i\omega t}\mathbf{v} + c.c. + h.o.t. \end{aligned}$$

where $\mathbf{v}$ is the linear eigenvector associated with the critical model. The expansion implies:

- If $\mathbf{y}(t)$ is a solution of the original equation then $A(T)$ is a solution of the amplitude equation and vice versa.
- The time-shifted function $\mathbf{y}(t + \Delta t)$ is represented in terms of the complex amplitude by $A(T)e^{i\omega\Delta t}$, i.e. shifting the fast time by Δt is equivalent to rotating the complex amplitude A by $e^{i\omega\Delta t}$.
- If $\mathbf{y}(t + \Delta t)$ is a solution of the original equation then $A(T)e^{i\omega\Delta t}$ is a solution of the amplitude equation and vice versa.

Using the fact that with $y(t)$ also $y(t+\Delta t)$ is a solution and that the amplitude equation itself (i.e. its coefficients) does not depend on t or Δt, one obtains the following commutative diagram

$y(t)$ solves the original equation	$\Leftrightarrow$	$A(T)$ solves the amplitude equation
$\Updownarrow$		$\Updownarrow$
$y(t+\Delta t)$ solves the original equation	$\Leftrightarrow$	$A(T)e^{i\omega\Delta t}$ solves the amplitude equation

One can think of it also this way: using the ansatz for $\mathbf{y}(t+\Delta t)$ in the expansion generates exactly the same expressions everywhere as they are obtained with $\mathbf{y}(t)$, except with $A(T)$ everywhere replaced by $A(T)e^{i\omega\Delta t}$, and therefore $A(T)e^{i\omega\Delta t}$ has to satisfy the same equation as does $A(T)$.

Notes:

- The implication that $y(t)$ solves the original equation, iff $A(T)$ solves the amplitude equation holds only in the limit of small amplitudes.
- This symmetry argument makes use of the multi-timing assumption that the slow time T and the fast time t are independent variables: the time translation is only applied to the fast time, where the dynamics are periodic, but not to the slow time where the dynamics may not be periodic and where shifts in time would not necessarily be equivalent to phase shifts. That is an approximation: the two time variables are not really independent and for solutions that are not strictly periodic, e.g. with a non-periodic time dependence of $A(T)$, the symmetry argument does not hold. In fact, in the center manifold reduction, which does not use multi-timing, additional terms arise in the amplitude equations. They can, however, be shifted to higher orders by *near-identity transformations* of the amplitude. The resulting *normal form* does have the phase-shift symmetry (58). The equations obtained with our symmetry arguments generate that normal form (in a non-rigorous way).

One says:

- Translations Δt in time induce an *action* on the amplitude:

  $$t \to t+\Delta t \quad \Rightarrow \quad A(T) \to A(T)e^{i\omega\Delta t}\,.$$

  In this case the action corresponds to a phase shift by an arbitrary amount $\Delta\phi = \omega\Delta t$.
- The amplitude equation must be *equivariant* under that action:
  all terms of the amplitude equation must transform the *same* way under that operation.

8.2.1 Selection rule

Since the amplitude equation arises in an expansion in terms of the complex amplitude it has the general form

$$\frac{d}{dT}A = \mathcal{F}(A, A^*) = \sum_{m,n} a_{mn} A^m A^{*n}\,. \tag{60}$$

If $A(T)$ is a solution to (60) so must be $A(T)e^{i\Delta\phi}$ for arbitrary $\Delta\phi$.

Thus

$$\frac{d}{dT}Ae^{i\Delta\phi} = \mathcal{F}(Ae^{i\Delta\phi}, A^*e^{-i\Delta\phi}) = \sum_{m,n} a_{mn}A^m A^{*n} e^{i(m-n)\Delta\phi}\,.$$

Inserting dA/dT from (60) we get

$$\sum_{m,n} a_{mn}A^m A^{*n}\, e^{i\Delta\phi} = \sum_{m,n} a_{mn}A^m A^{*n} e^{i(m-n)\Delta\phi}\,.$$

Equating like powers of A and A^* implies

$$a_{mn} = a_{mn}e^{i\Delta\phi(m-n-1)} \quad \text{for all} \quad \Delta\phi\,.$$

Thus, we get the selection rule

either $\quad m = n+1 \qquad$ or $\qquad a_{mn} = 0$

Alternatively, one can express this result also as:

The action induced by the time-translation symmetry transforms the terms in the expansion as

$$A^n A^{*m} \to A^n A^{*m} e^{i\Delta\varphi\,(n-m)}$$

- Equivariance of the amplitude equation under this action requires that for all terms in the amplitude equation the difference $n-m$ must be the same:
  thus
  $$n - m = k \quad \text{for some} \quad k \in \mathbb{N}.$$
- Since the amplitude equation has a term $\frac{d}{dT}A$ in it one has $k = 1$.
  Thus, the only terms allowed are of the form
  $$|A|^{2l}A \qquad 0 \le l \in \mathbb{N}\,.$$

8.2.2 Scaling

In the weakly nonlinear regime $\mathbf{y}$ is small. Since we do not know the proper scaling yet we do not introduce an explicit ϵ but rather assume that the amplitude $A(T)$ is small. To leading order in the amplitudes one therefore gets

$$\partial_T A = \mu A - \gamma|A|^2 A\,,$$

implying the scaling

$$\frac{d}{dT} \sim \mu \sim |A|^2\,.$$

Notes:

- The symmetry condition allows us to write down the form of the resulting amplitude equation without performing the nonlinear expansion in detail.
- Of course, to obtain the values of the coefficients one still has to do the algebra.
- But the algebra is simplified since the scaling of the various parameters can be determined before hand (no trial and error needed).

8.3 Non-resonant Forcing

Consider $\omega = \alpha\omega_0$ with α irrational.

To include this information it is useful to consider an *extended dynamical system* in which the forcing is considered an additional dynamical variable rather than an external force (somewhat similar to the extension of the dynamical system when deriving the center manifold away from the bifurcation point). Consider for simplicity and concreteness the forced Duffing oscillator,

$$\ddot{\hat{y}} + \hat{\beta}\dot{\hat{y}} + \omega_0^2\hat{y} + \alpha\hat{y}^3 - \hat{f} = 0 \tag{61}$$

$$\ddot{\hat{f}} + \omega^2\hat{f} = 0 \tag{62}$$

where we assume $\hat{\beta}$, and $\hat{y}$ are small. In addition, we assume *weak* forcing, $\hat{f}$ small. At this point it is not clear how these quantities scale with each other.

Expand now

$$\begin{aligned} \hat{y} &= A(T)e^{i\omega_0 t} + A(T)^* e^{-i\omega_0 t} + \dots \\ \hat{f} &= F(T)e^{i\omega t} + F(T)^* e^{-i\omega t} \end{aligned}$$

Note:

- the expansion for $\hat{f}$ does not have any higher order terms since its evolution (62) is linear and is not coupled to y.
- since the forcing is now part of the dynamical system this extended dynamical system is invariant under *any time translations*.

The expansion will lead to solvability conditions of the type

$$\begin{aligned} \partial_T A &= \mathcal{F}_{\mathcal{A}}(A, A^*, F, F^*) = \sum_{klmn} a_{klmn} A^k A^{*l} F^m F^{*n} \\ \partial_T F &= \mathcal{F}_F(F, F^*) = \sum_m f_{mn} F^m F^{*n} \end{aligned}$$

Note:

- The equation for F is actually linear.

Time translations act on the amplitude $A(T)$ and $F(T)$ as

$$t \to t + \Delta t \quad \Rightarrow \quad A \to Ae^{i\omega_0 \Delta t} \quad F \to Fe^{i\omega \Delta t}$$

Selection Rule

Considering the fact that $\omega = \alpha\omega_0$ with α irrational, the selection rule

$$k - l + \alpha\,(m - n) = 1$$

implies actually two equations

$$m = n \qquad \text{and} \qquad k = l + 1\,.$$

Since $m = n$ the forcing appears only in the form of $|F|^2$ and the lowest-order term is given by

$$|F|^2 A$$

Thus, the phase of the forcing does not play a role reflecting that there is no resonance between the oscillator and the forcing.

Balance saturation and forcing

$$A^3 \sim F^2 A \quad \Rightarrow \quad F \sim A$$

resulting in the amplitude equation

$$A' = \left(\mu + |F|^2\right) A - \gamma |A|^2 A \tag{63}$$

Notes:

- Non-resonant forcing does not introduce new terms in the equation of the unforced oscillator (at any order), it only modifies its coefficients. None of the terms are phase-sensitive.
  - in principle, all coefficients depend on $|F|^2$
  - the strongest effect of the forcing is on the bifurcation parameter μ because it is small
  - only for the linear term is the shift of the coefficient of the same order as the coefficient itself and therefore relevant at leading order

8.4 1:1 Forcing

Now consider forcing near the 1:1-resonance, i.e. there is a term in the basic equation that is periodic with time $2\pi/\omega$, with $|\omega - \omega_0| \ll 1$, where ω_0 is the Hopf frequency.

When $0 \neq |\omega - \omega_0| \ll 1$ is fixed, the ratio of the two frequencies is either irrational or 'almost irrational'. The analysis of the irrational case shows that this will not lead to any new terms. Therefore consider the exact resonance $\omega = \omega_0$.

With 1:1-resonance the system is not invariant for arbitrary time shifts any more, but still for shifts

$$t \to t + \frac{2\pi}{\omega}\,.$$

With the expansion

$$y(t,T) = A(T)e^{i\omega t} + A(T)^* e^{-i\omega t} + \mathcal{O}(\epsilon)$$

time translations induce the action

$$t \to t + \frac{2\pi}{\omega} \quad \Rightarrow \quad A \to Ae^{i\omega\frac{2\pi}{\omega}} = A$$

i.e. the amplitude is *unchanged* by such translations. One says, in this case the action of the symmetry is *trivial*.

Thus

- with 1:1-forcing *any* polynomial in A and A^* is allowed by symmetries

$$\partial_T A = a_{00} + a_{10}A + a_{01}A^* + a_{20}A^2 + \dots\,. \tag{64}$$

Scaling

We also assume *weak* forcing. The amplitude of the forcing has not entered our symmetry consideration leading to (64). Use again an extended dynamical system like (61,62) and expand again as

$$\begin{aligned} y &= A(T)e^{i\omega_0 t} + A(T)^* e^{-i\omega_0 t} + \dots \\ f &= F(T)e^{i\omega_0 t} + F(T)^* e^{-i\omega_0 t}\,. \end{aligned}$$

Note:

- Since the forcing is now part of the dynamical system this extended dynamical system is invariant under *any time translations*.

Again, the expansion will lead to solvability conditions of the type

$$\partial_T A = \sum_{klmn} a_{klmn} A^k A^{*l} F^m F^{*n} \qquad \partial_T F = \sum_{klmn} f_{klmn} A^k A^{*l} F^m F^{*n}\,.$$

Time translations act on the amplitude $A(T)$ and $F(T)$ as

$$t \to t + \Delta t \quad \Rightarrow \quad A \to Ae^{i\omega_0 \Delta t} \quad F \to Fe^{i\omega_0 \Delta t}\,,$$

i.e. for the 1:1-resonance, $\omega = \omega_0$, A and F transform the same way under time translations. The selection rule is therefore

$$k - l + m - n = 1\,.$$

In terms of powers in A and A^* this condition does not impose any restrictions, consistent with (64).

What are the lowest-order terms?

- It makes sense, to aim to retain a nonlinear term that is saturating even *without* forcing. This is, however, not strictly required. Generically, the leading-order saturating[7] term without forcing is $|A|^2A$.
- The leading-order forcing term is F, i.e. $k = l = n = 0$.
- To balance these two essential terms we have

$$F \sim A^3$$

$\Rightarrow$ to leading order the only term containing the forcing is F.

To leading order we then get

$$\partial_T A = a_{1000}A + a_{0010}F + a_{2100}|A|^2A\,. \tag{65}$$

Summary:

- The oscillation amplitude A and the forcing amplitude F are each associated with the common resonance frequency ω_0.
- In terms of Fourier modes all terms in the resulting amplitude equation have to correspond to the same frequency, which for the A-equation is ω_0.
- No attention has to be paid to the equation for the forcing amplitude since the equation (62) for the forcing is not coupled to the oscillation amplitude.

How can we capture a small detuning $\Delta\omega$ between ω and ω_0? So far we took F to be constant. However, it could also evolve on the slow time scale T,

$$F = \mathcal{F}e^{i\Omega T}\,,$$

which would correspond to a slightly different forcing frequency,

$$f(t) = \mathcal{F}e^{i(\omega_0 t+\Omega T)} + c.c.\,.$$

This modifies the amplitude equation (65),

$$\partial_T A = a_{1000}A + a_{0010}\mathcal{F}e^{i\Omega T} + a_{2100}|A|^2A\,. \tag{66}$$

Due to the form of the cubic nonlinearity the time-dependent coefficient can be absorbed into the amplitude A by writing

$$A = \mathcal{A}e^{i\Omega T}\,,$$

yielding

$$\partial_T \mathcal{A} = (a_{1000} - i\Omega)\,\mathcal{A} + a_{0010}\mathcal{F} + a_{2100}|\mathcal{A}|^2\mathcal{A}\,.$$

Note:

[7] Whether this term is actually saturating will depend on the sign of its coefficient.

- The imaginary part of the linear coefficient gives the *detuning* between the Hopf frequency and the forcing frequency.
- To capture near-resonant behavior in a weakly nonlinear approach one has to *expand* around the *exact* resonance, i.e. one has to consider the distinguished limit of $\omega - \omega_0 \propto A^2$.
- If the detuning is large compared to A^2, one obtains (63) and the forcing is essentially non-resonant in a weakly nonlinear asymptotic approach.
- Since the term $\mathcal{F}$ breaks the symmetry $A \to Ae^{i\varphi}$, the forcing will lead to *qualitatively new* phenomena.

8.5 3:1 Forcing

Consider now $\omega = 3\omega_0$ and use the expansion

$$\begin{aligned} y &= A(T)e^{i\omega_0 t} + A(T)^* e^{-i\omega_0 t} + \dots \\ \hat{f} &= F(T)e^{3i\omega_0 t} + F(T)^* e^{-3i\omega_0 t} \end{aligned}$$

It induces the action

$$t \to t + \Delta t \quad \Rightarrow \quad A \to Ae^{i\varphi} \quad F \to Fe^{3i\varphi} \quad \text{with } \varphi = \omega_0 \Delta t$$

$$A^k A^{*l} F^m F^{*n} \to A^k A^{*l} F^m F^{*n} e^{i\varphi(k-l+3(m-n))}$$

Selection Rule

for the equation for A

$$k - l + 3\,(m - n) = 1 \quad \Rightarrow \quad k - l = 1 - 3\,(m - n)$$

Identify the lowest-order terms in the forcing

F:

$$m - n = 1 \qquad \Rightarrow \qquad k - l = -2 \qquad k = 0 \quad l = 2 \quad \Rightarrow \quad FA^{*2}$$

F^*:

$$m - n = -1 \qquad \Rightarrow \qquad k - l = 4 \qquad k = 4 \quad l = 0 \quad \Rightarrow \quad F^* A^4$$

For any small A one has $FA^{*2} \gg F^* A^4$. Therefore use FA^{*2} to balance saturation and forcing

$$A^3 \sim FA^2 \quad \Rightarrow \quad F \sim A$$

Keeping only terms up to cubic order we get therefore the restriction

$$k + l + m + n \leq 3$$

We have already considered the case $m + n = 1$.

Consider now $m+n=2$:

$$\begin{aligned} m=2 \quad n=0 \qquad &\Rightarrow \qquad k-l=-5 \quad \Rightarrow \quad F^2A^{*5} \\ m=1 \quad n=1 \qquad &\Rightarrow \qquad k-l=1 \quad \Rightarrow \quad |F|^2A \\ m=0 \quad n=2 \qquad &\Rightarrow \qquad k-l=7 \quad \Rightarrow \quad F^{*2}A^7 \end{aligned}$$

For $m+n=3$ we get the condition

$$k=0=l \qquad \Rightarrow \qquad 3(m-n)=1 \qquad \text{cannot be satisfied}$$

To leading order symmetry and scaling show that the equation has to have the form

$$\partial_T A = \left(\mu+\beta|F|^2\right)A - \gamma|A|^2A + \delta F A^{*2} \tag{67}$$

Notes:

- Through the term $\beta|F|^2$ the forcing modifies the linear coefficient of the equation
  - depending on the sign of β_r the forcing can enhance or reduce the damping
  - through $\beta_i|F|^2$ the frequency of small-amplitude oscillations are modified by the forcing
- The forcing will lead to *qualitatively new* phenomena only through the new term involving A^{*2}, because only it is phase-sensitive and breaks the continuous phase symmetry $A \to Ae^{i\varphi}$ (there is a residual discrete phase-shift symmetry $\varphi \neq \frac{2\pi}{3}$).
- For consistent scaling we need $\mu=\mathcal{O}(A^2)$ and $F=\mathcal{O}(A)$.
- (67) captures the weakly nonlinear behavior of all generic, weakly forced oscillators near the 1:3 resonance. Different oscillators only differ in the values of the coefficients. If a specific system has additional symmetries, coefficients in (67) can vanish although they are allowed by the time translation symmetry.
- In the resonant cases - as those discussed before - all the coefficients depend also on $|F|^2$, but these effects are of higher order unless the corresponding coefficient vanishes (or is small) in the absence of the forcing.

Higher Resonances:

- Resonant forcing with higher resonances ($m:1$ with $m \geq 4$) does not lead to additional terms in the lowest order amplitude equation
  - but it introduces new *higher-order terms* that are *phase-sensitive*
  - to capture aspects of its impact on the system in a leading-order amplitude equation one may have to consider singular limits like $|\gamma| \ll 1$ to make the higher-order phase-sensitive terms of the same order as the formally lower-order nonlinear terms, i.e. consider higher singular points.

8.6 A Quadratic Oscillator with 3:1 Forcing

Consider the forced oscillator

$$\partial_{\tilde{t}}^2 y + \hat{\beta}\partial_{\tilde{t}} y + \omega_0^2 y + \alpha y^2 = \hat{f}(\tilde{t}) \tag{68}$$

with $\hat{f} \sim \cos\omega\tilde{t}$ where ω is close to $3\omega_0$.

We want to reduce this equation to an amplitude equation using multiple time scales.

For weak forcing and weak damping we expect for the oscillation amplitude A on symmetry grounds the equation

$$\partial_T A = \left(\mu + \beta|F|^2\right) A - \gamma|A|^2 A + \delta F A^{*2}\,.$$

All 5 terms will arise at the same order if the following scaling is satisfied

$$\partial_T = \mathcal{O}(\mu) = \mathcal{O}(A^2) \qquad F = \mathcal{O}(A)\,.$$

Notes:

- The oscillator equation has only a quadratic nonlinearity. How will the cubic nonlinearities be generated that the symmetry arguments predict?
- The forcing is non-parametric, i.e. it is given by an inhomogeneous term rather than a time-dependent coefficient. Why is there then no inhomogeneous term in the amplitude equation?

Introduce a small parameter ϵ explicitly via

$$y = \epsilon y_1 + \epsilon^2 y_2 + \qquad \ldots$$

and the rescaled variables

$$T = \epsilon^2 t, \qquad \hat{\beta} = \epsilon^2\beta, \qquad \hat{f} = \epsilon\mathcal{F}, \qquad \omega = 3\left(\omega_0 + \epsilon^2\Omega\right)$$

The amplitude equation is then expected to arise at $\mathcal{O}(\epsilon^3)$.

We get then

$$\partial_{\tilde{t}} = \partial_t + \epsilon^2\partial_T \qquad \partial_{\tilde{t}}^2 = \partial_t^2 + \epsilon^2 2\partial_t\partial_T + \mathcal{O}(\epsilon^4)\,.$$

$\mathcal{O}(\epsilon)$:

$$\partial_t^2 y_1 + \omega_0^2 y_1 = \mathcal{F}\cos\left(3\left(\omega_0 + \epsilon^2\Omega\right)t\right) = \frac{1}{2}\mathcal{F}\left\{e^{3i\omega_0 t + 3i\Omega T} + e^{-3i\omega_0 t - 3i\Omega T}\right\}$$

Thus, the structure of the leading-order equation is

$$\mathcal{L}y_1 \equiv \left(\partial_t^2 + \omega_0^2\right) y_1 = I_1,$$

i.e. the equation is inhomogeneous.

The linear operator $\mathcal{L}$ is singular: $\mathcal{L}e^{\pm i\omega_o t} = 0$ but $\mathcal{L}e^{ni\omega_o t} \neq 0$ for $n \neq 1, -1$.

The general solution at this order is therefore given by

$$y_1 = Ae^{i\omega_0 t} + Be^{3i\omega_0 t} + A^* e^{-i\omega_0 t} + B^* e^{-3i\omega_0 t}\,,$$

with A yet undetermined and

$$-8\omega_0^2 B = \frac{1}{2}\mathcal{F}e^{3i\Omega T} \qquad B = -\frac{1}{16\omega_0^2}\mathcal{F}e^{3i\Omega T}\,.$$

$\mathcal{O}(\epsilon^2)$:

$$\partial_t^2 y_2 + \omega_0^2 y_2 = -\alpha y_1^2 = -\alpha\left\{Ae^{i\omega_0 t} + Be^{3i\omega_0 t} + A^* e^{-i\omega_0 t} + B^* e^{-3i\omega_0 t}\right\}^2$$

The r.h.s. has no terms proportional $e^{i\omega_0 t}$ or $e^{-i\omega_0 t}$ $\Rightarrow$ no secular terms arise and we can solve for y_2 without any solvability condition arising.

$$y_2 = C + De^{2i\omega_0 t} + Ee^{4i\omega_0 t} + Fe^{6i\omega_0 t} + D^* e^{-2i\omega_0 t} + E^* e^{-4i\omega_0 t} + F^* e^{-6i\omega_0 t}$$

with

$$F = \frac{1}{35}\frac{\alpha B^2}{\omega_0^2} \qquad E = \frac{2}{15}\frac{\alpha AB}{\omega_0^2}$$

$$D = \frac{1}{3}\frac{\alpha}{\omega_0^2}\left\{A^2 + 2A^* B\right\} \qquad C = -\frac{2\alpha}{\omega_0^2}\left\{|A|^2 + |B|^2\right\}\,.$$

$\mathcal{O}(\epsilon^3)$:

$$\partial_t^2 y_3 + \omega_0^2 y_3 = -2\alpha y_1 y_2 - \beta\partial_t y_1 - 2\partial_t\partial_T y_1$$

Secular terms arise from the term $y_1 y_2$,

$$y_1 y_2 \sim \ldots AC + \ldots BD^* + \ldots A^* D + \ldots B^* E$$

Collecting these terms with Maple or Mathematica one gets

$$\partial_t^2 y_3 + \omega_0^2 y_3 = e^{i\omega_0 t}\left\{-\partial_T A - \frac{1}{2}\beta A - i\frac{6}{5}\frac{\alpha^2}{\omega_0^3}|B|^2 A - i\frac{5}{3}\frac{\alpha^2}{\omega_0^3}|A|^2 A + i\frac{\alpha^2}{\omega_0^3}BA^{*2}\right\} + \text{non-secular terms}\,.$$

Inserting B we get an equation of the form

$$\partial_T A = \left(\mu + \mu_2|\mathcal{F}|^2\right)A - \gamma|A|^2 A + \delta\mathcal{F}A^{*2}e^{3i\Omega T}\,.$$

Eliminate the time-dependence of the coefficent of the forcing via the rotation

$$A = \hat{A}e^{i\Omega T}\,,$$

which yields

$$\partial_T \mathcal{A} = \left(\mu + \mu_2|\mathcal{F}|^2\right)\mathcal{A} - \gamma|\mathcal{A}|^2\mathcal{A} + \delta\mathcal{F}\mathcal{A}^{*2} \tag{69}$$

with the coefficients

$$\mu = -\frac{1}{2}\beta - i\Omega \qquad \mu_2 = -\frac{3}{640}i\frac{\alpha^2}{\omega_0^7} \qquad \gamma = -\frac{5}{3}i\frac{\alpha^2}{\omega_0^3} \qquad \delta = -\frac{1}{16}i\frac{\alpha^2}{\omega_0^5}\,.$$

Notes:

- The nonlinearities of the amplitude equation are *not* directly determined by the nonlinearities of the underlying differential equation from which it is derived. Higher-order nonlinearities can always be generated by cycling through the lower-order nonlinearities.
- The determining factor for the form of the amplitude equation is the *action on the amplitude that is induced by the symmetries* of the underlying equation.
- The form of the nonlinearities of the underlying equation may determine aspects of its symmetries. For instance, if the underlying equation is odd in y it induces the *additional* action $A \to -A$ under which the amplitude equation must be equivariant as well. For (69) this would imply that $\delta = 0$.
- Symmetries can make coefficients zero but not non-zero.
- Among the coefficients, only μ is small, $\mu = \mathcal{O}(A^2) = \mathcal{O}(F^2)$.

Analysis of the Amplitude Equation

To get steady-state solutions it is better to write (69) in terms of magnitude and phase

$$\mathcal{A} = R(T)e^{i\phi(T)}e^{i\theta}$$

$$\partial_T R + iR\partial_T\phi = \left(\mu + \mu_2|F|^2\right) R - \gamma|R|^2R + \delta\, FR^2e^{-3i\phi}e^{-3i\theta}\,.$$

In general, δ and $\mathcal{F}$ are complex. However, one can absorb the argument of $\delta = \bar{\delta}e^{i\psi_\delta}$ and the phase of $\mathcal{F} = \bar{\mathcal{F}}e^{i\psi_F}$ by a constant phase shift $\theta = \frac{1}{3}(\psi_\delta + \psi_F)$. Thus, without loss of generality one can assume $\mathcal{F}$ and δ to be real and positive. In the following write F instead of $\mathcal{F}$, for ease of writing.

Introduce $m = \mu + \mu_2\,|F|^2 \equiv m_r + im_i$ etc.

$$\partial_T R = m_rR - \gamma_rR^3 + |\delta|FR^2\cos 3\phi \tag{70}$$

$$R\partial_T\phi = m_iR - \gamma_iR^3 + |\delta|FR^2\sin 3\phi \tag{71}$$

Analyze the fixed points (critical points) of (70,71).

There is always a fixed point

$$R_\infty^{(1)} = 0$$

i.e.

$$y = -\frac{f}{8\omega_0^2}\cos\left(3\left(\omega_0 + \epsilon^2\Omega\right)t\right) + h.o.t.$$

Its linear stability is determined by

$$\partial_T\mathcal{A} = \left(\mu + \mu_2|F|^2\right)\mathcal{A}$$

it is linearly stable for

$$\mu_r + \mu_{2r}\,|F|^2 < 0$$

with $\mu_{2r} = 0$ and $\mu_r = -\frac{1}{2}\beta$ this is always the case for the oscillator (68). Thus, if there is a nonlinear solution it does not arise via a bifurcation off this basic state.

For $R_\infty \neq 0$ one gets

$$|\delta|^2 F^2 R^2 = \left(m_r - \gamma_r R^2\right)^2 + \left(m_i - \gamma_i R^2\right)^2$$

i.e.

$$|\gamma|^2 R^4 - \left\{2\left(m_r\gamma_r + m_i\gamma_i\right) + |\delta|^2 F^2\right\} R^2 + |m|^2 = 0 \tag{72}$$

$$R^2 = \frac{\{\} \pm \sqrt{\Delta}}{2\left|\gamma\right|^2}\,.$$

We need positive solutions for R^2. Consider the discriminant Δ,

$$\Delta = \left\{2\left(m_r\gamma_r + m_i\gamma_i\right) + |\delta|^2 F^2\right\}^2 - 4|\gamma|^2|m|^2\,.$$

Because $|\gamma|^2|m|^2 > 0$, both solutions have either the same sign or they are complex.

Note:

- In particular, $R = 0$ cannot be a solution of this equation. This is consistent with the fact that the solution $R_\infty^{(1)} = 0$ is linearly stable for all F and therefore does not undergo a (local) bifurcation.

We need $R^2 > 0$. Consider $F^2 \to \infty$,

$$\{\} \to |\delta|^2 F^2 > 0\,.$$

Since the sign of the solution of the biquadratic equation (72) does not change, $R^2 > 0$ for all F for which R^2 is real (i.e. R^2 becomes complex before $\{\}$ becomes negative).

To get any steady-state solutions we need the discriminant to be non-negative

$$\begin{aligned}\Delta &= \left\{2\left(m_r\gamma_r + m_i\gamma_i\right) + |\delta|^2 F^2\right\}^2 - 4|\gamma|^2|m|^2 \\ &= |\delta|^4 F^4 + 4|\delta|^2 F^2\left(m_r\gamma_r + m_i\gamma_i\right) - 4\left(m_r\gamma_i - m_i\gamma_r\right)^2 \overbrace{\geq}^{!} 0\,.\end{aligned}$$

In our case: $\gamma_r = 0$ and $\mu_{2r} = 0$

$$\begin{aligned}\Delta &= |\delta|^4 F^4 + 4|\delta|^2 F^2 m_i\gamma_i - 4m_r^2\gamma_i^2 \\ &= \left(|\delta|^4 + 4|\delta|^2\mu_{2i}\gamma_i\right) F^4 + 4|\delta|^2 F^2\mu_i\gamma_i - 4\mu_r^2\gamma_i^2\,.\end{aligned}$$

In addition, $\gamma_i < 0$, $\mu_{2i} < 0$. Therefore

$$\begin{aligned}\Delta &< 0 \qquad \text{for } F^2 \to 0 \\ \Delta &> 0 \qquad \text{for } F^2 \to \infty\,.\end{aligned}$$

Thus,

$$\begin{aligned}F &< F_c \qquad \text{no solution} \\ F &> F_c \qquad \text{2 solutions}\end{aligned}$$

with F_c defined via

$$\Delta(F_c) = 0.$$

At $F = F_c > 0$ one has a saddle-node bifurcation. Solving $\Delta = 0$ for $\mu_i = -\Omega$ the line of bifurcations is given by

$$\Omega_{SN} = -\mu_i = \frac{|\delta|^2 + 4\mu_{2i}\gamma_i}{4\gamma_i} F^2 - \frac{\mu_r^2 \gamma_i}{|\delta|^2} \frac{1}{F^2}$$

with two solutions appearing for $\Omega > \Omega_{SN}$.

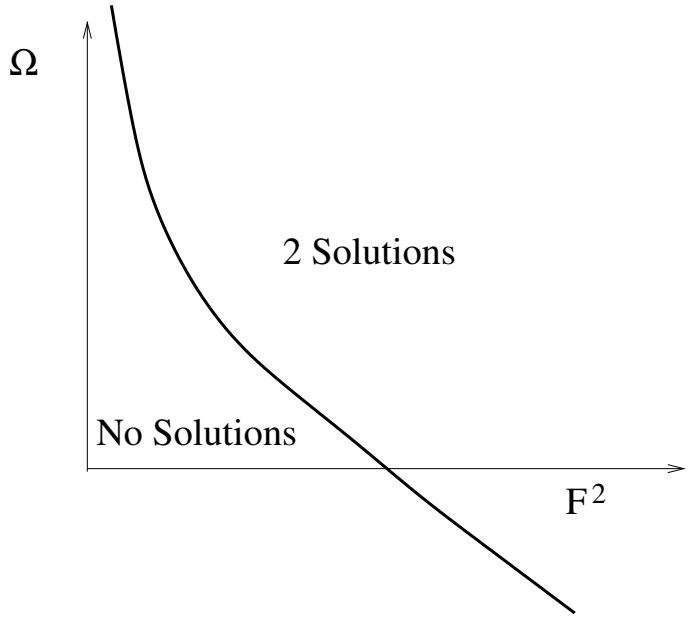


Figure 18: Phase diagram for quadratic oscillator with $\gamma_r = 0 = \mu_{2r}$ and $\gamma_i < 0$ and $\mu_{2i} < 0$. The line denotes a line of saddle-node bifurcations. Note: for all parameter values there is the additional solution $R_\infty^{(1)} = 0$.

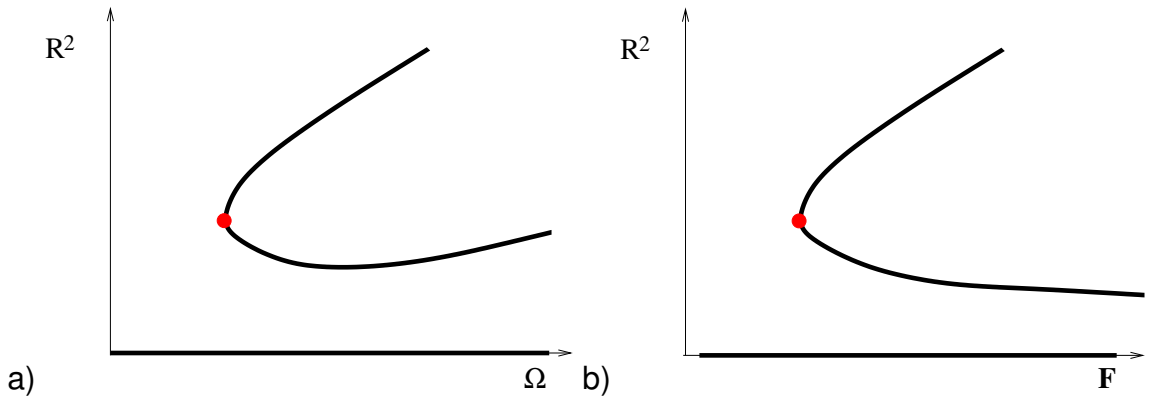


Figure 19: a) Bifurcation diagram for quadratic oscillator with $\gamma_r = 0 = \mu_{2r}$ obtained by cutting the phase diagram along a line of constant F. The circle denotes the saddle-node bifurcations. Note: for all parameter values there is the additional solution $R_\infty^{(1)} = 0$. b) Bifurcation diagram when varying F at fixed Ω.

Notes:

- For the 3:1-resonance the phase-locked forced solution does not arise in a bifurcation off the basic solution $y = -\frac{f}{8\omega_0^2}\cos\left(3\left(\omega_0 + \epsilon^2\Omega\right)t\right) + h.o.t.$; instead it arises at finite amplitude through a saddle-node bifurcation.
- The saddle-node bifurcation is systematically captured by this amplitude equation, since

  $$\Omega_{SN} = \mathcal{O}(F^2) = \mathcal{O}(\frac{\mu_r^2}{F^2}) = \mathcal{O}(\mu_r) \qquad R_{SN}^2 = \frac{\{\}}{2\left|\gamma\right|^2} = \mathcal{O}(\mu_r, F^2)$$

  without any assumption on the $\mathcal{O}(1)$-parameters of the original equation (68). I.e., as $\mu \to 0$ and $F \to 0$ the amplitude at the saddle-node bifurcation goes to 0.

9 Higher-Dimensional Center Manifolds: Mode Interaction

The dimension of the center manifold is given by the number of eigenvalues that cross the imaginary axis (or go through 0) *simultaneously* when the control parameter is changed. The behavior on the center manifold becomes then more complex, resulting from the interaction of these different modes.

Such a situation is often the result of symmetries, which may require that symmetrically related modes destabilize the basic state for the same control parameter value. For instance, in Rayleigh-Benard convection in a square container convection rolls can equally appear along the x-axis and along the y-axis.

9.1 Center-Manifold from PDE

Consider as a model system the PDE

$$\frac{\partial}{\partial t}\psi = R\psi - \left(\frac{\partial^2}{\partial x^2} + 1\right)^2\psi + \left(\frac{\partial}{\partial x}\psi\right)^2 \tag{73}$$

with periodic boundary condition in a 1-dimensional system of length L.

This equation is similar to the Kuramoto-Sivashinsky equation

$$\frac{\partial}{\partial t}\psi = -\frac{\partial^4}{\partial x^4}\psi - \frac{\partial^2}{\partial x^2}\psi + \left(\frac{\partial}{\partial x}\psi\right)^2, \tag{74}$$

which describes diffusive instabilities of laminar flames and other long-wave instabilities. Note, that the only parameter of the Kuramoto-Sivashinsky equation is the system size L. For our purposes it is therefore more useful to work with the modification (73), which has the additional control parameter R.

Simple fixed-point solution: $\psi = 0$.

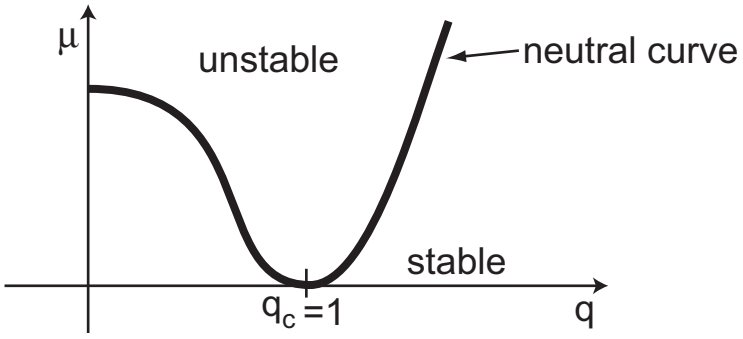
Linear stability:

Since the PDE has constant coefficients make a Fourier ansatz

$$\psi = Ae^{\sigma t}e^{iqx} \qquad \text{with} \qquad q = \frac{2\pi}{L}n\,,$$

where n is an integer. This results in the real growth rate

$$\sigma = R - \left(-q^2+1\right)^2\,.$$



Notes:

- The fixed point is unstable for $R > R_n(q) = (1-q^2)^2$.
- When increasing R the fixed point is first destabilized at $R \equiv R_c = 0$ with $q \equiv q_c = 1$.
- For $R > R_c$ the fixed point is unstable to a continuous range of modes $q_{min} \le q \le q_{max}$.
  In an infinite system the fixed point would be destabilized by infinitely many modes.
  In a finite system the destabilization is via a finite discrete set of modes.

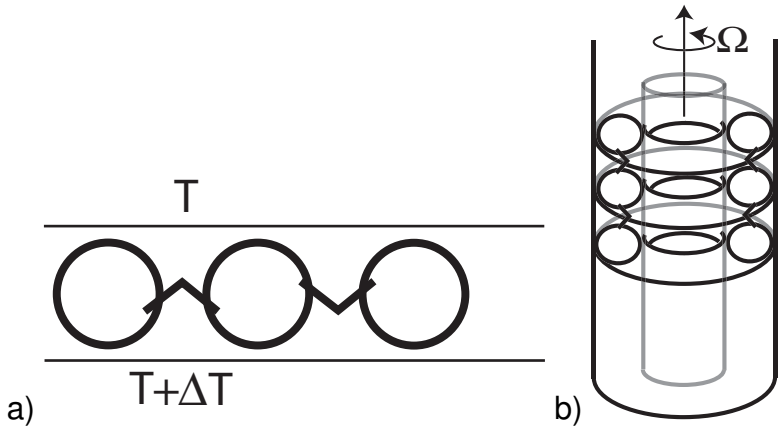


Figure 20: Examples of instabilities of a spatially homogeneous state to periodic structures. a) Rayleigh-Benard convection. b) Taylor vortex flow.

Weakly nonlinear analysis

Focusing on spatially periodic solutions we expand in Fourier modes,

$$\begin{aligned}\psi &= Ae^{iqx} + Be^{2iqx} + C + De^{3iqx} + Ee^{4iqx} + \ldots + c.c. \\ \frac{\partial}{\partial x}\psi &= iqAe^{iqx} + 2iqBe^{2iqx} + 3iqDe^{3iqx} + 4iqEe^{4iqx} + \ldots + c.c.\,,\end{aligned}$$

which leads to a coupled set of ODEs for the amplitudes

e^{iqx} :

$$\frac{d}{dt}A = \underbrace{RA - (1-q^2)^2 A}_{\mu A} - 4q^2 A^* B - 12q^2 B^* D + 0 \cdot AC + \ldots \tag{75}$$

e^{0iqx} :

$$\frac{d}{dt}C = RC - C - 2q^2|A|^2 - 8q^2|B|^2 - 18q^2|D|^2 + \ldots$$

e^{2iqx} :

$$\frac{d}{dt}B = RB - (1-4q^2)^2 B - q^2 A^2 - 6q^2 A^* D + \ldots$$

e^{3iqx} :

$$\frac{d}{dt}D = RD - (1-9q^2)^2 - 4q^2 AB + \ldots$$

Choose q and R on the neutral stability curve $R = R_n(q)$ close to the minimum:

- The Fourier mode A with wavenumber q spans the center eigenspace $E^{(c)}$.
- All other Fourier modes are damped: they are in the stable eigenspace $E^{(s)}$.

$\Rightarrow$ Perform a center-manifold reduction[8].

To include the dependence on $R = R_n(q) + \mu$ use a suspended system by including the differential equation for μ,

$$\frac{d}{dt}\mu = 0\,.$$

On the center manifold the amplitudes in the stable eigenspace are given by strictly nonlinear functions of the coordinates on the center eigenspace,

$$B = h_B(A, A^*, \mu) \qquad C = h_C(A, A^*\mu) \qquad D = h_D(A, A^*\mu)\ldots\,.$$

Thus, B, C, D etc. are at least quadratic in A, A^*, μ.

- There is no term of $\mathcal{O}(A^2)$ in (75).
- In this model the mode C does not contribute to (75).
- The contribution from D to the evolution equation for A is $\mathcal{O}(BD)$, which is at least a quartic term, and can be neglected.
- $\Rightarrow$ the lowest nonlinearity is cubic and we need only B and that only to $\mathcal{O}(A^2)$.

[8] For $q < 1$ further away from the minimum additional modes with wavenumber nq can destabilize $\psi = 0$.

Expand

$$h_B = \alpha_B A^2 + \beta_B |A|^2 + \gamma_B A^{*2} + \delta_B \mu^2 + \rho_B \mu A + \tau_B \mu A^* + \dots$$

and similarly for h_C and h_D.

Insert $h_{B,C,D}$ into the equation for $\frac{d}{dt}B$ (using $\frac{d}{dt}\mu = 0$):

$$\frac{\partial}{\partial A} h_B(A, A^*, \mu) \frac{d}{dt} A + \frac{\partial}{\partial A} h_B^*(A, A^*, \mu) \frac{d}{dt} A^* = (R_n + \mu) h_B - (1-4q^2)^2 h_B - q^2 A^2 - 6q^2 A^* h_D + \dots$$

Consider the terms at $\mathcal{O}(A^2)$:
All the terms on the left-hand side are at least of cubic order. To $\mathcal{O}(A^2)$ the only inhomogeneous term (in terms of h_B) is the term $-q^2A^2$ on the right-hand side: we need the term $\alpha_B A^2$ in h_B to balance it and we can set $\beta_B = 0$ and $\gamma_B = 0$,

$$0 = R_n \alpha_B - (1-4q^2)^2 \alpha_B - q^2$$

$$\alpha_B = \frac{1}{(1-q^2)^2 - (1-4q^2)^2} \qquad B = \frac{1}{(1-q^2)^2 - (1-4q^2)^2} A^2 \,.$$

For $q = 1$, which corresponds to the minimum of the neutral curve, one gets

$$\alpha_B = -\frac{1}{9} \qquad B = -\frac{1}{9} A^2 + \dots$$

Inserting B into (75) the evolution equation on the center manifold is given by

$$\frac{d}{dt} A = \mu A - \frac{4}{9} |A|^2 A + \dots \tag{76}$$

Notes:

- The equation has the same form as the normal form for a Hopf bifurcation $\Rightarrow$ the magnitude R of A decouples from the phase ϕ,

  $$A = R(t) e^{i\phi} \qquad \frac{d}{dt} R = \mu R - \frac{4}{9} R^3 \,.$$

  The pattern arises in a supercritical pitch-fork bifurcation.

- The phase ϕ is arbitrary: (76) is equivariant under

  $$A \to A e^{i\phi} \qquad \text{for arbitrary } \phi \,.$$

  The origin of this phase-shift symmetry is the spatial translation symmetry $x \to x + \Delta x$ of (73),

  $$\begin{aligned} \psi(x + \Delta x, t) &= A e^{iq(x+\Delta x)} + B e^{2iq(x+\Delta x)} + C + D e^{3iq(x+\Delta x)} + \dots \\ &= A e^{iq\Delta x} e^{iqx} + B e^{2iq\Delta x} e^{2iqx} + C + \dots \end{aligned}$$

  i.e. translations in x are equivalent to phase shifts in A, they induce the same action on the amplitudes as the time-translation symmetry does for the Hopf bifurcation.
  $\Rightarrow$ the amplitude equation (76) must be equivariant under arbitrary phase shifts.

- The phase-shift symmetry is exact for (76), whereas for the Hopf bifurcation the symmetry is only present in the normal form. The difference is associated with the use of two-timing in the Hopf bifurcation, while no multiple scales were introduced in the spatial case. With multi-timing the shifts in time were only applied on the fast time, where it lead to the phase-shift symmetry, but not on the slow time, where no specific symmetry would be induced because the dynamics in the slow time variable is allowed to be arbitrary, rather than periodic[9].

- In contrast to the equation for the Hopf bifurcation, the coefficients of (76) are real. This is not specific to the PDE (73). Rather, it is a consequence of the reflection symmetry $x \to -x$ of the original PDE (73), which induces complex conjugation as the action on A,

  $$x \to -x \qquad A \to A^* \,.$$

  $\Rightarrow$ (76) has to be equivariant under complex conjugation, implying the coefficients are real.

- (76) can also be derived using multiple scales.

9.2 Interaction of Stripes of Different Orientations: Stripes vs Squares

Consider the two-dimensional version of an extended KS-equation,

$$\frac{\partial}{\partial t}\psi = R\psi - (\Delta + 1)^2 \psi + a\,(\nabla\psi)^2 - \psi^3 \,,$$

with cubic nonlinearity and periodic boundary conditions.

Linear stability of basic state $\psi = 0$:

$$\psi = e^{\sigma t} e^{i\mathbf{q}\cdot\mathbf{r}} \qquad \sigma = R - \left(-\mathbf{q}^2 + 1\right)^2$$

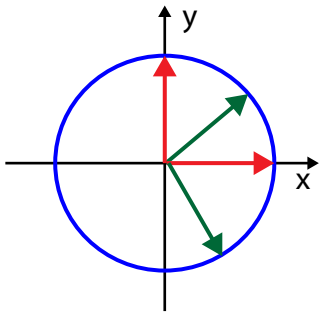


Figure 21: Isotropic neutral curve in 2 dimensions.

The growth rate does not depend on the orientation of the wave vector $\mathbf{q}$: the system is *isotropic.*
Infinitely many modes with different orientation destabilize the basic state simultaneously.

Consider a square system of size $L \times L$. The allowed wave vectors are then given by $\mathbf{q} = \frac{2\pi}{L}(m, n)$ with m, n integer.

For simplicity consider the case in which $q \equiv \frac{2\pi}{L}$ is close to the neutral curve, i.e. the growth rate σ is small. In that case we expect that an amplitude expansion of the form

$$\psi = A_1 e^{iqx} + A_2 e^{iqy} + B_{20} e^{2iqx} + B_{00} + B_{02} e^{2iqy} + B_{22} e^{2iqx+2iqy} + \ldots c.c.$$

[9] If the spatial dependence is not exactly periodic spatial translations do not induce an exact phase-shift symmetry. Thus, in situations in which slow spatial variations arise (see below), the phase-shift symmetry is also only approximate.

will be sufficient. The center eigenspace is given by $E^{(c)} = (A_1, A_2) \Rightarrow$ we expect two coupled equations for the evolution of A_1 and A_2 on the center manifold,

$$\frac{d}{dt}A_1 = f_1(A_1, A_1^*, A_2, A_2^*) \tag{77}$$

$$\frac{d}{dt}A_2 = f_2(A_1, A_1^*, A_2, A_2^*)\,. \tag{78}$$

Can we obtain the form of the equation without a detailed center-manifold reduction or multiple-scale analysis? Employ the symmetries of the system.

i) **Translation symmetries:**

$$\psi(x + \Delta x, y + \Delta y) = A_1 e^{iq(x+\Delta x)} + A_2 e^{iq(y+\Delta y)} + \ldots$$

The translation symmetry invokes the action

$$A_1 \to A_1 e^{iq\Delta x} \qquad A_2 \to A_2 e^{iq\Delta y}\,.$$

Thus: the resulting amplitude equations must be equivariant under *independent arbitrary phase shifts,*

$$A_1 \to A_1 e^{i\phi_1} \qquad A_2 \to A_2 e^{i\phi_2}\,.$$

Consider the transformation of (77) under the two phase shifts:

ϕ_1 :

$$e^{i\phi_1} \underbrace{\frac{d}{dt}A_1}_{f_1(A_1, A_1^*, A_2, A_2^*)} = f_1(A_1 e^{i\phi_1}, A_1^* e^{-i\phi_1}, A_2, A_2^*)$$

$$e^{i\phi_1} f_1(A_1, A_1^*, A_2, A_2^*) = f_1(A_1 e^{i\phi_1}, A_1^* e^{-i\phi_1}, A_2, A_2^*) \tag{79}$$

ϕ_2 :

$$\frac{d}{dt}A_1 = f_1(A_1, A_1^*, A_2 e^{i\phi_2}, A_2^* e^{-i\phi_2})$$

$$f_1(A_1, A_1^*, A_2, A_2^*) = f_1(A_1, A_1^*, A_2 e^{i\phi_2}, A_2^* e^{-i\phi_2})\,. \tag{80}$$

The amplitude equation is obtained in a Taylor expansion in $A_{1,2}$ and $A_{1,2}^*$,

$$f_1(A_1, A_1^*, A_2, A_2^*) = \sum \alpha^{(1)}_{mnm'n'} A_1^m A_1^{*m'} A_2^n A_2^{*n'}$$

$$f_1(A_1 e^{i\phi_1}, A_1^* e^{-i\phi_1}, A_2 e^{i\phi_2}, A_2^* e^{-i\phi_2}) = \sum \alpha^{(1)}_{mm'nn'} e^{i\phi_1(m-m')} e^{i\phi_2(n-n')} A_1^m A_1^{*m'} A_2^n A_2^{*n'}\,.$$

To satisfy (79,80) the only terms allowed in the sum are those with

$$m - m' = 1 \qquad \text{and} \qquad n - n' = 0$$

Thus

$$f_1(A_1, A_1^*, A_2, A_2^*) = A_1 \sum_{m\geq 0, n\geq 0} \alpha^{(1)}_{mn} |A_1|^{2m} |A_2|^{2n}$$

and analogously

$$f_2(A_1, A_1^*, A_2, A_2^*) = A_2 \sum_{m\geq 0, n\geq 0} \alpha^{(2)}_{mn} |A_1|^{2m} |A_2|^{2n}\,.$$

To lowest nonlinear order in A_i we get

$$\frac{d}{dt}A_1 = \alpha_{00}^{(1)}A_1 + A_1\left(\alpha_{10}^{(1)}|A_1|^2 + \alpha_{01}^{(1)}|A_2|^2\right) + \dots \quad (81)$$

$$\frac{d}{dt}A_2 = \alpha_{00}^{(2)}A_2 + A_2\left(\alpha_{10}^{(2)}|A_1|^2 + \alpha_{01}^{(2)}|A_2|^2\right) + \dots . \quad (82)$$

ii) **Reflections about the diagonal:**

Interchanging $x \leftrightarrow y$,

$$\psi(y,x) = A_1 e^{iqy} + A_2 e^{iqx} + \dots + c.c. = A_2 e^{iqx} + A_1 e^{iqy} + \dots + c.c.\,,$$

induces the action

$$A_1 \leftrightarrow A_2\,.$$

The amplitude equations (81,82) must therefore be in addition equivariant under the exchange $A_1 \leftrightarrow A_2$.

Acting on (81) yields

$$\frac{d}{dt}A_2 = \alpha_{00}^{(1)}A_2 + A_2\left(\alpha_{10}^{(1)}|A_2|^2 + \alpha_{01}^{(1)}|A_1|^2\right) + \dots$$

which is to be compared with (82). This requires

$$\alpha_{00}^{(1)} = \alpha_{00}^{(2)} \qquad \alpha_{10}^{(1)} = \alpha_{01}^{(2)} \qquad \alpha_{01}^{(1)} = \alpha_{10}^{(2)}\,.$$

Setting $\alpha_{00}^{(1)} = \mu$ and $a_{10}^{(1)} = a$ gives

$$\frac{d}{dt}A_1 = \mu A_1 - aA_1\left(|A_1|^2 + g|A_2|^2\right) + \dots \quad (83)$$

$$\frac{d}{dt}A_2 = \mu A_2 - aA_2\left(g|A_1|^2 + |A_2|^2\right) + \dots \quad (84)$$

with

$$a = -a_{10}^{(1)} = -a_{01}^{(2)} \qquad \textbf{and} \qquad g = \frac{\alpha_{01}^{(1)}}{\alpha_{01}^{(2)}}\,.$$

Notes:

- The growth rate μ has to be equal for the stripes of different orientation.
- μ is the bifurcation parameter.
- In addition to a there is another parameter, g, which captures the strength of the competition between the modes relative to the self-saturation.
- Because of the reflection symmetry the impact of A_1 on A_2 has to be equal to that of A_2 on A_1.

Because of the translation symmetry (phase-shift symmetry) the equations can again be separated into equations for the magnitudes and the phases: $A_i = R_i e^{i\phi_1}$,

$$\begin{aligned}
\frac{d}{dt}R_1 &= \mu R_1 - aR_1\left(R_1^2 + gR_2^2\right)\\
\frac{d}{dt}R_2 &= \mu R_2 - aR_2\left(gR_1^2 + R_2^2\right)\\
\frac{d}{dt}\phi_1 &= 0\\
\frac{d}{dt}\phi_2 &= 0\,.
\end{aligned}$$

Notes:

- The equations for the phases ϕ_i *decouple* from the equations for the amplitudes because of translation symmetry.
- The phases determine the position of the patterns in the system, which is arbitrary due to the translation symmetry of the system.
- The phases represent *soft modes*:
  there are two eigenvalues $\lambda_{\phi_{1,2}} = 0 \Rightarrow$ the patterns do not relax back to the original position after a shift in their position, i.e. after a perturbation in their phases
  $\Rightarrow$ the vanishing of these two eigenvalues points to *slow dynamics* that arises if the phases are perturbed and which can allow a further reduction of the system (cf. Sec.10.1).

Analysis of the Amplitude Equations:

Consider the various kinds of fixed points. For simplicity absorb a into the amplitudes $A_{1,2}$.

1. $A_1 = 0 = A_2$: basic state, it is stable for $\mu < 0$ and unstable for $\mu > 0$.
2. $A_1 \neq 0$, $A_2 = 0$:
   The non-trivial fixed point is given by
   $$A_1 = \sqrt{\mu}e^{i\phi}$$
   and arises in a super-critical pitchfork bifurcation as in the one-dimensional case. The phase ϕ is again arbitrary.
3. $A_1 = 0$, $A_2 \neq 0$:
   $$A_2 = \sqrt{\mu}e^{i\phi}$$
   analogous to case 2.
4. $A_1 = R_1 e^{i\phi_1} \neq 0$, $A_2 = R_2 e^{i\phi_2} \neq 0$:
   The two amplitudes satisfy
   $$\begin{aligned}
   0 &= \mu - R_1^2 - gR_2^2\\
   0 &= \mu - R_2^2 - gR_1^2\,,
   \end{aligned}$$
   thus $\mu = R_1^2 + gR_2^2 = R_2^2 + gR_1^2$, i.e.
   $$(1-g)R_1^2 = (1-g)R_2^2\,.$$

(a) If $g \neq 1$:

$$R_1^2 = R_2^2 = \frac{\mu}{1+g}$$

This solution corresponds to square patterns with equal amplitudes of the stripes in the x- and y-direction.
Within the subspace defined by $R_1(t) = R_2(t) \equiv R(t)$ ('square sub-space') one has

$$\frac{d}{dt}R = \mu R - (1+g)R^3 \,.$$

The bifurcation to squares is super-critical for $g > -1$ and is subcritical for $g < -1$.

(b) If $g = 1$:

$$\mu = R_1^2 + R_2^2 \,,$$

but R_1 and R_2 *cannot be determined* individually at this order:
One needs to go to higher order to resolve this *indeterminacy.*

Stability of fixed points:

1. Basic state: clear

2. Stripes
Linearize around the fixed point $(R_{10}, 0)$,

$$\begin{aligned} R_1 &= R_{10} + \epsilon R_{11} \\ R_2 &= \epsilon R_{21} \end{aligned}$$

with $R_{10}^2 = \mu$.
$\mathcal{O}(\epsilon)$:

$$\begin{aligned} \frac{d}{dt}R_{11} &= \mu R_{11} - 3R_{10}^2 R_{11} = -2\mu R_{11} \\ \frac{d}{dt}R_{21} &= \mu R_{21} - gR_{10}^2 R_{21} = \mu\,(1-g)R_{21} \end{aligned}$$

   - R_{11} never destabilizes the stripes (since $\mu > 0$), i.e. within the subspace corresponding to stripes this solution is linearly stable.
   - R_{21} destabilizes the stripes for $g < 1$.
   - Thus: stripes are linearly unstable for $g < 1$. This instability brings in the perpendicular mode and leads out of the stripe subspace.

3. Squares:
Linearize around (R_{10}, R_{20}),

$$\begin{aligned} R_1 &= R_{10} + \epsilon R_{11} \\ R_2 &= R_{20} + \epsilon R_{21} \,. \end{aligned}$$

$\mathcal{O}(\epsilon)$:

$$\begin{aligned}
\frac{d}{dt}R_{11} &= \mu R_{11} - 3R_{10}^2 R_{11} - gR_{20}^2 R_{11} - 2gR_{10}R_{20}R_{21} \\
&= \left(\mu - 3R_{10}^2 - gR_{20}^2\right) R_{11} - 2gR_{10}R_{20}R_{21} \\
\frac{d}{dt}R_{21} &= \mu R_{21} - 3gR_{20}^2 R_{21} - gR_{10}^2 R_{21} - 2gR_{10}R_{20}R_{11} \\
&= -2gR_{10}R_{20}R_{11} + \left(\underbrace{\mu - 3R_{20}^2 - gR_{10}^2}_{\mu\frac{1+g-3-g}{1+g}} \right) R_{21}
\end{aligned}$$

using $R_{10}^2 = R_{20}^2 = \frac{\mu}{1+g}$

$$\begin{pmatrix} \frac{d}{dt}R_{11} \\ \frac{d}{dt}R_{21} \end{pmatrix} = -\mu\frac{2}{1+g}\begin{pmatrix} 1 & g \\ g & 1 \end{pmatrix}\begin{pmatrix} R_{11} \\ R_{21} \end{pmatrix}.$$

The eigenvalues are given by

$$(1-\lambda)^2 - g^2 = 0 \qquad \lambda_{1,2} = 1 \pm g$$

with coresponding eigenvectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The growth rates are then given by

$$\begin{aligned}
\sigma_1 &= -2\mu < 0 \qquad \text{for } \mu > 0 \\
\sigma_2 &= -2\mu\,\frac{1-g}{1+g}\,.
\end{aligned}$$

(a) $1 < g : \sigma_1 < 0, \sigma_2 > 0$
The mode $\mathbf{v}_2$ destabilizes the squares and makes the amplitudes R_1 and R_2 different. Suspect an instability towards stripes.

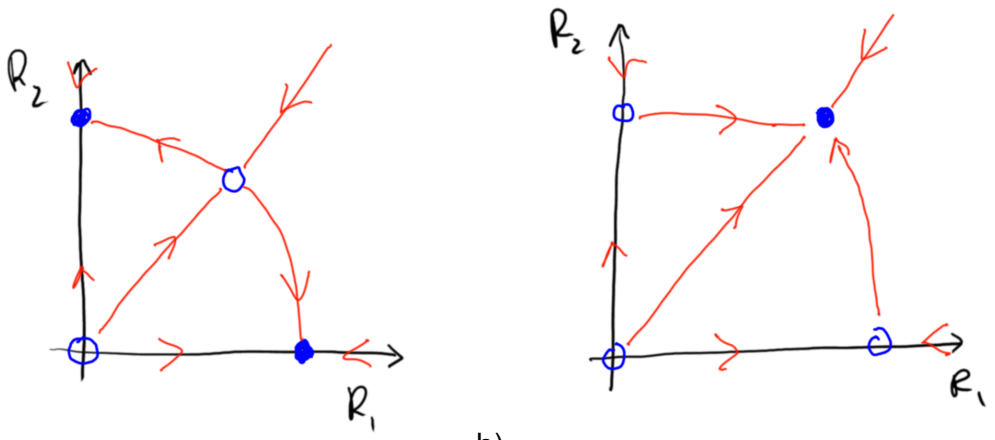
(b) $-1 < g < 1 : \sigma_1 < 0, \sigma_2 < 0$
Squares are linearly stable.

(c) $g < -1 : \sigma_1 > 0, \sigma_2 < 0$
$\sigma_1 > 0$ since for $g < -1$ these squares exist only for $\mu < 0$ (subcritical).
Squares are linearly unstable with respect to mode $\mathbf{v}_1$: the instability suggests a transition to large-amplitude squares. **But:** the stability of is determined by *higher-order terms* and the weakly nonlinear analysis may not be sufficient, unless g is close to -1.

Thus:

- For $g \neq 1$ *either squares or stripes* are linearly stable.

- For $|g-1| \ll 1$ we need to keep higher-order terms to determine stabilities: one finds mixed-modes in which $0 \neq R_1 \neq R_2 \neq 0$.



a) b)

Figure 22: a) $1 < g$. b) $-1 < g < +1$.

Symmetries play a powerful role:

- Symmetries restrict the form of the terms that can arise in the amplitude equations: they allow to predict the *form of the equations* based solely on
  - the results of the linearization (critical eigenvalue and eigenvectors) and
  - the *action of the symmetries* on the destabilizing modes.
- Symmetries simplify the analysis of the amplitude equations
  - e.g., squares have reflection symmetry $R_1 \leftrightarrow R_2$, one can show that therefore the eigenvectors of the linearization around the squares have to be either *even* or *odd* with respect to that reflection,

$$\mathbf{v}_e = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \mathbf{v}_o = \begin{pmatrix} 1 \\ -1 \end{pmatrix} .$$

    The diagonalization of the matrix is therefore much easier: symmetries reduce the order of the matrix that needs to be diagonalized, here from 2×2 to 1×1. This can be a great advantage for higher-dimensional center manifolds.
- Symmetries allow to identify which coefficients of the equations are relevant to answer a certain question: one can restrict the evalution to only those coefficients.

9.3 Interaction of Stripes of Different Orientations: Stripes vs Hexagons

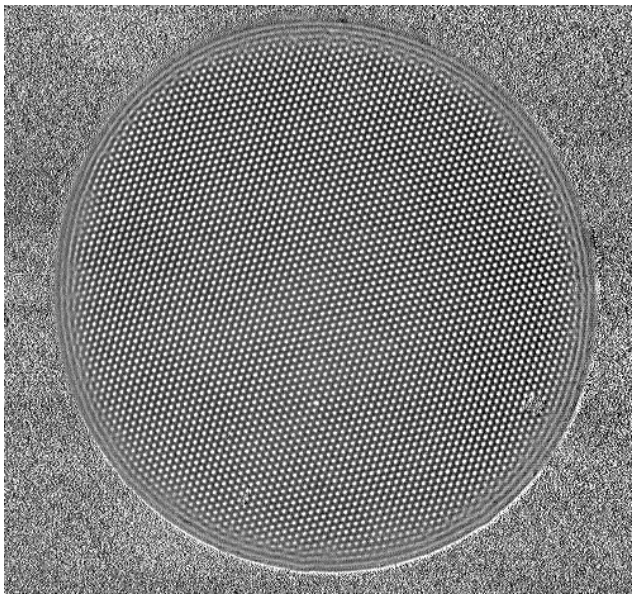


Figure 23: Hexagon patterns in non-Boussinesq Rayleigh-Benard convection. Top view. (Bodenschatz et al., 1991)

In many physical systems the patterns that arise immediately above the onset of the instability are hexagonal, not stripes or squares. Consider therefore two-dimensional systems in which three Fourier modes destabilize the unpatterned state simultaneously in a steady bifurcation,

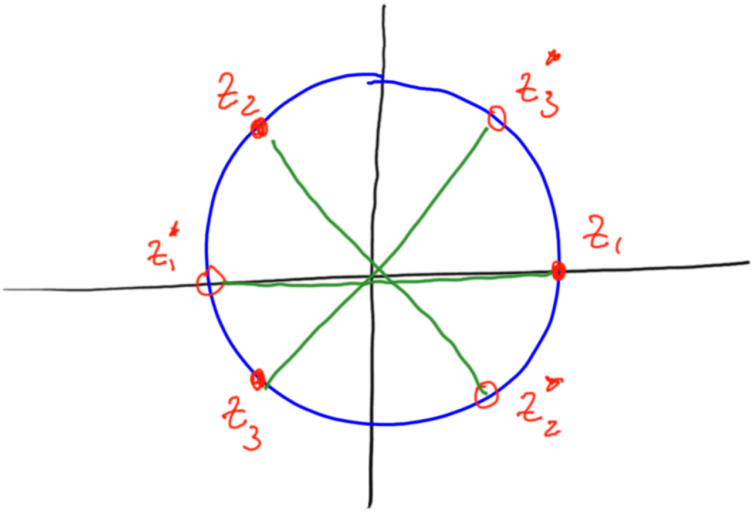
$$\psi = z_1(t)\, e^{i\mathbf{k}_1\cdot\mathbf{r}} + z_2(t)\, e^{i\mathbf{k}_2\cdot\mathbf{r}} + z_3(t)\, e^{i\mathbf{k}_3\cdot\mathbf{r}} + c.c + h.o.t.$$

with wave vectors

$$\mathbf{k}_1 = (1,0) \qquad \mathbf{k}_2 = (-\frac{1}{2}, \frac{\sqrt{3}}{2}) \qquad \mathbf{k}_3 = (-\frac{1}{2}, -\frac{\sqrt{3}}{2})$$

and growth rates

$$\sigma_1 = \sigma_2 = \sigma_3 = 0 \qquad \text{at the bifurcation point } \mu = 0.$$



Since now three amplitude equations will arise from the solvability conditions the analysis can become more complicated. It is a good example to introduce concepts that are useful for more complicated systems.

We expect equations of the form

$$\dot{z}_i = F_i(z_1, z_2, z_3, z_1^*, z_2^*, z_3^*) \qquad i = 1, 2, 3\,,$$

where three functions F_i on the right-hand side constitute a mapping from $\mathbb{C}^3$ to $\mathbb{C}^3$. Consider systems that exhibit a number of symmetries

- translation symmetry in x- and y-direction
- rotation symmetry (isotropy)
- reflection symmetries in the plane $x \to -x$ and $y \to -y$

The amplitude equations therefore need to be equivariant with respect to *all* these symmetries of the system.

The symmetry operations γ of any object form a **group** Γ, i.e.

- The composition of two symmetry operations is again a symmetry operation

$$\gamma_{1,2} \in \Gamma \quad \Rightarrow \quad \gamma_1 \circ \gamma_2 \in \Gamma\,.$$

- There is a symmetry operation that leaves the object unchanged

$$1 \in \Gamma\,.$$

- For each symmetry operation γ there is an inverse γ^{-1}, i.e. an operation that brings the object back into the state before γ was applied

$$\gamma^{-1} \circ \gamma = 1\,.$$

The invariance/equivariance of the equations needs to hold for *all* elements $\gamma \in \Gamma$. However, typically the elements of the group can be obtained from a smaller number of generators. In that case it is sufficient to show the invariance/equivariance for the generators.

Consider the example of the symmetries of a hexagon,

- six rotations $\mathcal{R}$ by 60°
- 3 reflections κ_1 about the axes defined by $\mathbf{k}_i$
- 3 reflections κ_2 about the axes rotated by 30° relative to the $\mathbf{k}_i$.

They form a dihedral group denoted by $\mathbf{D}_6$. One can show that all these operations can be generated by the 4 generators

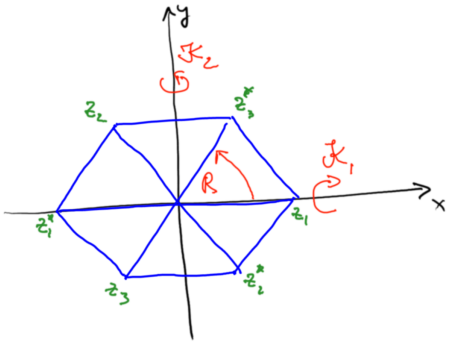


Figure 24: Symmetries of a hexagon.

- the permutations of (z_1, z_2, z_3):

$$\mathcal{P}_{12}(z_1, z_2, z_3) = (z_2, z_1, z_3) \quad (85)$$
$$\mathcal{P}_{23}(z_1, z_2, z_3) = (z_1, z_3, z_2) \quad (86)$$
$$\mathcal{P}_{13}(z_1, z_2, z_3) = (z_3, z_2, z_1) \quad (87)$$

- the point inversion $\mathcal{I}$:

$$(z_1, z_2, z_3) \to (z_1^*, z_2^*, z_3^*) \quad (88)$$

For instance, a rotation by 60° anti-clockwise is given by $P_{13} \circ \mathcal{I}$

We still need to identify the action of $\mathbf{T}^2$ on the amplitudes z_i

$$\mathbf{T}_{\Delta x}(z_1, z_2, z_3) = (e^{ik\Delta x} z_1, e^{-\frac{1}{2}ik\Delta x} z_2, e^{-\frac{1}{2}ik\Delta x} z_3) \quad (89)$$
$$\mathbf{T}_{\Delta y}(z_1, z_2, z_3) = (z_1, e^{\frac{\sqrt{3}}{2}ik\Delta y} z_2, e^{-\frac{\sqrt{3}}{2}ik\Delta y} z_3) \quad (90)$$

In the following we consider the representation (85,86,87,88,89,90) of $\mathbf{T}^2 \dot{+} \mathbf{D}_6$ on $\mathbb{C}^3$.

In our previous analysis we found that each term in the amplitude could be thought of as consisting of an equivariant term multiplied by an invariant function, e.g. in

$$\partial_T A = \left(\mu + \beta|F|^2\right) A - \gamma|A|^2 A + \delta F A^{*2}$$

the term $(\mu + \beta|F|^2)\, A$ represents a mapping $\mathbb{C} \to \mathbb{C}$ that is equivariant under the phase shift $A \to Ae^{i\phi}$. It is comprised of the basic equivariant term A and a function $(\mu + \beta|F|^2)$ of the phase-invariant quantity $|F|^2$. It is therefore in general useful to identify ways to generate all invariant functions and all equivariant functions.

For our problem at hand the symmetries are given by translations in the plane $\mathbf{T}^2$ combined with $\mathbf{D}_6$.

Terms invariant under $\mathbf{T}^2$ are $u_i \equiv |z_i|^2$ and $z_1 z_2 z_3$. Basic invariants under $\mathbf{T}^2 \dot{+} \mathbf{D}_6$ are then

$$\sigma_1 = u_1 + u_2 + u_3 \qquad \sigma_2 = u_1 u_2 + u_1 u_3 + u_2 u_3 \qquad \sigma_3 = u_1 u_2 u_3$$
$$q = z_1 z_2 z_3 + z_1^* z_2^* z_3^*$$

Note:

- $u_i = \mathcal{O}(z^2)$, $\sigma_i = \mathcal{O}(z^{2i})$, $q = \mathcal{O}(z^3)$.

Basic equivariant terms are

$$(z_1, z_2, z_3) \qquad (z_2^* z_3^*, z_1^* z_3^*, z_1^* z_2^*)$$

For example operating with $\mathcal{P}_{13}$ on each z_i yields

$$\mathcal{P}_{13}(z_2^* z_3^*, z_1^* z_3^*, z_1^* z_2^*) = (z_2^* z_1^*, z_1^* z_3^*, z_3^* z_2^*)\,,$$

which is equivalent to interchanging the first and third element in $(z_2^* z_3^*, z_1^* z_3^*, z_1^* z_2^*)$.

One can then show (Buzano and Golubitsky, 1983):

All mappings $g : \mathbb{C}^3 \to \mathbb{C}^3$ that are equivariant under this representation of $\mathbf{T}^2 \dot{+} \mathbf{D}_6$ have the form

$$g = l_1 \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} + l_2 \begin{pmatrix} u_1 z_1 \\ u_2 z_2 \\ u_3 z_3 \end{pmatrix} + l_3 \begin{pmatrix} u_1^2 z_1 \\ u_2^2 z_2 \\ u_3^2 z_3 \end{pmatrix} + m_1 \begin{pmatrix} z_2^* z_3^* \\ z_1^* z_3^* \\ z_1^* z_2^* \end{pmatrix} + m_2 \begin{pmatrix} u_1 z_2^* z_3^* \\ u_2 z_1^* z_3^* \\ u_3 z_1^* z_2^* \end{pmatrix} + m_3 \begin{pmatrix} u_1^2 z_2^* z_3^* \\ u_2^2 z_1^* z_3^* \\ u_3^2 z_1^* z_2^* \end{pmatrix}$$

where $l_i = l_i(\sigma_i, q)$ and $m_i = m_i(\sigma_i, q)$ are $\mathbf{T}^2 \dot{+} \mathbf{D}_6-$invariant functions.

Giving the terms up to cubic order explicitly one gets

$$\dot{z}_1 = \left(l_{10} + l_{11} \underbrace{\sigma_1}_{u_1+u_2+u_3} \right) z_1 + l_{20} u_1 z_1 + m_{10} z_2^* z_3^* + \mathcal{O}(z_i^4)$$

and the corresponding cyclic permutations. Written explicitly, we get

$$\begin{aligned} \dot{z}_1 &= \mu z_1 + \alpha x_2^* z_3^* + \beta z_1 \left(|z_1|^2 + \gamma \left(|z_2|^2 + |z_3|^2 \right) \right) + \mathcal{O}(z_i^4) \\ \dot{z}_2 &= \mu z_2 + \alpha x_3^* z_1^* + \beta z_2 \left(|z_2|^2 + \gamma \left(|z_3|^2 + |z_1|^2 \right) \right) + \mathcal{O}(z_i^4) \\ \dot{z}_3 &= \mu z_3 + \alpha x_1^* z_2^* + \beta z_3 \left(|z_3|^2 + \gamma \left(|z_2|^2 + |z_1|^2 \right) \right) + \mathcal{O}(z_i^4) \end{aligned}$$

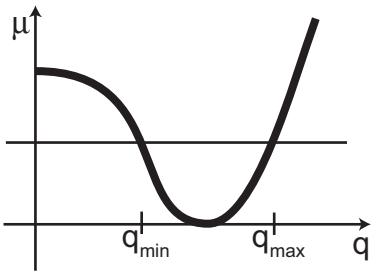
The analysis of all fixed points of these equations and their stability can be quite involved. Group theory can be very useful in identifying what kind of fixed points are actually possible and what the form of the linearization around those fixed points is (Buzano and Golubitsky, 1983; Golubitsky et al., 1988).

For the cubic terms in the amplitude equations to be kept consistently along with the quadratic terms the coefficient α has to be small. In that case

- Hexagons arise subcritically via a transcritical bifurcation
- Stripes appear in pitchfork bifurcation, which is supercritical for $\beta < 0$. They are unstable to the hexagons.
- Depending on γ, hexagons become unstable to stripes for larger amplitudes or they remain stable.
- The instabilities of the hexagons and the stripes is mediated by a branch of a steady solution in which all modes z_i have different magnitudes.

10 Steady Spatial Patterns: Real Ginzburg-Landau Equation

So far we have only considered patterns that are strictly periodic, with $q_{\min} < q < q_{\max}$ (cf. Swift-Hohenberg equation (73)). Effectively, we have considered systems of size $L = \frac{2\pi}{q}$, which contain only a single wavelength.



What about large systems that include many wavelengths? What about patterns that have different wavelengths in different parts of the system? What is their dynamics? How can we describe them?

Close to the minimum of the neutral curve the range of wavenumbers that destabilize the homogeneous state is small: $q = q_c + \epsilon\, Q$, all wavenumbers correspond to *small deviations* from critical wavenumber q_c, We expanded ψ as

$$\begin{aligned} \psi(x,t) &= \epsilon A(T) e^{iqx} + \mathcal{O}(\epsilon^2) \qquad \text{with } q_{\min} < q < q_{\max} \\ &= \epsilon \underbrace{A(T) e^{i(q-q_c)x}}_{A(X,T)} e^{iq_c x} + \mathcal{O}(\epsilon^2) \end{aligned}$$

$\Rightarrow$ we can introduce a slow space variable $X = \epsilon x$ and incorporate the small changes in the wavelength in the slow spatial variation of A.

Note:

- If the slow spatial variation of A is not periodic, one obtains patterns with wavenumbers that vary in space.

Expansion:

$$\psi = \epsilon A(X,T) e^{ix} + \epsilon^2 B(X,T) e^{2ix} + \epsilon^2 C(X,T) + \epsilon^3 D(X,T) e^{3ix} + \ldots + c.c. \qquad (91)$$

with $T = \epsilon^2 t$ and $\mu = \epsilon^2 \mu_2$

Note:

- This amounts to a true multi-scaling in space: the usual spatial variable x is replaced by two spatial variables

$$\psi(x) \to \psi(\xi, X) \qquad \partial_x \to \partial_\xi + \partial_x X \, \partial_X \, .$$

How to pick the scaling for the slow spatial variable?

Consider a reflection-symmetric systems like (73) or (74) and use symmetry arguments:

- As before the symmetry $x \leftrightarrow -x$ implies that the coefficients of the amplitude equation are real.
- In addition, it implies the equivariance of the resulting amplitude equation under $X \leftrightarrow -X$.
- The first term involving $\frac{\partial}{\partial X}$ and A that is allowed is therefore $\partial_X^2 A$.
- Balance $\frac{\partial^2 A}{\partial X^2} \sim |A|^2 A \Rightarrow \partial_X = \mathcal{O}(\epsilon)$. Thus, we have

$$\partial_x \to \partial_x + \epsilon \partial_X \, .$$

  Since the fast spatial variable ξ corresponds to the original spatial variable x it is customary not to introduce a new symbol for it.

We therefore expect an equation of the form

$$\partial_T A = \delta \partial_X^2 A + \mu A + \gamma |A|^2 A \, .$$

Apply the expansion (91) to the Swift-Hohenberg equation,

$$\frac{\partial}{\partial t}\psi = R\psi - \left(\frac{\partial^2}{\partial x^2} + 1\right)^2 \psi - \psi^3 \, , \tag{92}$$

using with $R = \epsilon^2 \mu_2$. For the derivatives we have then

$$\frac{\partial^2}{\partial x^2} \to \frac{\partial^2}{\partial x^2} + 2\epsilon \frac{\partial^2}{\partial x \partial X} + \epsilon^2 \frac{\partial^2}{\partial X^2}$$

$$\frac{\partial^4}{\partial x^4} \to \frac{\partial^4}{\partial x^4} + 4\epsilon \frac{\partial^4}{\partial x^3 \partial X} + 6\epsilon^2 \frac{\partial^4}{\partial x^2 \partial X^2}$$

i) $\mathcal{O}(\epsilon)$:

$$0 \;\; = \;\; 0 \, .$$

Formally we have $L_0 = -(\partial_x^2 + 1)^2$. This operator is singular since $L_0 e^{ix} = 0$
$\Rightarrow$we expect solvability conditions at higher orders.

ii) $\mathcal{O}(\epsilon^2)$:

$$0 = -(4(-i)\partial_X A + 2 \cdot 2i \partial_X A)$$

is already satisfied.

Note:

- One can check that this condition is automatically satisfied when one expands around the **minimum** of the neutral curve. Expansions around other wave numbers lead to non-trivial terms at $\mathcal{O}(\epsilon^2)$.

iii) $\mathcal{O}(\epsilon^3)$:

$$\begin{aligned} e^{ix}: \qquad \partial_T A &= \mu_2 A - (6(-1)\partial_X^2 A + 2\partial_X^2 A) - 3|A|^2 A \\ e^{3ix}: \qquad 0 &= 64D - A^3 \qquad \Rightarrow \qquad D = \frac{-A^3}{64} \end{aligned}$$

Thus, the solvability condition is given by

$$\partial_T A = 4\partial_X^2 A + \mu_2 A - 3|A|^2 A\,. \tag{93}$$

Note:

- This equation is called the **real Ginzburg-Landau equation**. This refers to the fact that it is similar in spirit to the equation that was first introduced in the context of phase transitions by Vitaly Ginzburg building on the equation without spatial derivative proposed by Lev Landau. In that equation A is an order parameter like the density in a liquid-gas transition or the magnetization in ferromagnetism. It is real.
- The coefficients in (93) are real due to the reflection symmetry $x \to -x$.

Because of the form of the nonlinearity it allows simple spatially periodic solutions,

$$A = Re^{iQX} \qquad \text{with} \qquad R^2 = \frac{1}{3}(\mu_2 - 4Q^2)\,,$$

which yield periodic solutions with wavenumber $q = 1 + \epsilon Q$,

$$\psi = \epsilon R e^{iQX} e^{ix} + \cdots = \epsilon R e^{i(1+\epsilon Q)x} + \ldots\,.$$

Note:

- The real Ginzburg-Landau equation allows also more complicated solutions in which the magnitude and/or the wavenumber depend on space and time. In that way it captures a wide range of phenomena seen in simulations of (92) and much more complex PDEs.

10.1 Phase Dynamics: Slow Dynamics Through the Breaking of a Continuous Symmetry

The breaking of a continuous symmetry leads in general to slow dynamics, which allow a reduction of the system.

Consider a pattern in a large, translation-invariant system

The wave number can vary slowly in space:

- will the pattern relax to a perfectly periodic pattern?
- can the dynamics be described in simple terms?

Spatial translation symmetry:

- The pattern can be shifted by arbitrary distances, there is no restoring force that would move the pattern back to the original position.
- Mathematically: the linearization of the equations around that steady pattern has a 0 eigenvalue, which implies a separation of time scales.

For concreteness, consider a steady solution $\psi_0(x)$ to the Swift-Hohenberg equation

$$\partial_t \psi = R\psi - \left(\partial_x^2 + 1\right)^2 \psi - \psi^3 \,.$$

Any shifted solution $\psi_0(x + \Delta x)$ is also a steady solution, i.e.

$$R\psi_0(x + \Delta x) - \left(\partial_x^2 + 1\right)^2 \psi_0(x + \Delta x) - \psi_0^3(x + \Delta x) = 0 \qquad \text{for any } \Delta x \,.$$

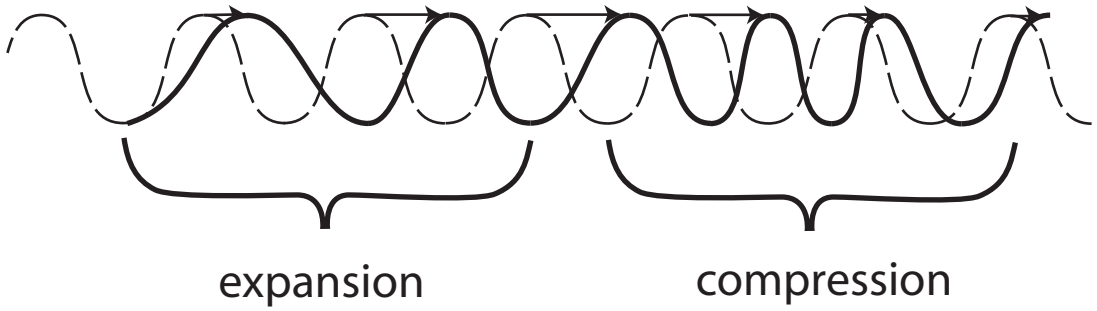
Taking a derivative with respect to Δx yields

$$R\psi_0'(x) - \left(\partial_x^2 + 1\right)^2 \psi_0'(x) - 3\psi_0^2(x)\psi_0'(x) = \mathcal{L}\psi_0' = 0 \tag{94}$$

where

$$\mathcal{L} \equiv R - \left(\partial_x^2 + 1\right)^2 - 3\psi_0^2$$

is the operator arising from the linearization around $\psi_0(x)$.



Expect:

- Dynamics arise only from spatial variations (gradients) in the translation. They lead to expansions and compressions, which amount to gradients in the wavenumber.

- If the expansions/compressions occur on longer and longer spatial scales, the relaxation will become slower and slower.
- Mathematically: for long-wave perturbations the 0-eigenvalue is only perturbed slightly: small eigenvalue $\Rightarrow$ slow dynamics

Thus: the long-wave dynamics will be slow $\Rightarrow$ there is a separation of time scales $\Rightarrow$ a reduction in the dynamics should be possible.

For simplicity, consider the real Ginzburg-Landau equation rather than the Swift-Hohenberg equation as a basic system,

$$\partial_t A = \partial_x^2 A + \mu A - |A|^2 A\,,$$

because we have exact periodic solutions for it. Since we will introduce yet super-slow time and space variables, write x and t as fast variables.

Rewrite it in terms of magnitude and phase: $A = Re^{i\phi}$,

$$\partial_t R = \partial_x^2 R - (\partial_x \phi)^2 R + \mu R - R^3 \tag{95}$$

$$\partial_t \phi = \partial_x^2 \phi + 2\partial_x \phi \,\frac{\partial_x R}{R}\,. \tag{96}$$

Consider first patterns with constant wavenumber

$$\phi = qx \qquad R = \sqrt{\mu - q^2}\,.$$

Thus, the pattern amplitude depends on the wavenumber and goes to 0 at the neutral curve where $\sigma(q) = 0$.

Morevover, $q = \frac{\partial \phi}{\partial x}$. Allowing the wavenumber to vary in space, $q = q(x)$, the equation for the phase becomes

$$\partial_t \phi = \partial_x q + 2q\,\frac{\partial_x R}{R}\,.$$

Notes:

- For spatially periodic solutions, which have constant magnitude and wavenumber, $\partial_t \phi = 0$.
- For long-wave perturbations the dynamics are slow: $\partial_t \phi \to 0$ as $\partial_x R \to 0$ and $\partial_x^2 \phi = \frac{\partial q}{\partial x} \to 0$.
- A spatially varying wavenumber cannot be implemented as $A = R(q(x))e^{iq(x)\cdot x}$, i.e. $\phi(x) \neq q(x) \cdot x$. The phase $\phi(x)$ 'counts' the number of periods from the origin $x = 0$ to the position x. This number of periods depends not only on the wavenumber at x, but on the wavenumber *everywhere* between $x = 0$ and x. Therefore

$$\phi(x) = \int_0^x q(x')dx' \qquad \text{or} \qquad q = \frac{\partial \phi}{\partial x}.$$

We want slow variations of q in space and time,

$$q = q(X,T)\,.$$

To capture this in the phase, introduce a phase Φ that depends on super-slow scales,

$$\frac{1}{\epsilon}\Phi(X,T) = \phi(x,t) \qquad \text{with} \qquad X = \epsilon x \quad T = \epsilon^2 t\,.$$

We have then

$$q = \frac{\partial\phi}{\partial x} = \epsilon\frac{\partial\phi}{\partial X} = \epsilon\frac{1}{\epsilon}\frac{\partial}{\partial X}\Phi(X,T) = \frac{\partial\Phi}{\partial X} = \mathcal{O}(1)$$

and

$$\frac{\partial q}{\partial x} = \epsilon^2\frac{\partial^2}{\partial X^2}\phi = \epsilon\frac{\partial^2\Phi}{\partial X^2} = \mathcal{O}(\epsilon)\,.$$

The scaling of X and T is motivated by the diffusive character of the equation (96) for the phase ϕ.

Note:

- This expansion is similar to the WKBJ-expansion, since $A = R(X,T)e^{\frac{1}{\epsilon}i\Phi(X,T)}$, but here the equations are nonlinear.

Allowing variations in the wavenumber, $q = q(X,T)$, we expand the solution,

$$\begin{aligned} A(x,t) &= A_0(\phi(x,t),X,T) + \epsilon A_1(\phi(x,t),X,T) + \mathcal{O}(\epsilon^2) \\ &= A_0(\frac{1}{\epsilon}\Phi(X,T),X,T) + \epsilon A_1(\frac{1}{\epsilon}\Phi(X,T),X,T) + \mathcal{O}(\epsilon^2)\,. \end{aligned}$$

We need to rewrite the derivatives in terms of the phase and the slow variables,

$$\begin{aligned} \frac{\partial}{\partial t} &\to \epsilon\frac{\partial\Phi}{\partial T}\frac{\partial}{\partial\phi} + \epsilon^2\frac{\partial}{\partial T} \\ \frac{\partial}{\partial x} &\to \frac{\partial\Phi}{\partial X}\frac{\partial}{\partial\phi} + \epsilon\frac{\partial}{\partial X} = q(X,T)\frac{\partial}{\partial\phi} + \epsilon\frac{\partial}{\partial X} \\ \frac{\partial^2}{\partial x^2} &\to \left(q(X,T)\frac{\partial}{\partial\phi} + \epsilon\frac{\partial}{\partial X}\right)\left(q(X,T)\frac{\partial}{\partial\phi} + \epsilon\frac{\partial}{\partial X}\right) \\ &= q^2\frac{\partial^2}{\partial\phi^2} + 2\epsilon q\frac{\partial}{\partial\phi}\frac{\partial}{\partial X} + \epsilon\frac{\partial q}{\partial X}\frac{\partial}{\partial\phi} + \epsilon^2\frac{\partial^2}{\partial X^2} \\ &= q^2\frac{\partial^2}{\partial\phi^2} + 2\epsilon q\frac{\partial}{\partial\phi}\frac{\partial}{\partial X} + \epsilon\frac{\partial q}{\partial X}\frac{\partial}{\partial\phi} + \epsilon^2\frac{\partial^2}{\partial X^2}\,. \end{aligned}$$

Note, the derivatives with respect to ϕ are with respect to the first argument of A.

Inserting the expansion into the Ginzburg-Landau equation we obtain at the various orders in ϵ

$\mathcal{O}(\epsilon^0)$:

$$0 = \mu A_0 + q^2\frac{\partial^2}{\partial\phi^2}A_0 - |A_0|^2 A_0\,. \tag{97}$$

This is the original Ginzburg-Landau equation, rewritten in terms of the phase. The solution is

$$A_0 = \sqrt{\mu - q(X,T)^2}e^{i\phi}\,.$$

Note:

- Through the wavenumber $q = q(X,T)$, the amplitude A_0 becomes *explicitly* dependent on X and T.

$\mathcal{O}(\epsilon)$:

$$\frac{\partial\Phi}{\partial T}\frac{\partial A_0}{\partial\phi} = \mu A_1 + q^2\frac{\partial^2}{\partial\phi^2}A_1 - 2|A_0|^2A_1 - A_0^2A_1^* + 2q\frac{\partial^2 A_0}{\partial\phi\,\partial X} + \frac{\partial^2\Phi}{\partial X^2}\frac{\partial A_0}{\partial\phi}\,.$$

The X-dependence of A_0 is via $q(X,T)$

$$\frac{\partial^2 A_0}{\partial\phi\,\partial X} = \frac{\partial}{\partial\phi}\frac{\partial A_0}{\partial q}\frac{\partial q}{\partial X} = \frac{\partial q}{\partial X}\frac{\partial}{\partial q}\frac{\partial A_0}{\partial\phi} = \frac{\partial^2\Phi}{\partial X^2}\frac{\partial}{\partial q}\frac{\partial A_0}{\partial\phi}$$

Reorder to get

$$\mathcal{L}A_1 = -\frac{\partial\Phi}{\partial T}\frac{\partial A_0}{\partial\phi} + \frac{\partial^2\Phi}{\partial X^2}\frac{\partial A_0}{\partial\phi} + 2q\frac{\partial^2 A_0}{\partial\phi\,\partial X} \tag{98}$$

with

$$\mathcal{L}A_1 = \mu A_1 + q^2\frac{\partial}{\partial\phi^2}A_1 - 2|A_0|^2A_1 - A_0^2\,\mathcal{C}A_1 \qquad \text{where} \quad \mathcal{C}A_1 = A_1^*\,.$$

Is $\mathcal{L}$ invertible or singular?

The starting point for this analysis is the fact that when the pattern is shifted rigidly it does not return to the original position, since the system is translation invariant. Thus, such a perturbation has a vanishing growth rate, which must also be described by this linearization. The spatial translation symmetry is reflected in the fact that the coefficients in the Ginzburg-Landau equation - and therefore also in (97) - are space-independent. In our rescaled coordinates this amounts to an independence of ϕ. Consider therefore the ϕ-derivative of (97),

$$\begin{aligned} 0 = \frac{\partial}{\partial\phi}\left\{\mu A_0 + q^2\frac{\partial^2}{\partial\phi^2}A_0 - |A_0|^2A_0\right\} &= \mu\frac{\partial A_0}{\partial\phi} + q^2\frac{\partial^2}{\partial\phi^2}\frac{\partial A_0}{\partial\phi} - 2|A_0|^2\frac{\partial A_0}{\partial\phi} - A_0^2\frac{\partial A_0^*}{\partial\phi} \\ &= \mathcal{L}\frac{\partial A_0}{\partial\phi}\,. \end{aligned}$$

Thus, $\mathcal{L}$ is singular and its zero-eigenvector is $\frac{\partial A_0}{\partial\phi}$ (cf. (94)).

Note:

- Quite generally, the *translation mode* $\frac{\partial A}{\partial\phi}$ characterizes the linear perturbation associated with a small spatial shift of a pattern $A(\phi)$:

$$A(\phi+\Delta\phi) = A(\phi) + \frac{\partial A}{\partial\phi}\Delta\phi + \mathcal{O}(\Delta\phi^2)\,.$$

- The existence of the translation mode requires that the system has a *continuous symmetry* and that this symmetry is *broken* by the solution around which we expand (A_0).
  - In order for the derivative with respect to the shift $\Delta\phi$ make sense, there must be a continuous family of solutions differing in $\Delta\phi$.
  - If A does not brake the continuous symmetry, then $\frac{\partial A}{\partial \phi} = 0$ and the eigenvector is a zero-vector, i.e. $\frac{\partial A}{\partial \phi}$ is not an eigenvector at all.

To solve the equation at $\mathcal{O}(\epsilon)$ we therefore have to satisfy a solvability condition associated with the translation mode $\frac{\partial A_0}{\partial \phi}$. It involves the eigenvector of the adjoint of $\mathcal{L}$ or the left eigenvector of $\mathcal{L}$. To define an adjoint of $\mathcal{L}$, one has to define first a scalar product in the space of solutions. The adjoint depends in general on the scalar product chosen. In this case, one can define a scalar product that makes $\mathcal{L}$ self-adjoint.

For 2π-periodic complex functions $\psi_{1,2}$ consider the usual scalar product

$$\langle \psi_1, \psi_2 \rangle = \int_0^{2\pi} \psi_1^* \psi_2 \, d\phi \, .$$

Then, integrating by parts as usually, yields

$$\begin{aligned} \langle \psi_1, \mathcal{L}\psi_2 \rangle &= \int_0^{2\pi} \psi_1^* \, (\mathcal{L}\psi_2) \, d\phi \\ &= \int \psi_1^* \left(\mu\psi_2 + q^2\partial_\phi^2\psi_2 - 2|A_0|^2\psi_2 - A_0^2 \underbrace{\mathcal{C}\psi_2}_{\psi_2^*} \right) d\phi \\ &= \int \left(\mu\psi_1^* + q^2\partial_\phi^2\psi_1^* - 2|A_0|^2\psi_1^* \right) \psi_2 \, d\phi - \int A_0^2 \, \psi_1^*\psi_2^* \, d\phi \, . \end{aligned}$$

Compare this with

$$\begin{aligned} \langle \mathcal{L}\psi_1, \psi_2 \rangle &= \int_0^{2\pi} (\mathcal{L}\psi_1^*) \, \psi_2 d\phi \\ &= \int \left(\mu\psi_1^* + q^2\partial_\phi^2\psi_1^* - 2|A_0|^2\psi_1^* \right) \psi_2 \, d\phi - \int \underbrace{A_0^{*2} \, \psi_1\psi_2}_{\left(A_0^2\psi_1^*\psi_2^*\right)^*} d\phi \, . \end{aligned}$$

Thus, the term involving $\mathcal{C}$ is not the same in the two expressions and $\mathcal{L}$ is not self-adjoint.

However, the two terms involving $\mathcal{C}$ are complex conjugates of each other. One can therefore define a new scalar product by taking only the real part,

$$\langle \psi_1, \psi_2 \rangle_\Re = \Re \left(\int_0^{2\pi} \psi_1^* \psi_2 \, d\phi \right) \, .$$

With this scalar product $\mathcal{L}$ is self-adjoint,

$$\langle \psi_1, \mathcal{L}\psi_2 \rangle_\Re = \langle \mathcal{L}\psi_1, \psi_2 \rangle_\Re \, ,$$

and the left and right eigenvectors are the same. We therefore project (98) onto the left eigenvector $\partial_\phi A_0$,

$$0 = \langle \partial_\phi A_0, \mathcal{L} A_1 \rangle_\Re = \left\langle \partial_\phi A_0, -\frac{\partial \Phi}{\partial T}\frac{\partial A_0}{\partial \phi} + \frac{\partial^2 \Phi}{\partial X^2}\frac{\partial A_0}{\partial \phi} + 2q\frac{\partial^2 \Phi}{\partial X^2}\frac{\partial}{\partial q}\frac{\partial A_0}{\partial \phi} \right\rangle_\Re ,$$

i.e.

$$\Re \int_0^{2\pi} \partial_\phi A_0^* \left(-\frac{\partial \Phi}{\partial T}\frac{\partial A_0}{\partial \phi} + \frac{\partial^2 \Phi}{\partial X^2}\frac{\partial A_0}{\partial \phi} + 2q\frac{\partial^2 \Phi}{\partial X^2} \underbrace{\frac{\partial}{\partial q}\frac{\partial A_0}{\partial \phi}}_{\frac{\partial}{\partial \phi}\frac{\partial A_0}{\partial q}} \right) d\phi = 0 .$$

Here $A_0 = \sqrt{\mu - q(X,T)^2}\, e^{i\phi}$ and

$$\frac{\partial A_0}{\partial q} = \frac{-q}{\sqrt{\mu - q^2}} e^{i\phi} = -\frac{q}{\mu - q^2} A_0 .$$

We therefore get for the solvability condition

$$\Re \int_0^{2\pi} \underbrace{\partial_\phi A_0^* \partial_\phi A_0}_{|\partial_\phi A_0|^2} d\phi \left(-\frac{\partial \Phi}{\partial T} + \frac{\partial^2 \Phi}{\partial X^2} - \frac{\partial^2 \Phi}{\partial X^2}\frac{2q^2}{\mu - q^2} \right) = 0$$

implying

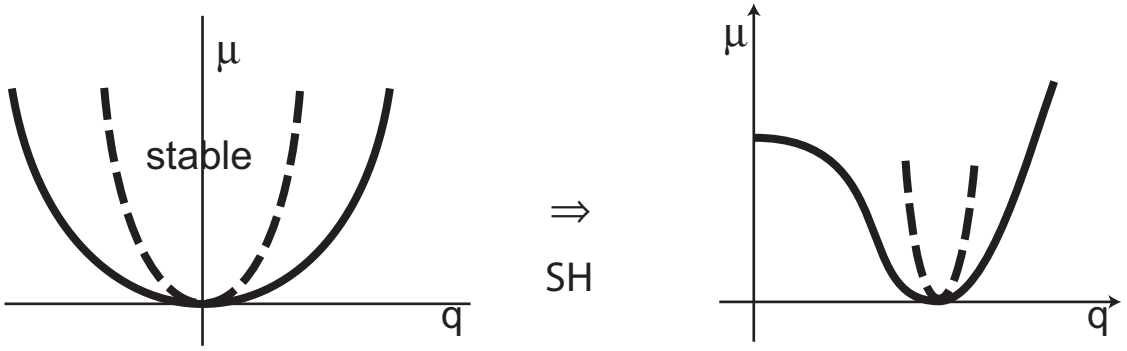
$$\frac{\partial \Phi}{\partial T} = D(q)\frac{\partial^2 \Phi}{\partial X^2} \qquad \text{with} \qquad D(q) = \frac{\mu - 3q^2}{\mu - q^2} . \tag{99}$$

Notes:

- The phase satisfies a diffusion equation: wavenumber gradients relax diffusively. This is due to the fact that the system is reflection-symmetric in space, but not in time.
- The diffusion coefficient $D(q)$ depends on the wavenumber $q = \partial_X \Phi$, which makes the phase diffusion equation *nonlinear.*
- The pattern exists over the whole range of wavenumbers $-\sqrt{\mu} < q < +\sqrt{\mu}$. However, the diffusion coefficient changes sign already at

  $$q_{Eckhaus} = \pm\sqrt{\frac{\mu}{3}} .$$

  For $D(q) < 0$ the periodic pattern is *unstable* to modulations. This instability was first identified by Eckhaus in terms of a side-band instability (Eckhaus, 1965).



- This approach does not make use of any small amplitudes; it is based solely on the assumption of a slowly varying wavenumber. Therefore it can be applied directly to the equations describing the system on the original (fast) scales, e.g. the Swift-Hohenberg equation or Navier-Stokes equation etc. $\rightarrow$ e.g. experiments in Taylor vortex flow (cf. Fig.26).
- The Eckhaus instability is a *universal* instability of steady one-dimensional patterns.

The Eckhaus instability is a long-wave instability. Consider small modulations around a wavenumber q_0, which allows to ignore the q-dependence of $D(q)$,

$$\frac{\partial \Phi}{\partial T} = D(q_0)\,\frac{\partial^2 \Phi}{\partial X^2}\,. \tag{100}$$

Using $\Phi = \Phi_0 e^{\sigma t + iPX}$ we get

$$\sigma = -D(q_0)\,P^2\,.$$

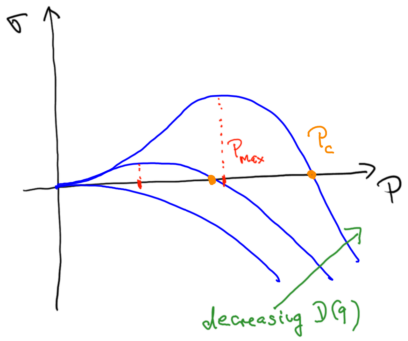


Figure 25: Growth rate for the Eckhaus instability.

For $D(q_0) < 0$ the diffusion equation is ill-posed, the fastest growing modes would have infinite modulation wavenumber P. We therefore need to regularize the equation. For $|D(q_0)| \ll 1$ one can go to higher order in the expansion and obtains due to reflection symmetry

$$\frac{\partial \Phi}{\partial T} = D(q_0)\,\frac{\partial^2 \Phi}{\partial X^2} - D_4(q_0)\frac{\partial^4 \Phi}{\partial X^4}\,.$$

Then

$$\sigma(P) = -D(q_0)\,P^2 - D_4(q_0)P^4\,.$$

The fastest growing mode has a finite perturbation wavenumber P_{max}

$$P_{max}^2 = -\frac{D(q_0)}{2D_4(q_0)}\,.$$

For $D(q_0) = 0$, $P_{max} = 0$, corresponding to an infinite wavelength of the perturbation. Therefore, in a system of finite size L the onset of the instability is delayed and occurs for

$$D(q_0) = -D_4(q_0)\,\left(\frac{2\pi}{L}\right)^2 < 0\,.$$

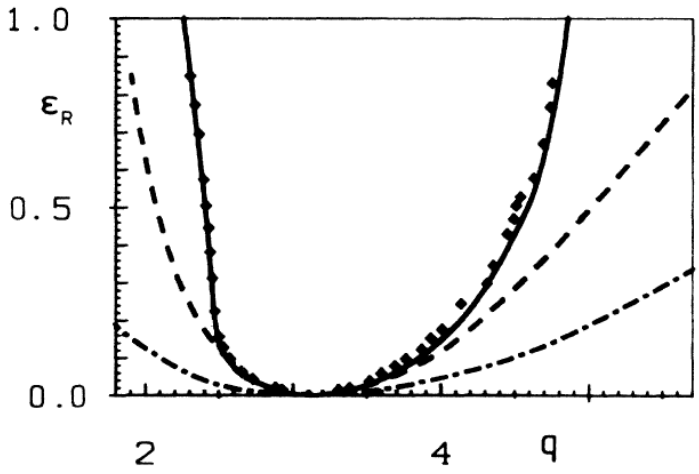


Figure 26: Stability limits for axisymmetric vortices in Taylor vortex flow. Here ϵ_R measures the distance from the onset of formation of patterns. Symbols: experimental results (Dominguez-Lerma et al., 1986), dashed line: stability limit based on the cubic amplitude equation (cf. (99)), solid line: $D(q) = 0$ for the fully nonlinear fluid pattern (Riecke and Paap, 1986).

Notes:

- Nonlinear evolution of Eckhaus instability:
  - For $D < 0$ the solution to (100) blows up $\Rightarrow$ the assumption (101) of small gradients breaks down.
  - No saturation of the instability: the modulation drives the wavenumber locally further into the unstable regime making phase-diffusion equation leads to singularity in finite time
  - Phase slip $\Rightarrow$ change in wave number

10.1.1 Easier Derivation of the Linear Phase Diffusion Equation

If one limits oneself to small changes in the wavenumber, one can derive a *linear* diffusion equation somewhat more easily. Consider a pattern with almost constant wavenumber,

$$\begin{aligned} \phi &= qx + \epsilon\Phi(X,T), \qquad \underbrace{X = \epsilon x, T = \epsilon^2 t}_{\text{superslow scales}} \\ R &= R_0 + \epsilon^2 r(X,T)\,. \end{aligned} \tag{101}$$

Insert

$$\frac{\partial R}{\partial t} = \epsilon^4 \frac{\partial r}{\partial T} \qquad \frac{\partial R}{\partial x} = \epsilon^3 \frac{\partial r}{\partial X}$$

and

$$\frac{\partial \phi}{\partial t} = \epsilon^3 \frac{\partial \Phi}{\partial T} \qquad \frac{\partial \phi}{\partial x} = q + \epsilon^2 \frac{\partial \Phi}{\partial X} \qquad \frac{\partial^2 \phi}{\partial x^2} = \epsilon^3 \frac{\partial^2 \Phi}{\partial X^2}$$

into the Ginzburg-Landau equation.

$\mathcal{O}(\epsilon^0)$:

$$0 = (\mu - q^2)R_0 - R_0^3 \qquad \Rightarrow \qquad R_0 = \sqrt{\mu - q^2}\,.$$

This recovers the usual steady pattern.

$\mathcal{O}(\epsilon^2)$:

$$\begin{aligned} 0 &= -2q\partial_X\Phi\, R_0 - q^2 r + \mu r - 3R_0^2\, r \\ &= -2q\partial_X\Phi R_0 - 2R_0^2 r \\ r &= -\frac{q}{R_0}\,\partial_X\Phi \end{aligned}$$

Thus, the amplitude changes with the change $\partial_X\Phi$ of the wavenumber.

$\mathcal{O}(\epsilon^3)$:

$$\begin{aligned} \partial_T\Phi &= \partial_X^2\Phi + 2\frac{q}{R_0}\partial_X r \\ &= \partial_X^2\Phi + \frac{2q}{R_0}\left(-\frac{q}{R_0}\right)\partial_X^2\Phi \\ &= \partial_X^2\Phi\left\{1 - \frac{2q^2}{\mu - q^2}\right\} \end{aligned}$$

Thus:

$$\frac{\partial\Phi}{\partial T} = D(q)\,\frac{\partial^2\Phi}{\partial X^2} \qquad \text{with} \qquad D(q) = \frac{\mu - 3q^2}{\mu - q^2}\,, \tag{102}$$

which corresponds to (99) for fixed q.

11 Oscillations: Complex Ginzburg-Landau Equation

Consider oscillations in a spatially extended system. Will the oscillations at different locations be synchronized? Can the system support traveling waves? Are such waves stable? Close to a Hopf bifurcation we can derive a weakly nonlinear equation to address these questions.

Consider systems that are reflection symmetric and have translation symmetry in space.

In a weakly nonlinear description of a Hopf bifurcation the oscillations will be given in the form

$$\psi(x,t) = \epsilon A(X,T)\, e^{i\omega_h t} + \epsilon A(X,T)^*\, e^{-i\omega_h t} + h.o.t.\ldots.$$

Symmetries:

- The reflection symmetry $x \leftrightarrow -x$ acts trivially on A,

  $$A \to A\,,$$

  and therefore does not impose any condition on the coefficients of the resulting amplitude equation. But it does require the amplitude equation to be equivariant under $X \leftrightarrow -X$, i.e. the spatial derivatives have to be even.

- The translation symmetry $x \to x + \Delta x$ also acts trivially on A. On the slow scale it implies that the coefficients of the equation are independent of X.

Scaling:

- Since only even spatial derivatives are possible we get

$$\partial_X^2 \sim \mathcal{O}(A^2) \qquad \Rightarrow \qquad X = \epsilon x \,.$$

We expect therefore

$$\partial_T A = (d_r + i d_i)\, \partial_X^2 A + (\mu_r + i\mu_i)\, A - (g_r + i g_i)\, |A|^2 A \,,$$

i.e. all the coefficients will in general be complex.

By rescaling the amplitude, time, and space variables and by transforming $A \to e^{i\mu_i t} A$ we can simplify the equation to

$$\partial_T A = (1 + ib)\, \partial_X^2 A + A - (1 + ic)\, |A|^2 A \tag{103}$$

with only 2 parameters, b and c.

Note:

- In analogy to the real Ginzburg-Landau equation (93), this equation is called the *complex Ginzburg-Landau equation* (CGL). It provides the universal description of weakly nonlinear oscillations in spatially homogeneous and reflection symmetric systems.
- In the rescaling leading to (103) it is assumed that the Hopf bifurcation is supercritical (i.e. $g_r > 0$).
- Through this rescaling the bifurcation parameter is fixed to $\mu_r = +1$; in particular it is taken to be positive.
  If one wants to have the ability to vary the control parameter across the Hopf bifurcation, one simply retains the parameter μ_r.

The CGL allows simple periodic solutions,

$$A = R e^{iQX + i\Omega T} \qquad \text{with} \quad R^2 = 1 - Q^2 \quad \text{and} \quad \Omega = -cR^2 - bQ^2 = -c - (b - c)Q^2 \,,$$

implying for the original microscopic system

$$\psi(x,t) = \epsilon R e^{i\epsilon Q x + i(\omega_h + \epsilon^2 \Omega)t} + c.c. + h.o.t. \,.$$

Notes:

- For $Q \neq 0$ the phase of the oscillation is not the same at all locations; they are not synchronized. The solution represents a traveling wave,

$$\psi(x,t) = \psi(qx + \omega t) \,.$$

- The frequency of the oscillations depends
  - on the amplitude through c.
  - on the wavenumber through b.
- The phase velocity, which is the velocity with which the extrema of the wave travel, is given by
$$v_\phi = \frac{\omega}{q} = \frac{\omega_h + \epsilon^2\Omega}{\epsilon Q}\,.$$
- The group velocity, which is the velocity of wave packets, is given by
$$v_g = \frac{d\omega}{\epsilon dQ} = -2\epsilon(b+c)\,Q\,.$$

11.1 Phase Dynamics for Oscillations

Analogous to the derivation of the phase equation for spatial patterns (99) one can also derive a phase equation for oscillations. However, while the slow dynamics of the phase of spatial patterns arises from the breaking of the continuous translation symmetry in space, for the oscillations the origin of the slow phase dynamics is the breaking of the continuous translation symmetry in time:

- The phase of the spatially homogeneous oscillations is arbitrary
$$A = Re^{i\omega t}e^{i\phi} \qquad \phi \in \mathbb{R}$$
and a spatially homogeneous shift in the phase does not experience a 'restoring force'.
- If the phase is shifted differently at different locations, the phase differences lead to an interaction between the oscillations at different locations that changes the phase.
- If the variation of the phase occurs on a long spatial scale, the interaction leads to a slow temporal evolution. This evolution can be captured by the reduction to a phase equation.

Consider as example the complex Ginzburg-Landau equation[10],

$$\partial_t A = (1+ib)\,\partial_x^2 A + A - (1+ic)\,|A|^2 A\,, \tag{104}$$

again writing the variables as fast variables, since we will introduce super-slow variables.

Rewrite in terms of $A = Re^{i\phi}$,

$$\partial_t R + iR\partial_t\phi = (1+ib)\,\left(\partial_x^2 R + 2i\partial_x\phi\partial_x R + R\left(i\partial_x^2\phi - (\partial_x\phi)^2\right)\right) + R - (1+ic)\,R^3\,.$$

[10] An analogous analysis could be performed directly on the microscopic equations, even for oscillations at finite oscillation amplitude.

Separating real and imaginary parts yields

$$\partial_t R = \partial_x^2 R - R\left(\partial_x \phi\right)^2 - b\left(2\partial_x\phi\,\partial_x R + R\partial_x^2\phi\right) + R - R^3 \tag{105}$$

$$\partial_t \phi = b\left(\frac{1}{R}\partial_x^2 R - \left(\partial_x\phi\right)^2\right) + \frac{2}{R}\partial_x\phi\,\partial_x R + \partial_x^2\phi - cR^2\,. \tag{106}$$

Note:

- In contrast to the case of the real Ginzburg-Landau equation, the phase is coupled to the amplitude and its evolution depends on the wavenumber $\partial_x\phi$ itself and not only on wavenumber gradients.

To derive an equation for the evolution of the wavenumber or the phase consider for simplicity *small* perturbations of the phase around the uniform oscillations, $A = Re^{i\Omega t}$,

$$\phi = -ct + \epsilon\Phi(X,T) \qquad \text{and} \qquad R = R_0 + \epsilon^\alpha r(X,T)\,. \tag{107}$$

As was the case in the analysis leading to (102), this restricts the analysis to small changes in the local wavenumber, which is not the case for the more involved WKBJ-like approach used in the derivation of (99). Here, X and T are super-slow scales, yet slower than the scales of the complex Ginzburg-Landau equation. The case of traveling waves ($Q \neq 0$) can be treated analogously.

Since a spatially uniform change of the phase does not have any effect on the dynamics, we expect a phase equation of the form

$$\partial_T\Phi = F(\partial_X\Phi, \partial_X^2\Phi, \ldots)\,,$$

with no dependence on the phase Φ itself. Due to the spatial reflection symmetry, which does not affect the temporal phase ϕ itself, all terms in the phase equation need to have an even number of spatial derivatives. We therefore expect the balance $\partial_T \sim \partial_X^2$. This suggests the scaling

$$X = \epsilon x \qquad T = \epsilon^2 t\,.$$

Insert (107) in (105,106):

$\mathcal{O}(\epsilon^0)$:

$$0 = R_0 - R_0^3 \qquad \Rightarrow \qquad R_0 = 1\,.$$

The first non-trivial contributions arise at $\mathcal{O}(\epsilon^3) \Rightarrow$ choose $\alpha = 3$.

$\mathcal{O}(\epsilon^3)$:

$$0 = -bR_0\partial_X^2\Phi + r - 3R_0^2 r \qquad \Rightarrow \qquad r = -\frac{1}{2}b\partial_X^2\Phi$$

$$\partial_T\Phi = \partial_X^2\Phi - 2cR_0 r\,.$$

This yields

$$\partial_T\Phi = (1+bc)\,\partial_X^2\Phi\,.$$

Notes:

- The phase Φ evolves *diffusively* with a diffusion coefficient given by

$$D = 1 + bc\,.$$

- The diffusion coefficient can be **negative**:
  For $bc < -1$ homogeneous oscillations are **unstable to long-wave perturbations**. This instability is called the Benjamin-Feir instability.

- One can also derive the phase equation for traveling wave solutions ($Q \neq 0$),

$$A = Re^{i(QX+\Omega T)}e^{i\phi}\,.$$

  The resulting equation shows that for $bc > -1$ the waves are stable in a wavenumber band $[-Q_m, +Q_m]$ around $Q = 0$. This situation is similar to that of the Eckhaus instability (99) for steady patterns. However, the stable band closes at $bc = -1$ and for $bc < -1$ these waves are unstable for *all wavenumbers.* Thus, for $bc < -1$ (103) has *no* stable traveling wave solutions.

- For $|D| \ll 1$ one can derive a nonlinear phase equation, which is the Kuramoto-Sivashinsky equation,

$$\partial_T\phi = D\partial_x^2\phi + g\partial_x^4\phi + h(\partial_x\phi)^2\,.$$

Numerical Simulations (Chaté, 1994):

- The Benjamin-Feir instability leads to different chaotic states depending on b and c
  - Phase turbulence: persistently evolving modulation of the wavenumber and associated with it of the amplitude, but the amplitude remains bounded away from 0. Thus there are no phase slips and the total phase is conserved.
  - Amplitude turbulence: phase slips, during which the magnitute $|A|$ goes to 0, occur irregularly and persistently, changing the local wavenumber.

- Turbulence can also arise in the regime in which some plane waves are linearly stable (Benjmain-Feir-stable regime).

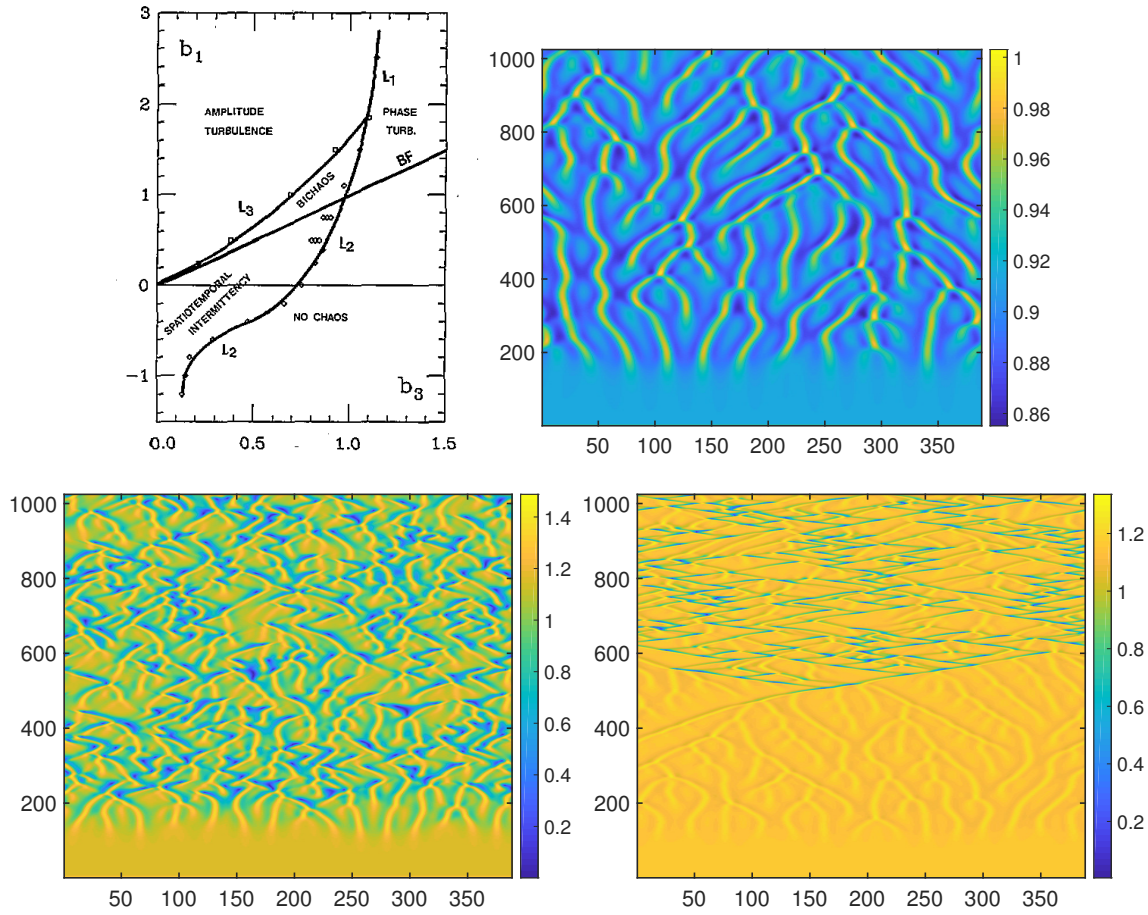


Figure 27: Phase diagram for 1-dimensional CGL (Chaté, 1994). Space-time diagrams for phase turbulence, amplitude turbulence, and bichaos (depending on initial conditions one obtains phase turbulence or amplitude turbulence).

The complex Ginzburg-Landau equation can easily be extended to 2 dimensions

$$\partial_t A = (1 + ib)\,\Delta A + A - (1 + ic)\,|A|^2 A\,.$$

- The Benjamin-Feir instability arises also in this case. As in 1 dimension, it can lead to phase turbulence or amplitude turbulence.
- Amplitude turbulence is characterized by the appearance of *defects* at which the magnitude $|A|$ vanishes. They correspond to rotating spirals.
- The spirals are topologically stable: the integral over the phase along *any* contour that encircles a spiral core is either 2π or -2π, depending on the 'charge' of the spiral; the spiral core (the zero of the magnitude of the complex amplitude) cannot simply disappear by itself, since it would require unwinding the phase everywhere. Spiral cores can only disappear through a collision with a spiral core of opposite charge.
- There are analytical approaches that treat the spirals as discrete objects and allow to obtain evolution equations for their position and how they interact (Aranson and Kramer, 2002).

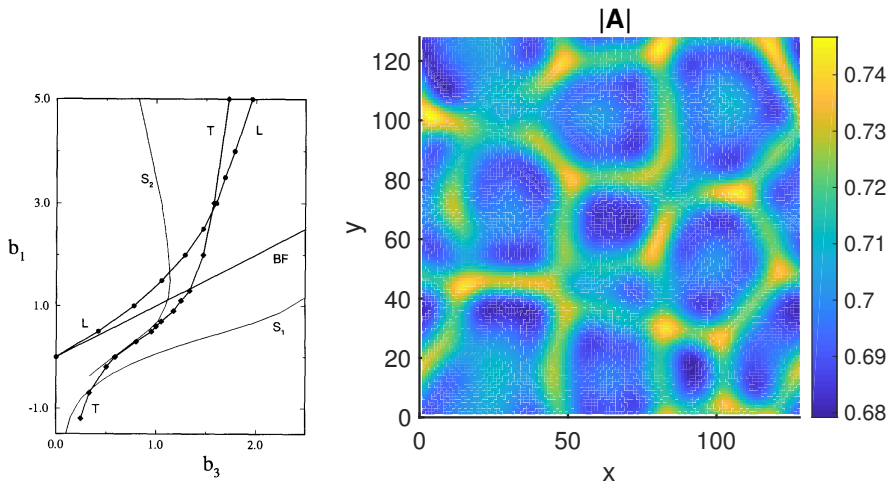


Figure 28: Phase diagram for 2-dimensional CGL (Chaté, 1994). Snapshot of phase turbulence. Note that the magnitude of A is bounded away from 0: no defects.

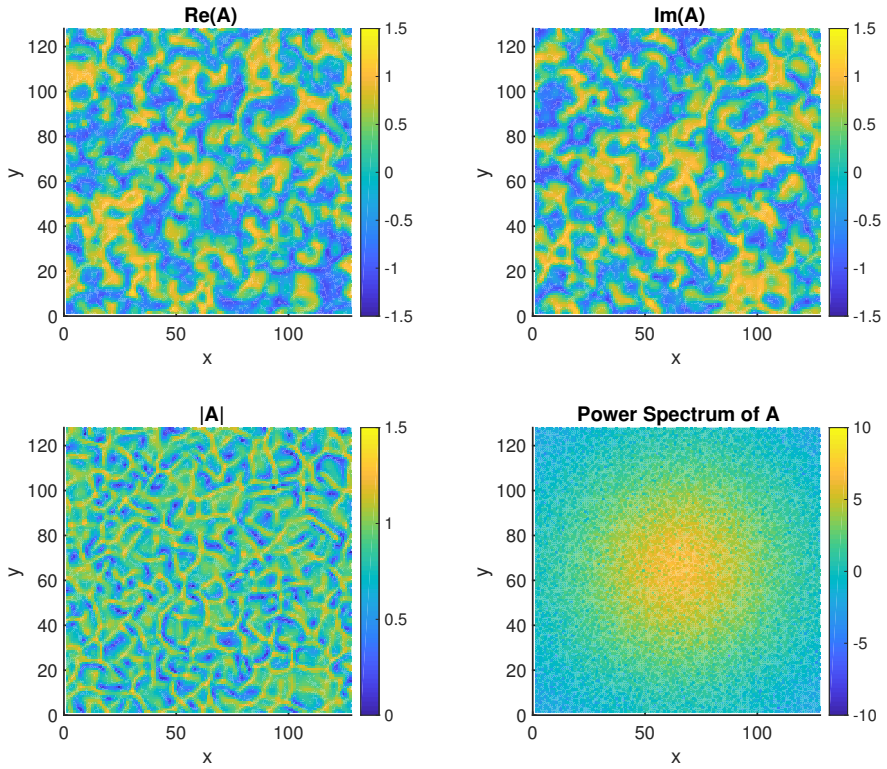


Figure 29: Snapshot of defect turbulence in the two-dimensional CGL ($b_1 = 2$, $b_3 = 1$ (Chaté, 1994)).

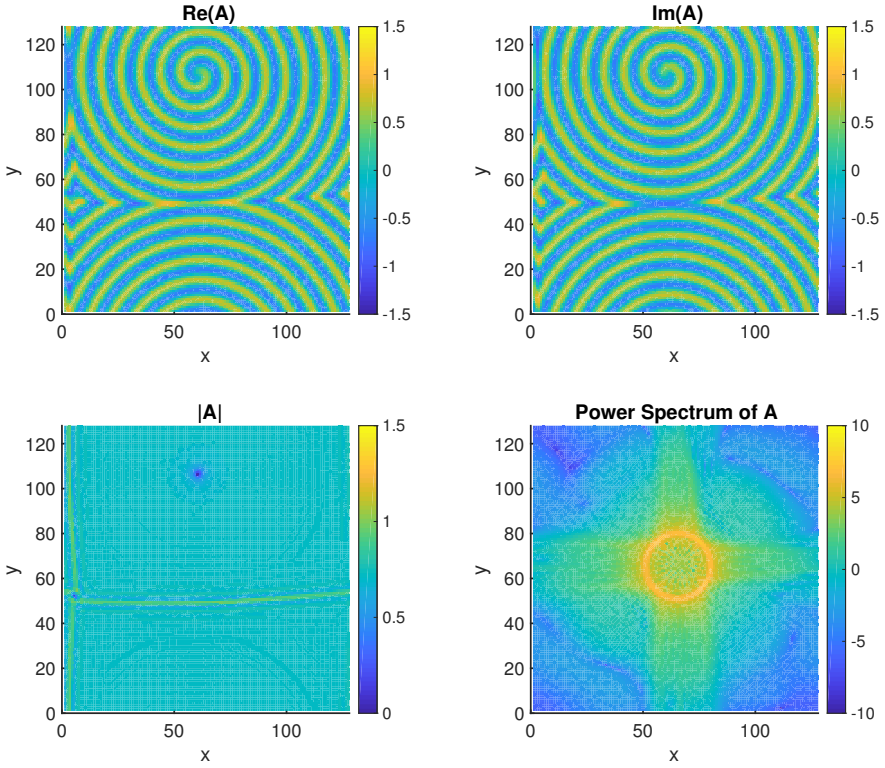


Figure 30: Snapshot of spirals in the two-dimensional CGL ($b_1 = 2$, $b_3 = 1.51$ (Chaté, 1994)).

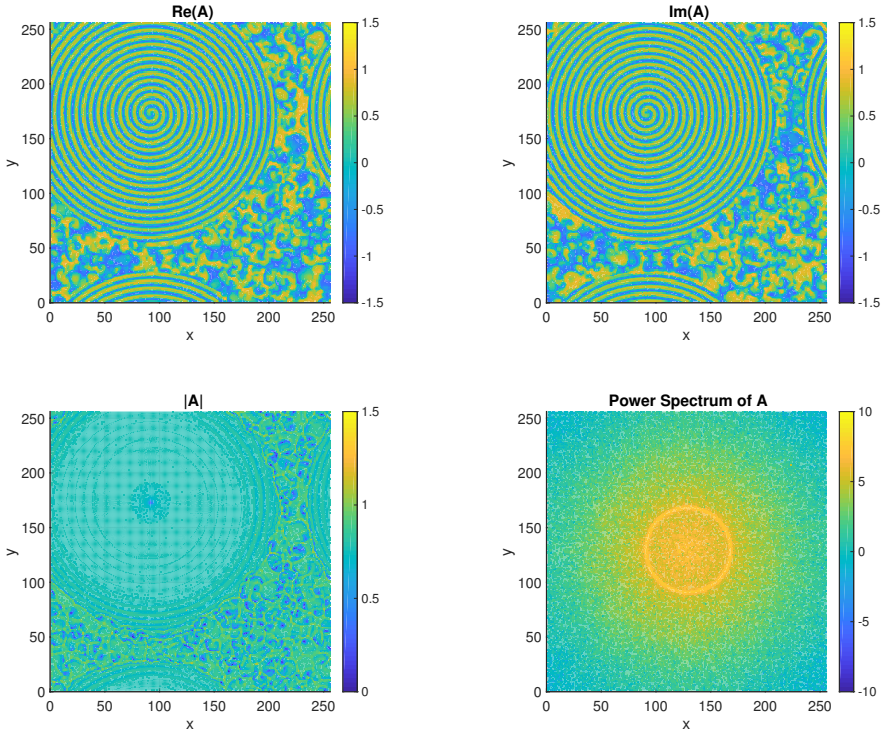


Figure 31: Snapshot of spirals of finite size in the turbulent background in the two-dimensional CGL ($b_1 = 2$, $b_3 = 1.33$ (Chaté, 1994)). The initial condition was a state of spirals obtained for $b_3 = 1.51$.

12 Fronts and Their Interaction

Consider nonlinear PDEs with spatial translation symmetry that have multiple stable spatially homogeneous solutions

- $\Rightarrow$ there must be also solutions that connect the stable states: fronts or kinks
- these fronts are *heteroclinic* in space: they connect two different fixed points for $x \to \pm\infty$. They are topologically stable: they cannot disappear except at infinity or by collision with 'anti-fronts'.
- This is to be compared to *homoclinic* solutions which connect to the same fixed point for $x \to \pm\infty$, i.e. localized 'humps' (like the solitons). They are not topologically stable since they can disappear, e.g., due to a sufficiently large perturbation.

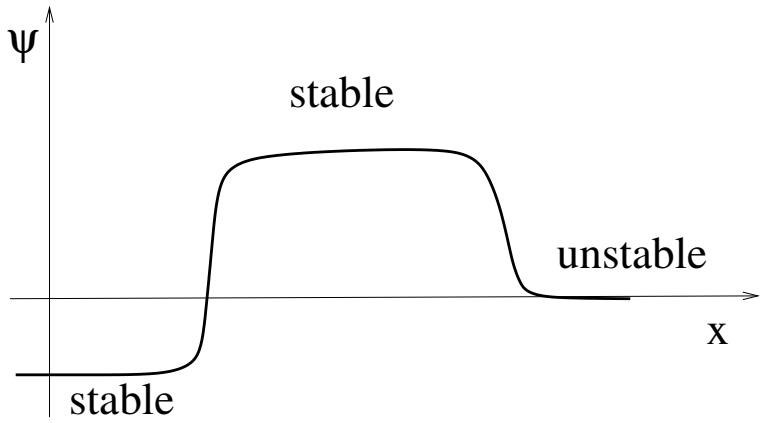
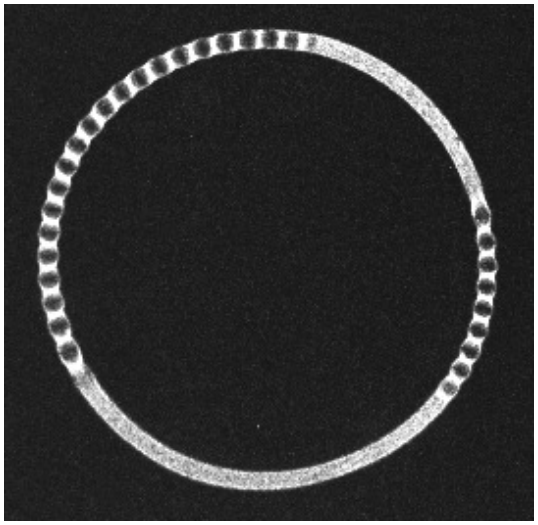


Figure 32: Fronts connecting two stable and one unstable spatially homogeneous state.

Questions:

- Do such fronts travel? What determines their speed?
- How do the fronts interact? Can they from stable bound states: localized domains?

a)



b)

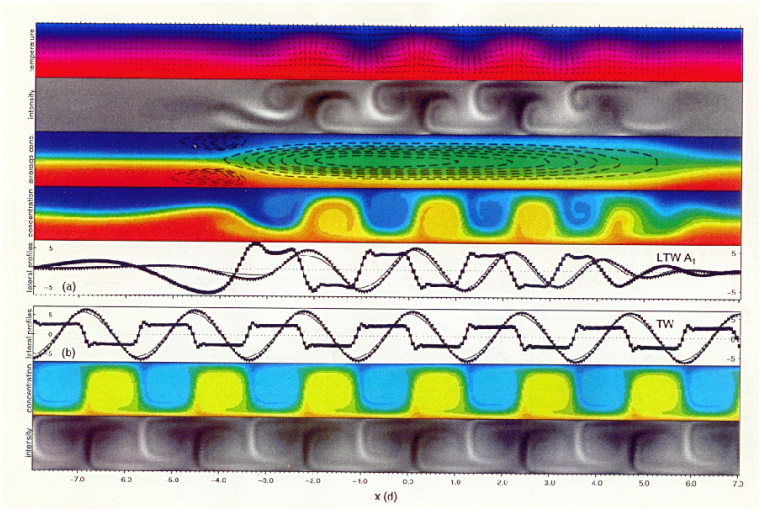


Figure 33: Localized wave trains in convection of water-alcohol mixtures. a) Top view of annular convection cell. In this regime the localized waves are spatially more extended, resembling bound states of fronts. Two slowly drifting, stable localized wave trains are seen Kolodner et al. (1988). b) Numerical simulations of localized states and of extended traveling waves Barten et al. (1991, 1995).

12.1 Single Fronts Connecting Stable States

Consider a simple nonlinear diffusion equation

$$\partial_t \psi = \partial_x^2 \psi + f(\psi) \equiv \partial_x^2 \psi - \partial_\psi V(\psi;\lambda)$$

where λ is a control parameter of the system.

This equation can be written in variational form

$$\partial_t \psi = -\frac{\delta \mathcal{V}\{\psi\}}{\delta \psi} \qquad \text{with} \qquad \mathcal{V}\{\psi\} = \int \frac{1}{2}\left(\partial_x \psi\right)^2 + V(\psi;\lambda) dx$$

Assume $V(\lambda;\psi)$ has two minima at $\psi = \psi_{1,2}$, corresponding to stable, spatially homogeneous solutions.

Look for 'wave solution', i.e. a steadily propagating front solution

$$\psi = \psi(\zeta) \qquad \text{with} \quad \zeta = x - vt$$

which satisfies

$$\partial_\zeta^2\psi + v\partial_\zeta\psi = +\partial_\psi V(\psi) \equiv -\partial_\psi \hat{V}(\psi) \qquad \text{with} \qquad \hat{V}(\psi;\lambda) = -V(\psi;\lambda).$$

Notes:

- this equation can be read as describing the position ψ of a particle in the potential $\hat{V}(\psi)$ and experiencing friction with coefficient v.
- we are interested in solutions that start at ψ_1 and end at ψ_2

$$\psi(\zeta) \to \psi_1 \quad \text{for} \quad \zeta \to -\infty \qquad \psi(\zeta) \to \psi_2 \quad \text{for} \quad \zeta \to +\infty$$

- since in terms of $\hat{V}(\psi)$ the 'positions' $\psi_{1,2}$ are actually maxima, the 'friction' v must be tuned exactly such that the particle, starting at one maximum, stops at the other maximum:
  $\Rightarrow$ the velocity is uniquely determined.
- depending on the relative heights of the maxima the 'friction' may be negative.
- for fronts connecting a stable state with an unstable state the velocity ('friction') is not uniquely determined: the unstable state corresponds to the minimum of the potential for the 'particle' and the 'particle' will end up in that minimum for a wide range of friction values. The velocity selection in this situation is an interesting problem (e.g. van Saarloos (1988)).

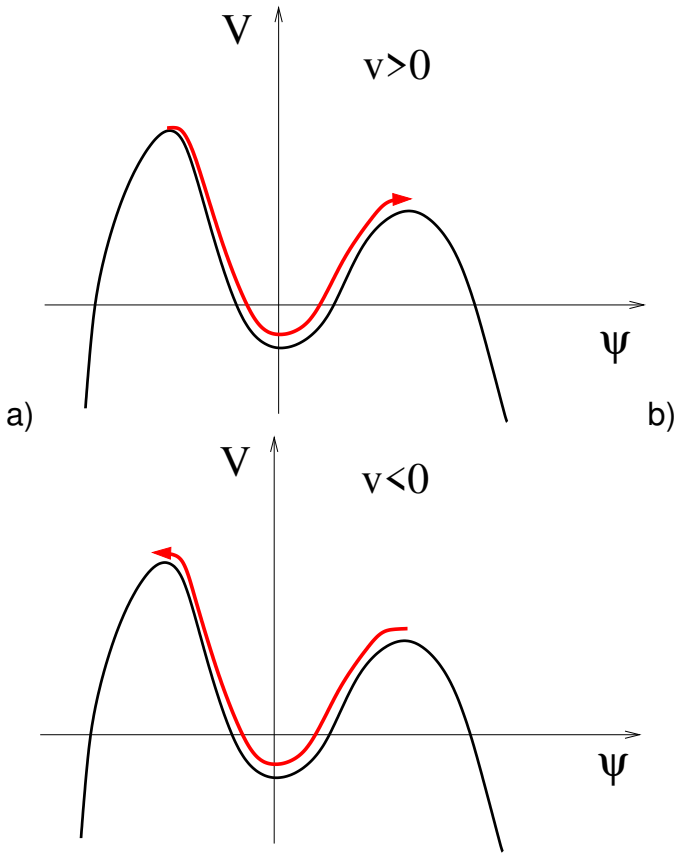


Figure 34: Fronts correspond to a particle moving in a potential with friction. a) friction positive. b) friction negative.

12.1.1 Perturbation Calculation of the Front Velocity

Assume there is a parameter value, $\lambda = \lambda_0 \equiv 0$, for which the front solution $\psi(x;\lambda=0)$ is stationary. If one has access to that solution one can obtain the front velocity for close-by parameter values perturbatively.

Expand

$$\lambda = \epsilon\lambda_1 \qquad v = \epsilon v_1 + \epsilon^2 v_2 + h.o.t. \qquad \psi = \psi_0(x) + \epsilon\psi_1 + h.o.t.$$

$\mathcal{O}(\epsilon^0)$:

$$\partial_x^2\psi_0 + \partial_\psi\hat{V}(\psi_0;0) = 0$$

yields the equation for the stationary front

$\mathcal{O}(\epsilon)$:

$$\underbrace{\partial_x^2\psi_1 + \partial_\psi^2\hat{V}(\psi_0;0)\Big|}_{\mathcal{L}}\psi_1 = -v_1\partial_x\psi_0 - \lambda_1\,\partial_\lambda\partial_\psi\hat{V}(\psi;\lambda)_{\lambda=0,\psi=\psi_0} \tag{108}$$

Can we invert the operator $\mathcal{L}$ and solve directly for ψ_1?
The system is invariant under spatial translations: take the x-derivative of the equation at $\mathcal{O}(\epsilon^0)$:

$$\partial_x\left[\partial_x^2\psi_0 + \partial_\psi\hat{V}(\psi;0)\right] = \partial_x^2\partial_x\psi_0 + \partial_\psi^2\hat{V}(\psi;0)\Big|_{\psi=\psi_0}\partial_x\psi_0 = \mathcal{L}\,\partial_x\psi_0$$

Since $\psi_0(x)$ breaks the continuous translation symmetry $\partial_x\psi_0$ does not vanish and is a proper eigenvector of $\mathcal{L}$ with eigenvalue 0.
Thus, $\mathcal{L}$ is singular and the eigenvector associated with the 0 eigenvalue is the translation mode $\partial_x\psi_0$.

Note:

- if ψ_0 did not break the translation symmetry, $\partial_x\psi_0$ would vanish and not represent an eigenvector and there would be no 0 eigenvalue associated with the translation symmetry and $\mathcal{L}$ could be invertible.

$\mathcal{L}$ is self-adjoint $\Rightarrow \partial_x\psi_0$ is also its left 0-eigenvector.

Project (108) on $\partial_x\psi_0$

$$0 = \int_{-\infty}^{+\infty} \partial_x\psi_0 \left[-v_1\partial_x\psi_0 - \lambda_1\partial_\lambda\partial_\psi \, \hat{V}(\psi;\lambda)\Big|_{\lambda=0,\psi=\psi_0} \right] dx$$

$$v_1 \int_{-\infty}^{\infty} (\partial_x\psi_0)^2 \, dx = -\lambda_1\partial_\lambda \int_{-\infty}^{+\infty} \underbrace{\partial_\psi\hat{V}(\psi_0;\lambda)\,\partial_x\psi_0}_{\partial_x\hat{V}(\psi_0(x);\lambda)} \, dx$$

Thus

$$v_1 \int_{-\infty}^{\infty} (\partial_x\psi_0)^2 \, dx = -\lambda_1 \, \partial_\lambda \left[V(\psi_0(x);\lambda)\right]|_{\lambda=0}|_{x=-\infty}^{+\infty} \approx - \, V(\psi_0(x);\lambda_1)|_{x=-\infty}^{+\infty} \tag{109}$$

Notes:

- The l.h.s of the equation can be read as the amount of work performed by the friction

$$\int_{-\infty}^{+\infty} \left(\beta\frac{dx}{dt}\right) \underbrace{\frac{dx}{dt}dt}_{dx}$$

- The r.h.s of the equation can be read as the difference in potential energy between initial and final state
- Important: the perturbation method does not rely on the existence of a potential $\Rightarrow$ it works also when there are multiple coupled components $\psi_j(x,t)$ satisfying nonlinear PDEs that cannot be derived from a potential and the front motion cannot be interpreted in terms work performed and change in the potential.

12.2 Interaction between Fronts

Consider fronts of the nonlinear diffusion equation

$$\partial_t\psi = \partial_x^2\psi - \psi + c\psi^3 - \psi^5 \tag{110}$$

Notes:

- the coefficients of $\partial_x^2\psi$, ψ, and of ψ^5 can be chosen to have magnitude 1 by rescaling of space, time and ψ.
- the coefficient of ψ is chosen negative: $\psi = 0$ is linearly stable
- the coefficient of ψ^5 is chosen negative: saturation at large values of ψ

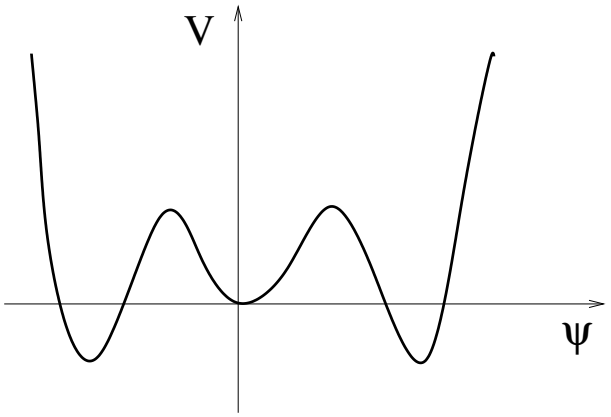


Figure 35: Potential with minima at ψ_0 and $\pm\psi_0$.

Homogeneous stationary states:

linearly stable

$$\psi = 0 \qquad \text{or} \qquad \psi_0^2 = \frac{c + \sqrt{c^2 - 4}}{2}$$

linearly unstable

$$\psi_u^2 = \frac{c - \sqrt{c^2 - 4}}{2}$$

Consider two fronts that connect $\psi = 0$ with $\psi = \psi_0$

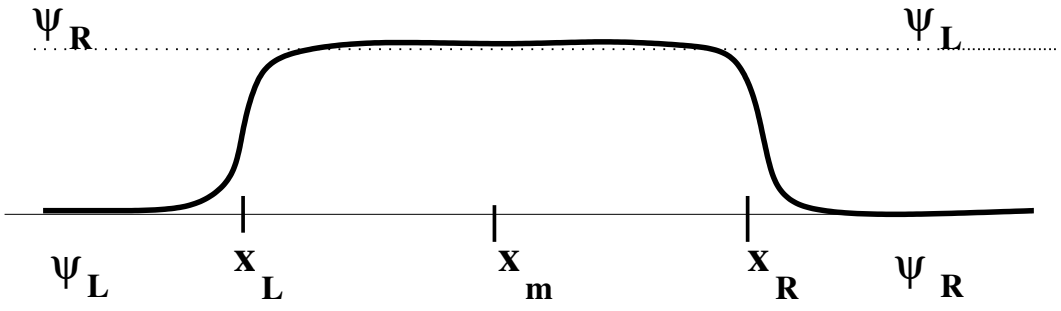


Figure 36: Front positions.

Goal:
Obtain evolution equations for the positions x_L and x_R. These equations would describe the interaction between the two fronts and reduce the PDE to coupled ODEs.

For such a reduction we need a separation of time scales

- The relaxation of ψ to the equilibrium values ψ_0 and $\psi = 0$ should be much faster than the motion of the individual fronts.
- For the motion to be slow in the presence of the interaction between the fronts the interaction must be weak:
  consider widely separated fronts, $x_R - x_L$ large, then the fronts deform each other only weakly.

Note:

- Since the fronts approach their asymptotic value ex2frontxlxr.fig 2frontxlxr.pdfponentially fast, it turns out that the interaction is exponentially weak in the distance between the fronts.

Ansatz:

$$\psi = \psi_L + \psi_R \qquad \underbrace{-\psi_0}_{\text{subtract common part}} \qquad +\epsilon\psi_1 + \dots$$

with

$$\psi_L = \psi_F\left(x - x_L(T)\right) \qquad \psi_R = \psi_F\left(x_R(T) - x\right)$$

$$T = \epsilon t \qquad c = c_0 + \epsilon c_1 \qquad c_0 = \frac{4}{\sqrt{3}}$$

$$\psi_F(\zeta) = \psi_0\sqrt{\frac{1}{2}\left(1 + \tanh\zeta\right)} \qquad \psi_0 = 3^{\frac{1}{4}}$$

Notes:

- c_0 can be determined as the point where the potential $V(\psi) = +\frac{1}{2}\psi^2 - \frac{1}{4}c\psi^4 + \frac{1}{6}\psi^6$ has the same value at the two minima $\psi = 0$ and $\psi = \psi_0$ (cf. Sec.12.1.1).
- The scaling of T relative to $c - c_0$ can be gleaned from (109).

Denote

$$\psi_L' \equiv \left.\frac{d\psi_F(\xi_L)}{d\xi_L}\right|_{\xi_L = x - x_L} \qquad \psi_R' \equiv \left.\frac{d\psi_F(\xi_R)}{d\xi_R}\right|_{\xi_R = x_R - x}$$

Then

$$\partial_t\psi_L = \psi_L'\frac{d\xi_L}{dt} = -\epsilon\psi_L'\partial_T x_L \qquad \partial_t\psi_R = \psi_R'\frac{d\xi_R}{dt} = +\epsilon\psi_R'\partial_T x_R$$

Note: $\frac{d\psi_L}{dx} = \psi_L'$, but $\frac{d\psi_R}{dx} = -\psi_R'$.

Insert expansion

$$\begin{aligned} 0 \;=\; & \epsilon\left\{\psi_L'\,\partial_T x_L - \psi_R'\,\partial_T x_R\right\} + \psi_L'' + \psi_R'' + \epsilon\psi_1'' - \psi_L - \psi_R + \psi_0 - \epsilon\psi_1 \\ & + (c_0 + \epsilon c_1)\left\{\psi_L + \psi_R - \psi_0 + \epsilon\psi_1\right\}^3 - \left\{\psi_L + \psi_R - \psi_0 + \epsilon\psi_1\right\}^5 \end{aligned}$$

For $x < x_m \equiv \frac{1}{2}(x_L + x_R)$ we have $x_R - x \gg 1$:

$$\begin{aligned}
\psi_R - \psi_0 &= \psi_0 \left\{ \sqrt{\frac{1}{2}\left(1 + \tanh\left(x_R - x\right)\right)} - 1 \right\} \\
&= \psi_0 \left\{ \sqrt{\frac{1}{2}\frac{e^{x_R - x} + e^{x - x_R} + e^{x_R - x} - e^{x - x_R}}{e^{x_R - x} + e^{x - x_R}}} - 1 \right\} \\
&= \psi_0 \left\{ \frac{1}{\sqrt{1 + e^{-2(x_R - x)}}} - 1 \right\} \to -\frac{1}{2}\psi_0 e^{-2(x_R - x)} \qquad \text{for} \quad x_R - x \to \infty
\end{aligned}$$

Analogously for $x > x_m$ we have $x - x_L \gg 1$:

$$\psi_L - \psi_0 \to -\frac{1}{2}\psi_0 e^{-2(x - x_L)}$$

Consider now the expansion separately for $x < x_m$ and $x > x_m$.

For $x < x_m$:

$$\left\{\psi_L + (\psi_R - \psi_0) + \epsilon\psi_1\right\}^3 = \psi_L^3 + 3\psi_L^2\left(\psi_R - \psi_0\right) + +3\epsilon\psi_1\psi_L^2 + \mathcal{O}\left(\left(\psi_R - \psi_0\right)^2, \epsilon\left(\psi_R - \psi_0\right), \epsilon^2\right)$$

and

$$\left\{\psi_L + (\psi_R - \psi_0) + \epsilon\psi_1\right\}^5 = \psi_L^5 + 5\psi_L^4\left(\psi_R - \psi_0\right) + +5\epsilon\psi_1\psi_L^4 + \mathcal{O}\left(\left(\psi_R - \psi_0\right)^2, \epsilon\left(\psi_R - \psi_0\right), \epsilon^2\right)$$

Using that $\psi_{L,R}$ satisfy the $\mathcal{O}(\epsilon^0)$ equations

$$\psi_L'' - \psi_L + c_0\psi_L^3 - \psi_L^5 = 0 \qquad \psi_R'' - \psi_R + c_0\psi_R^3 - \psi_R^5 = 0$$

we get for $x < x_m$

$$\begin{aligned}
-\epsilon \underbrace{\left\{\psi_1'' - \psi_1 + 3c_0\psi_L^2\psi_1 - 5\psi_L^4\psi_1\right\}}_{\mathcal{L}_L\psi_1} &= \psi_R'' + \left(\psi_R - \psi_0\right)\left\{-1 + 3c_0\psi_L^2 - 5\psi_L^4\right\} + \\
&\quad + \epsilon\left\{c_1\psi_L^3 + \partial_T x_L\psi_L' - \partial_T x_R\psi_R'\right\}
\end{aligned}$$

with

$$\mathcal{L}_L = \partial_x^2 - 1 + 3c_0\psi_L^2 - 5\psi_L^4$$

For $x > x_m$

$$\begin{aligned}
-\epsilon\mathcal{L}_{\mathcal{R}}\psi_1 &= \psi_L'' + \left(\psi_L - \psi_0\right)\left\{-1 + 3c_0\psi_R^2 - 5\psi_R^4\right\} + \\
&\quad + \epsilon\left\{c_1\psi_R^3 + \partial_T x_L\psi_L' - \partial_T x_R\psi_R'\right\}
\end{aligned}$$

with

$$\mathcal{L}_R = \partial_x^2 - 1 + 3c_0\psi_R^2 - 5\psi_R^4$$

To obtain evolution equations for x_L and x_R we need *two solvability conditions*

Translation symmetry:

- single front: $\partial_x\psi_{L,R}$ is a 0-eigenvector

- two interacting fronts: there is only one exactly vanishing eigenvalue with the eigenvector arising from the double-front solution $\partial_x \psi_{LR}$, for which the $x_R - x_L$ is not growing or shrinking

Note:

- the double-front solution is stationary for c slightly different than c_0 due to the interaction between the two fronts.

How do we get a second solvability condition?

We want the perturbation expansion to remain well-ordered in the limit $x_R - x_L \to \infty$. Thus, even if the fronts are infinitely far apart, we want ψ_1 to remain small compared to $\psi_L + \psi_R - \psi_0$

- for any finite L: only 1 translation mode, which is (approximately) $\partial_x (\psi_L + \psi_R - \psi_0)$
- for $L = \infty$: 2 independent fronts $\Rightarrow$ expect 2 translation modes

$$\mathcal{L}_L \partial_x \psi_L = 0 \qquad \mathcal{L}_R \partial_x \psi_R = 0$$

Project in the two domains $x < x_m$ and $x > x_m$ separately onto the two translation modes $\partial_x \psi_{L,R}$, respectively.

$x < x_m$:

$$\begin{aligned} -\epsilon \int_{-\infty}^{x_m} \psi_L' \mathcal{L}_L \psi_1 dx &= \epsilon \partial_T x_L \int_{-\infty}^{x_m} \psi_L'^2 dx - \epsilon \partial_T x_R \int_{-\infty}^{x_m} \psi_L' \psi_R' dx + \int_{-\infty}^{x_m} \psi_L' \psi_R'' dx + \\ &+ \int_{-\infty}^{x_m} \psi_L' \left\{ -1 + 3c_0 \psi_L^2 - 5\psi_L^4 \right\} (\psi_R - \psi_0)\, dx + \epsilon c_1 \int_{-\infty}^{x_m} \psi_L' \psi_L^3 dx \end{aligned}$$

For $x_m \to \infty$ the operator $\mathcal{L}_L$ is self-adjoint and we could roll it over to ψ_L' and the l.h.s. would vanish. For finite x_m boundary terms arise.
Integrate the l.h.s by parts

$$\int_{-\infty}^{x_m} \psi_L' \mathcal{L}_L \psi_1 dx = \psi_L' \psi_1' \big|_{-\infty}^{x_m} - \underbrace{\int_{-\infty}^{x_m} \psi_L'' \psi_1' dx}_{\psi_L'' \psi_1 \big|_{\infty}^{x_m} - \int_{-\infty}^{x_m} \partial_x^2 (\partial_x \psi_L) \psi_1 dx} + \int_{-\infty}^{x_m} \psi_L' \left\{ -\psi_1 + 3c_0 \psi_L^2 \psi_1 - 5\psi_L^4 \psi_1 \right\} dx$$

ψ_L' and ψ_L'' are exponentially small at x_m and for $x \to -\infty \Rightarrow$ boundary terms are exponentially small and can be ignored at this order since they are already multiplied by ϵ. What remains after the integration by parts is $\psi_1 \mathcal{L}_L \partial_x \psi_L$, which vanishes since $\mathcal{L}_L \partial_x \psi_L = 0 \Rightarrow$ we obtain a solvability condition.

To estimate and evaluate the integrals rewrite in terms of

$$s = e^{x - x_L} \qquad \text{and} \qquad L = x_R - x_L$$

$$dx = \frac{1}{s} ds \qquad \int_{-\infty}^{x_m} \ldots dx = \int_0^{e^{\frac{L}{2}}} \ldots \frac{1}{s} ds$$

$$\psi_L = \psi_0\sqrt{\frac{1}{2}\left(1+\frac{s-\frac{1}{s}}{s+\frac{1}{s}}\right)} = \psi_0\sqrt{\frac{1}{2}\frac{1+s^2+s^2-1}{1+s^2}} = \psi_0\frac{s}{\sqrt{1+s^2}}$$

$$\psi_R - \psi_0 \to -\frac{1}{2}\psi_0 e^{-2(x_R-x)} = -\frac{1}{2}\psi_0 e^{-2x_R}e^{2(x-x_L)}e^{2x_L} = -\frac{1}{2}\psi_0 e^{-2L}\, s^2$$

$$\psi_R' \to -\psi_0 s\frac{ds}{dx}e^{-2L} = -\psi_0 s^2 e^{-2L} \qquad \psi_R'' \to -2\psi_0 s^2 e^{-2L}$$

$$\partial_s\psi_L = \psi_0\left\{\frac{1}{\sqrt{1+s^2}} - \frac{s^2}{\sqrt{1+s^2}^3}\right\} = \psi_0\frac{1}{\sqrt{1+s^2}^3} \quad \Rightarrow \quad \psi_L' = \psi_0\frac{s}{\sqrt{1+s^2}^3}$$

We get then

$$\begin{aligned} 0 \;=\; & \epsilon\partial_T x_L\psi_0^2\int_0^{e^{\frac{L}{2}}}\frac{s^2}{(1+s^2)^3}\frac{1}{s}ds + \epsilon\partial_T x_R\psi_0^2\int_0^{e^{\frac{L}{2}}}\frac{s}{\sqrt{1+s^2}^3}\left(-s^2\right)e^{-2L}\frac{1}{s}ds + \\ & +\psi_0^2\int_0^{e^{\frac{L}{2}}}\frac{s}{\sqrt{1+s^2}^3}\left(-2s^2\right)e^{-2L}\frac{1}{s}ds + \\ & +\psi_0^2\int_0^{e^{\frac{L}{2}}}\frac{s}{\sqrt{1+s^2}^3}\left\{-1+3c_0\psi_0^2\frac{s^2}{1+s^2}-5\psi_0^4\frac{s^4}{(1+s^2)^2}\right\}\frac{-s^2}{2}e^{-2L}\frac{1}{s}ds \\ & +\epsilon c_1\frac{1}{4}\underbrace{\psi_L^4\big|_{x=-\infty}^{x=x_m}}_{\psi_0^4+h.o.t.} \end{aligned} \tag{111}$$

Analogously for $x > x_m$:

$$\begin{aligned} 0 \;=\; & \epsilon\partial_T x_L\int_{x_m}^{\infty}\psi_L'\psi_R'dx - \epsilon\partial_T x_R\int_{x_m}^{\infty}\psi_R'^2dx + \int_{x_m}^{\infty}\psi_R'\psi_L''dx + \\ & +\int_{x_m}^{\infty}\psi_R'\left\{-1+3c_0\psi_R^2-5\psi_R^4\right\}(\psi_L-\psi_0)\,dx + \epsilon c_1\int_{x_m}^{\infty}\psi_R'\psi_R^3dx \end{aligned} \tag{112}$$

Rewrite these integrals in terms of

$$u = e^{x_R-x}$$

$$dx = -\frac{1}{u}du \quad \frac{du}{dx} = -u \qquad \psi_R = \psi_0\frac{u}{\sqrt{1+u^2}} \qquad \partial_u\psi_R = \psi_0\frac{1}{\sqrt{1+u^2}^3}$$

$$\psi_R' = -\frac{d\psi_R}{dx} = u\frac{d\psi_R}{du} = \psi_0\frac{u}{\sqrt{1+u^2}^3}$$

$$\psi_L - \psi_0 \to -\frac{1}{2}\psi_0 u^2 e^{-2L} \qquad \psi_L' \to \psi_0 u^2 e^{-2L} \qquad \psi_L'' \to -2\psi_0 u^2 e^{-2L}$$

Since

$$\int_{x_m}^{\infty}\ldots dx \to \int_{e^{x_R-x_m}}^{0}\ldots\left(-\frac{1}{u}\right)du = \int_0^{e^{\frac{L}{2}}}\ldots\frac{1}{u}du$$

each integral in the expression for $x > x_m$ has a corresponding integral for $x < x_m$ and their magnitudes are the same.

Add (112) and (111)

$$\begin{aligned}
0 &= \epsilon\left(\partial_T x_L - \partial_T x_R\right)\psi_0^2\left\{\int_0^{e^{\frac{L}{2}}}\frac{s^2}{\left(1+s^2\right)^3}\frac{1}{s}ds - \int_0^{e^{\frac{L}{2}}}\frac{s}{\sqrt{1+s^2}^3}\left(-s^2\right)e^{-2L}\frac{1}{s}ds\right\} + \\
&+2e^{-2L}\psi_0^2\int_0^{e^{\frac{L}{2}}}\frac{s}{\sqrt{1+s^2}^3}\left(-2s^2\right)\frac{1}{s}ds + \\
&+2e^{-2L}\psi_0^2\int_0^{e^{\frac{L}{2}}}\frac{s}{\sqrt{1+s^2}^3}\left\{-1+3c_0\psi_0^2\frac{s^2}{1+s^2}-5\psi_0^4\frac{s^4}{\left(1+s^2\right)^2}\right\}\frac{-s^2}{2}\frac{1}{s}ds + \\
&+\epsilon c_1\frac{1}{2}\psi_0^4
\end{aligned}$$

For large s all integrands decay at least as $\frac{1}{s} \Rightarrow$ the integrals are at most $\mathcal{O}(\ln s) = \mathcal{O}(L)$:

- We can therefore ignore the second integral with respect to the first integral in the first term
- ϵ must be exponentially small in L to balance the terms

Relevant integrals:

$$\begin{aligned}
\int_0^{e^{\frac{L}{2}}}\frac{s}{\left(1+s^2\right)^3}ds &= \frac{1}{4}\left(1-\frac{1}{\left(1+e^L\right)^2}\right) \\
\int_0^{e^{\frac{L}{2}}}\frac{s^2}{\sqrt{1+s^2}^3}ds &= \frac{1}{2}L+\ln 2-1+\mathcal{O}(e^{-L}) \\
\int_0^{e^{\frac{L}{2}}}\frac{s^4}{\sqrt{1+s^2}^3}ds &= \frac{1}{2}L+\ln 2-\frac{4}{3}+\mathcal{O}(e^{-L}) \\
\int_0^{e^{\frac{L}{2}}}\frac{s^6}{\sqrt{1+s^2}^3}ds &= \frac{1}{2}L+\ln 2-\frac{23}{15}+\mathcal{O}(e^{-L})
\end{aligned}$$

Thus

$$\partial_T L = -16\frac{e^{-2L}}{\epsilon} + 2\sqrt{3}c_1 \tag{113}$$

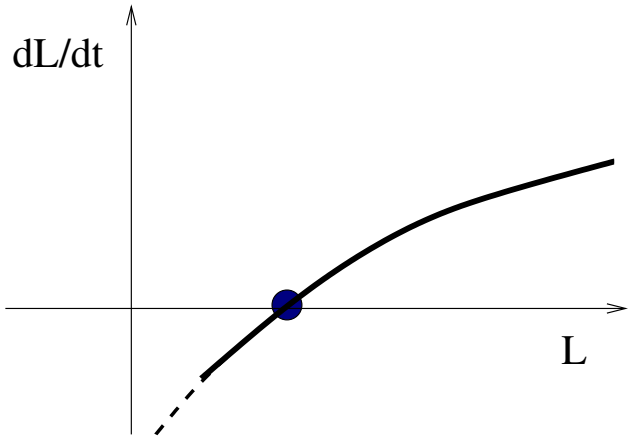


Figure 37: Dependence of growth rate of domain on domain size L.

Notes:

- Interaction
  - attactive $\Rightarrow$ fixed point $L = L_0$ for $c > c_0$, i.e. without interaction the fronts would be drifting apart
  - decays with distance
  - $\Rightarrow$ localized state is *unstable*:
    for $L > L_0$ the attraction is insufficient and the fronts drift apart
    for $L < L_0$ the attraction is too strong and the fronts annihilate each other.
- This localized state corresponds to a *critical droplet* in a first-order phase transition
  - $\psi = 0$ corresponds to the gas phase, say, and $\psi = \psi_0$ to the liquid phase
  - $L = 0$ corresponds to a pure gas phase, $L \to \infty$ to a pure liquid phase.
  - the localized state separates these two stable phases $\Rightarrow$ if there is only one such localized state it has to be unstable.
- The interaction between the fronts is exponential and *monotonic*
- In a more general system the interaction could be non-monotonic, e.g.,

$$\frac{dL}{dt} = a \cos \kappa L \, e^{-\alpha L} + bc_1$$

  then there are multiple localized states, alternating stable and unstable

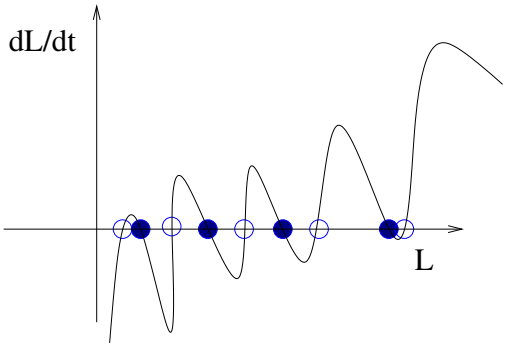


Figure 38: Oscillatory interaction between fronts would allow multiple localized states, stable and unstable.

For oscillatory interaction fronts can 'lock' into each other at multiple positions and arrays of fronts can actually be spatially chaotic (Coullet et al., 1987). This occurs, for instance, in a nonlinear diffusion equation that includes fourth spatial derivatives,

$$\partial_t \psi = \mu \partial_x^2 \psi - \partial_x^4 \psi - \psi - \psi^3 \,.$$

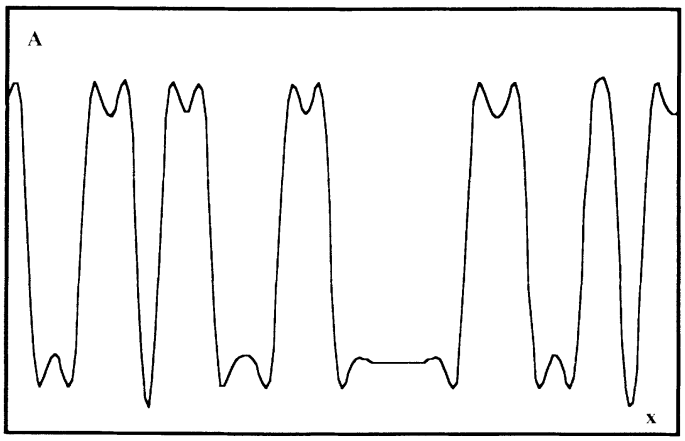


Figure 39: Spatially chaotic array of fronts and 'anti-fronts' connecting the states $\psi = \pm 1$ (Coullet et al., 1987).

- Localized states stabilized by the oscillary interaction between fronts arise in the subcritical Swift-Hohenberg equation

$$\partial_t \psi = R\psi - \left(\partial_x^2 + 1\right)^2 \psi + \psi^3 - \psi^5 \,. \tag{114}$$

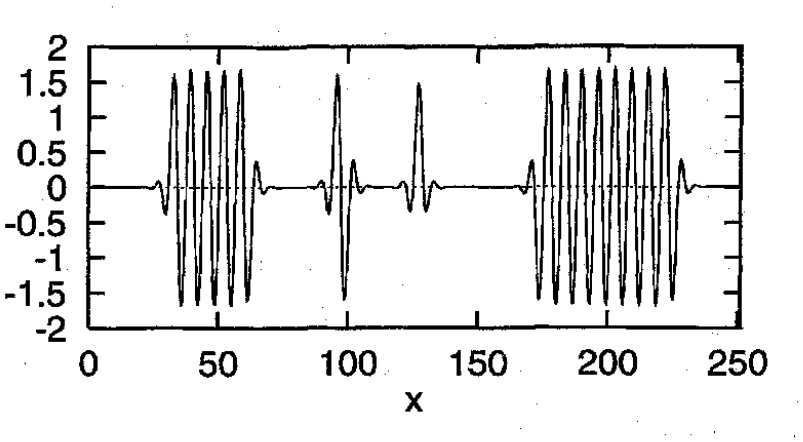


Figure 40: Localized states of various lengths in the subcritical Swift-Hohenberg equation (Sakaguchi and Brand, 1996)

The oscillatory interaction arises from the interaction between the slow modulation of the pattern amplitude that constitutes the front and the fast oscillations of the underlying pattern.

- The oscillatory interaction does not arise in the real Ginzburg-Landau equation; it is beyond all orders (exponentially small) in the multi-scale expansion leading to the Ginzburg-Landau equation. Advanced asymptotic techniques can capture this interaction and show the complexity of the bifurcation diagrams (Chapman and Kozyreff, 2009).

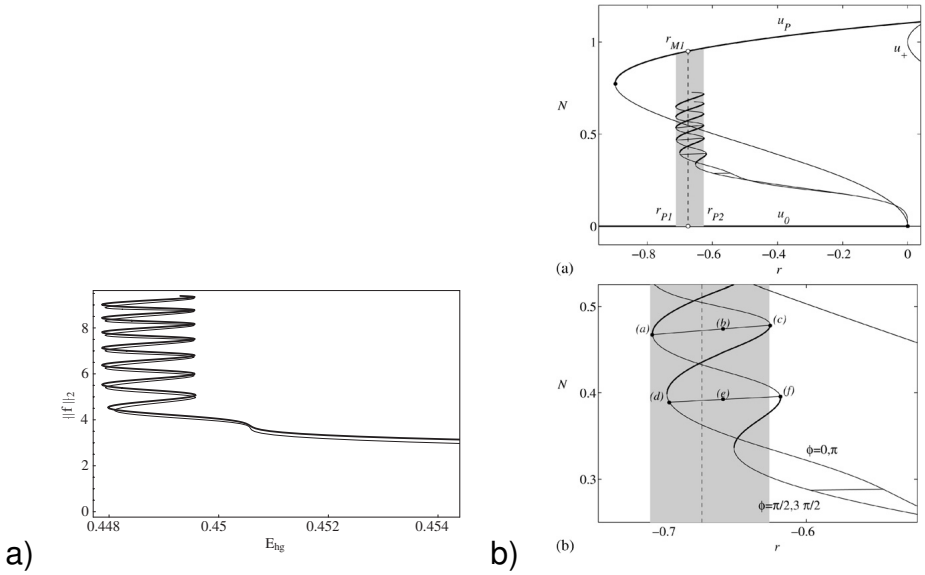


Figure 41: a) Analytical prediction (thick) compared with numerical results (thin) for the snaking bifurcations in (114) (Kozyreff and Chapman, 2006). b) Numerically determined bfurcation diagram for (114) showing the relationship between the periodic and localized solutions (Burke and Knobloch, 2007).

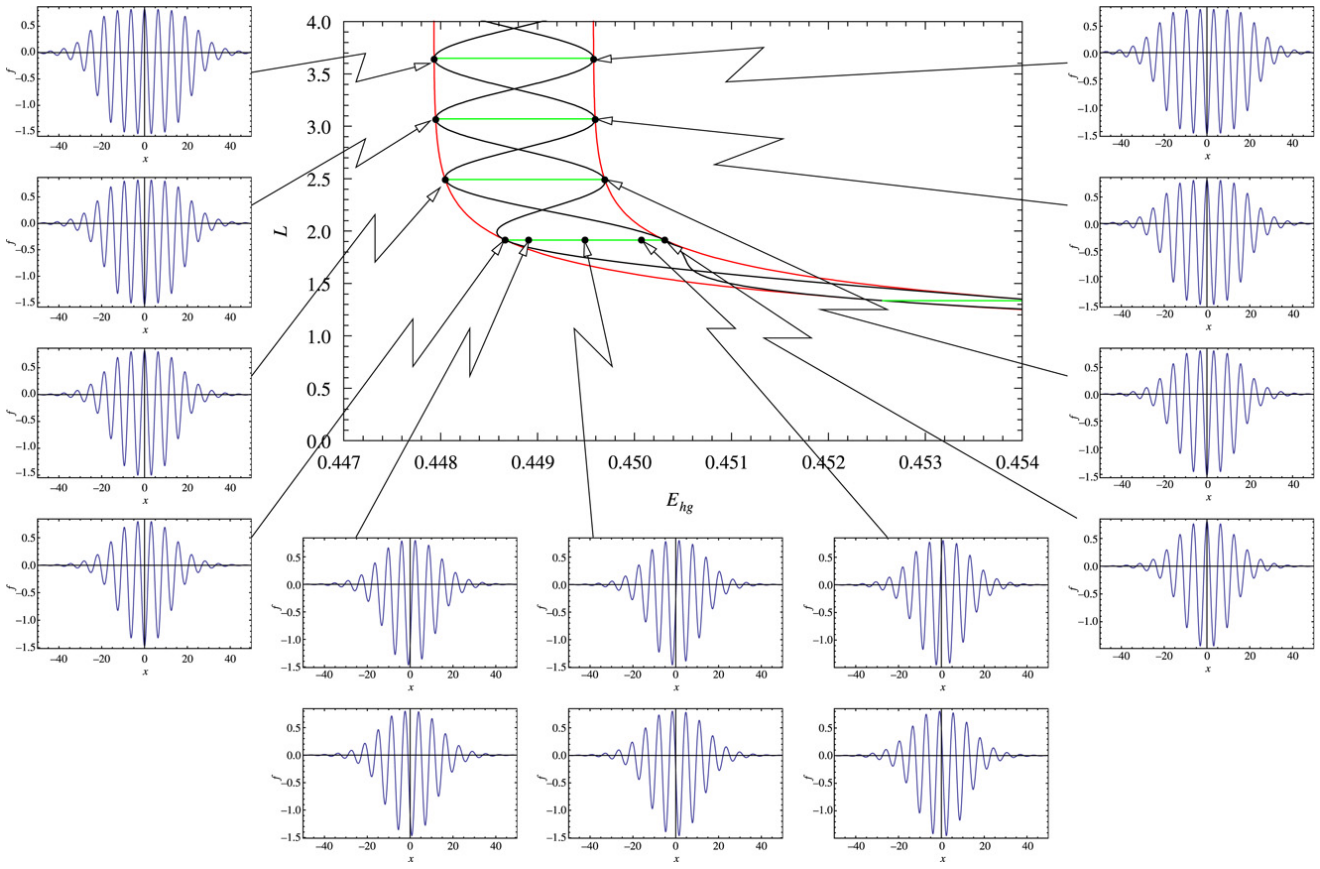


Figure 42: Relationship between localized solutions of (114) on the various bifurcation branches, obtained by asymptotic analysis (Chapman and Kozyreff, 2009).

- A non-monotonic interaction does arise in the complex Ginzburg-Landau equation.
  - There the front velocity depends on the wavenumber of the waves, which in turn is selected by the front and modified by the interaction between the fronts. For weak dispersion perturbation theory gives stable localized waves (Malomed and Nepomnyashchy, 1990).
  - For strong dispersion localized waves can be obtained via a perturbation analysis of the solitons of the nonlinear Schrödinger equation (Fauve and Thual, 1990) (cf. Sec.13).

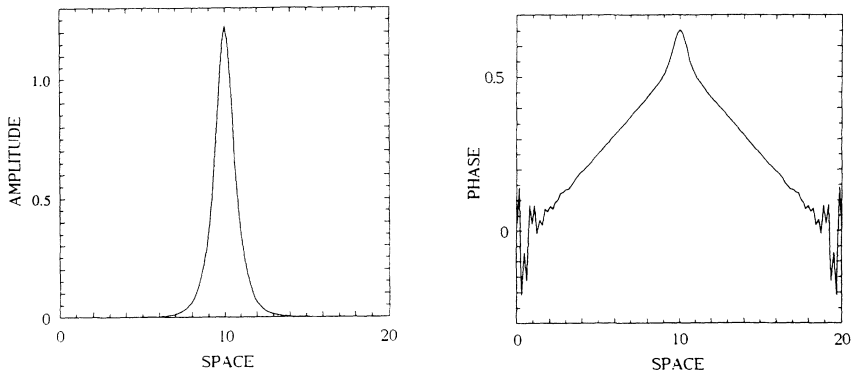


Figure 43: Dissipatively perturbed NLS-soliton. a) Amplitude. b) Phase, displaying the opposite wavenumbers selected by the two 'fronts' and the transition layer in the center (Fauve and Thual, 1990) (cf. (Malomed and Nepomnyashchy, 1990)).

In either case there is only a single stable localized state for given parameters (no snaking).

13 Nonlinear Schrödinger Equation

Consider oscillations in a nonlinear conservative system, i.e. a system without dissipation.

Classic example: pendulum of length L without damping:

$$\partial_t^2\psi = -\omega_0^2 \sin\psi \qquad \text{with} \qquad \omega = \frac{g}{L}$$

More generally, the right-hand side could be any function $f(\psi)$ (with $f(0) = 0$)

Consider nonlinear oscillations in a continuum (many coupled pendula)

$$\partial_t^2\psi - c^2\partial_x^2\psi + \omega_0^2 \sin\psi = 0 \tag{115}$$

Note: this nonlinear equation is called the sine-Gordon equation in analogy to the linear Klein-Gordon equation

$$\partial_t^2\psi - c^2\partial_x^2\psi + \omega_0^2\psi = 0$$

The Klein-Gordon equation allows simple traveling waves

$$\psi = Ae^{iqx-i\omega t} + A^*e^{-iqx+i\omega t} \qquad \text{with} \qquad \omega^2 = \omega_0^2 + c^2q^2$$

Weakly nonlinear regime: expect traveling waves with slightly different frequency and slightly different wave form.

Aim: weakly nonlinear theory for such waves that allows also spatially slow modulations of the waves, e.g. spatially varying wavenumbers or *wave packets*

Note:

- expect similarities between these traveling waves and the waves arising from a Hopf bifurcation, i.e. expect analogy to complex Ginzburg-Landau equation
  $\Rightarrow$ use similar approach and try multiple scales with similar scaling
- **but:** in contrast to the CGL conservative systems *do not* have attractors

Multiple-Scale Analysis:

Right-traveling wave

$$\psi = \epsilon A(X, T_1, T_2, \ldots)e^{iqx-i\omega t} + \epsilon A^*(X, T_1, T_2, \ldots)e^{-iqx+i\omega t} + \epsilon^2\psi_2 + \epsilon^3\psi_3 + \ldots \tag{116}$$

with $T_1 = \epsilon t$, $T_2 = \epsilon^2 t$, $X = \epsilon x$.

For a wave packet one would have $A \to 0$ for $X \to \pm\infty$.

For small ψ we can expand the $\sin\psi = \psi - \frac{1}{6}\psi^3 + \ldots$ and we consider

$$\partial_t^2\psi - c^2\partial_x^2\psi + \omega_0^2\psi = \frac{1}{6}\omega_0^2\psi^3$$

$\mathcal{O}(\epsilon)$: recover linear dispersion relation

$$\underbrace{\partial_t^2\psi_1 - c^2\partial_x^2\psi_1 + \omega_0^2\psi_1}_{\mathcal{L}\psi_1} = 0 \qquad \Rightarrow \qquad \omega^2 = \omega_0^2 + c^2q^2$$

$\Rightarrow \mathcal{L}$ is singular with $\mathcal{L}e^{iqx-i\omega t} = 0$

$\mathcal{O}(\epsilon^2)$:

$$\underbrace{\partial_t^2\psi_2 - c^2\partial_x^2\psi_2 + \omega_0^2\psi_2}_{\mathcal{L}\psi_2} = \left(2i\omega\partial_{T_1}A + 2iq\partial_X A\right)e^{iqx-i\omega t} + c.c.$$

since $\mathcal{L}$ is singular we get a solvability condition:
coefficients of $e^{iqx-i\omega t}$ and $e^{-iqx+i\omega t}$ must vanish

$$2i\omega\partial_{T_1}A + 2iqc^2\partial_X A = 0$$

$$\partial_{T_1}A = -v_g\partial_X A \qquad \text{with} \qquad v_g = \frac{d\omega(q)}{dq} = c^2\frac{q}{\omega}$$

Thus: the wave packet travels to leading order with the group velocity v_g

$$A = A(X - v_gT_1, T_2)$$

Then

$$\psi_2 = 0$$

$\mathcal{O}(\epsilon^3)$:

$$\begin{aligned}\partial_t^2\psi_3 - c^2\partial_x^2\psi_3 + \omega_0^2\psi_3 &= \frac{1}{6}\omega_0^2A^3e^{3iqx-3i\omega t} + \\ &\left(-\partial_{T_1}^2A + 2i\omega\partial_{T_2}A + c^2\partial_X^2A + \frac{1}{6}\,3\omega_0^2|A|^2A\right)e^{iqx-i\omega t} + c.c.\end{aligned}$$

Use the result from $\mathcal{O}(\epsilon^2)$

$$\partial_{T_1}^2A = \partial_{T_1}\left(-v_g\partial_X A\right) = v_g^2\partial_X^2A$$

and get

$$\partial_{T_2}A = \frac{-1}{2i\omega}\left((c^2 - v_g^2)\partial_X^2A + \frac{1}{2}\omega_0^2|A|^2A\right)$$

Consider $d^2\omega/q^2$ using implicit differentiation:

$$\begin{aligned}\omega^2 &= \omega_0^2 + c^2q^2 \\ 2\omega\frac{d\omega}{dq} &= 2c^2q \quad \Rightarrow \quad \frac{d\omega}{dq} = v_g = c^2\frac{q}{\omega} \\ \left(\frac{d\omega}{dq}\right)^2 + \omega\frac{d^2\omega}{dq^2} &= c^2 \quad \Rightarrow \quad \frac{d^2\omega}{dq^2} = \frac{c^2}{\omega^3}\left(\omega^2 - c^2q^2\right) = \frac{c^2}{\omega^3}\left(\omega^2 - \frac{v_g^2\omega^2}{c^2}\right)\end{aligned}$$

Thus:

$$\partial_{T_2}A = i\,\frac{1}{2}\frac{d^2\omega}{dq^2}\,\partial_X^2A + i\,\frac{1}{4}\frac{\omega_0^2}{\omega}|A|^2A$$

Notes:

- this equation is the *nonlinear Schrödinger equation* (NLS)
- the NLS has the same form as the CGL, except that all coefficients are purely imaginary
  real parts of the coefficients correspond to dissipation (i.e. diffusion and approach to an attractor on which the oscillation amplitude is fixed, $R = |A|$)
- the NLS is the generic description for small-amplitude waves in non-dissipative media

Use of Symmetries:

The sine-Gordon equation (115) is invariant under

$$\begin{aligned} \text{reflections in space:} \quad & x \to -x \\ \text{reflections in time:} \quad & t \to -t \\ \text{translations in space:} \quad & x \to x + \Delta x \\ \text{translations in time:} \quad & t \to t + \Delta t \end{aligned}$$

As in the case of the CGL the translations imply that the equation for A is equivariant under $A \to Ae^{i\phi}$ for arbtrary ϕ.

Action of the reflections:

- under spatial reflections (and under reflections in time) a right-traveling wave is transformed into a left-traveling wave
- (116) includes only a right-traveling wave: pure spatial reflections cannot be represented within the class of functions (116)
- *combined* reflections in time and space, however, have a simple action on the amplitude A in the ansatz (116)

$$x \to -x \quad \text{combined with} \quad t \to -t \quad \text{induces} \quad T_2 \to -T_2, \quad X \to -X, \quad A \to A^*$$

Apply to general evolution equation

$$\partial_T A = aA + d\partial_X^2 A + c|A|^2 A \quad \to \quad -\partial_T A^* = aA^* + d\partial_X^2 A^* + c|A|^2 A^*$$

taking the complex conjugate implies

$$-\partial_T A = a^* A + d^* \partial_X^2 A + c^* |A|^2 A$$

Thus:

all coefficients are purely imaginary: $a = -a^*$, $d = -d^*$, $c = -c^*$

13.1 Some Properties of the NLS

Consider the NLS in the form

$$\partial_t \psi = \frac{i}{2}\partial_x^2 \psi + is|\psi|^2\psi \qquad \text{with} \quad s = \pm 1$$

Note:

- the magnitude of the coefficients can be absorbed into the amplitude and the spatial scale
- the overall sign of the r.h.s. can be absorbed by running time backward $t \to -t$
- the relative sign s between $\partial_x^2\psi$ and $|\psi|^2\psi$ cannot be changed by scaling or coordinate transformations
  - $s = +1$: focussing case (spatially homogeneous oscillations linearly unstable, cf. Benjamin-Feir instability of CGL).
  - $s = -1$: defocussing case.

The NLS does not have a Lyapunov functional but is a Hamiltonian system with Hamiltonian (energy) functional

$$\mathcal{H}\{\psi, \psi^*\} = \frac{1}{2}\int |\partial_x\psi|^2 - |\psi|^4 dx$$

i.e.

$$\partial_t \psi = -i\frac{\delta \mathcal{H}\{\psi, \psi^*)}{\delta \psi^*} \tag{117}$$

since using integration by parts and employing the basic property of functional derivatives[11],

$$\frac{\delta\psi(x)}{\delta\psi(x')} = \delta(x - x'),$$

one gets

$$\frac{\delta \mathcal{H}\{\psi, \psi^*)}{\delta \psi^*} = \frac{\delta}{\delta\psi^*}\frac{1}{2}\int -\partial_x^2\psi\,\psi^* - \psi^2\psi^{*2}\,dx = -\frac{1}{2}\partial_x^2\psi - \psi^2\psi^*$$

Note:

- because of the factor i in (117) the energy of the system does not decrease with time as it does in systems with a Lyapunov functional. Instead it is conserved.

Conserved Quantities:

[11] $\psi(x)$ can be thought of as a vector with x labeling its component. The functional derivative is then analogous to $\frac{dv_i}{dv_j} = \delta_{ij}$ for a vector $\mathbf{v} = (v_1, \ldots, v_n)$.

- L_2-norm of ψ: $\mathcal{N} = \int |\psi|^2 dx$

$$\begin{aligned} \frac{d}{dt}\mathcal{N} &= \int \partial_t\psi\,\psi^* + \psi\,\partial_t\psi^* dx = \\ &= \int \left(\frac{i}{2}\partial_x^2\psi + i|\psi|^2\psi\right)\psi^* - \psi\left(\frac{i}{2}\partial_x^2\psi^* + i|\psi|^2\psi^*\right) dx \\ &\underbrace{=}_{\text{integration by parts}} \int \frac{i}{2}\partial_x^2\psi\,\psi^* - \left(\frac{i}{2}\partial_x^2\psi\right)\psi^* dx = 0 \end{aligned} \tag{118}$$

- total energy $\mathcal{H}$
  to compute $\frac{d}{dt}\mathcal{H}$ note that (118) can be written as

$$\int \partial_t\psi\,\psi^* + \psi\partial_t\psi^* dx = \int \partial_t\psi\,\frac{\delta\mathcal{N}\{\psi,\psi^*)}{\delta\psi(x)} + \frac{\delta\mathcal{N}\{\psi,\psi^*)}{\delta\psi^*(x)}\,\partial_t\psi^*\,dx$$

  i.e. think of $\psi(x,t)$ as a vector with components labeled by x, each of which depends on t

$$\begin{aligned} \frac{d}{dt}\mathcal{H} &= \int \frac{\delta\mathcal{H}\{\psi,\psi^*)}{\delta\psi}\,\partial_t\psi + \frac{\delta\mathcal{H}\{\psi,\psi^*)}{\delta\psi^*}\,\partial_t\psi^*\,dx = \\ &= \int -i\partial_t\psi^*\,\partial_t\psi + i\partial_t\psi\,\partial_t\psi^*\,dx = 0 \end{aligned}$$

Significance of conserved quantities:

Example:

- Newton's equation of motion conserves total energy

$$m\frac{d^2}{dt^2}x = F(x) = -\frac{d}{dx}V(x)$$

  multiply by $\frac{d}{dt}x$

$$\begin{aligned} m\frac{d}{dt}x\frac{d^2}{dt^2}x &= -\frac{d}{dt}x\,\frac{d}{dx}V(x) \\ \frac{1}{2}m\frac{d}{dt}\left(\left(\frac{d}{dt}x\right)^2\right) &= -\frac{d}{dt}V(x) \end{aligned}$$

  i.e.

$$\frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2 + V(x)\right) = 0$$

$$\frac{1}{2}m\dot{x}^2 + V(x) = E = const.$$

$$\dot{x} = \sqrt{\frac{2}{m}\left(E - V(x)\right)} \qquad \Rightarrow \quad t = \int \frac{dx}{\sqrt{\frac{2}{m}\left(E - V(x)\right)}}$$

  Thus:

- using energy conservation reduces the order of the differential equation: expresses $\dot{x}$ as a function of x
- solution can be obtained by simple integration (quadrature): the system is called *integrable*
- for two interacting particles $x_1(t)$ and $x_2(t)$ energy conservation alone leads to a single relation between the two velocities

$$\frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 + V(x_1, x_2) = E = const.$$

to express each velocity $\dot{x}_i$ in terms of the positions x_i would need a second equation: a second conserved quantity

- in general: for Newton's equations of motion with N degrees of freedom to be integrable one needs N independent, conserved quantities.

Note:

- Hamiltonian systems with N degress of freedom are integrable if they have N independent conserved quantities.
- the NLS has infinitely many degrees of freedom and *infinitely many* conserved quantities.
  It can be shown to be integrable. Exact solutions can be obtained by the inverse scattering transform (well beyond this class).

13.2 Soliton Solutions of the NLS

For $s = +1$ the NLS has exact localized solution of the form

$$\psi(x,t) = \lambda \frac{1}{\cosh \rho x} e^{i\omega t}$$

Inserting into NLS yields

$$\begin{aligned} \partial_t \psi - \frac{i}{2}\partial_x^2 \psi - si|\psi|^2\psi &= \lambda \left(i\omega \frac{1}{\cosh \rho x} - \frac{i}{2}\rho^2 \frac{\cosh^2 \rho x - 2}{\cosh^3 \rho x} - si\lambda^2 \frac{1}{\cosh^3 \rho x} \right) e^{i\omega t} = \\ &= \lambda \frac{i}{\cosh^3 \rho x} \left(\cosh^2 \rho x \left(\omega - \frac{1}{2}\rho^2 \right) + \rho^2 - s\lambda^2 \right) e^{i\omega t} \end{aligned}$$

Thus we need $s = +1$ and

$$\rho = \lambda \qquad \omega = \frac{1}{2}\lambda^2 \quad \Rightarrow \quad \psi(x,t) = \lambda \frac{1}{\cosh \lambda x} e^{i\frac{1}{2}\lambda^2 t} \tag{119}$$

Note:

- the parameter λ is *arbitrary*: there is a one-parameter family of solutions with different amplitudes and associated different frequencies and width

- the soliton solution is *not* an attractor: small perturbations *do not* relax and the solution does not come back to the unperturbed solution
- in fact, (119) corresponds already to a three-parameter family due to translation symmetry in space and time

NLS also allows transformations into moving frames of reference (*boosts*): $u = x - ct$

Consider

$$\tilde{\psi}(x,t) = \psi(t, x - ct)e^{iqx+i\omega t}$$

where $\psi(t,x)$ is a solution. For $\tilde{\psi}$ to be a solution, as well, q and ω have to satisfy certain conditions. Insert $\tilde{\psi}$ into NLS

$$\begin{aligned}\partial_t\tilde{\psi} - \frac{i}{2}\partial_x^2\tilde{\psi} - si|\tilde{\psi}|\tilde{\psi} &= \left(\partial_t\psi - c\partial_u\psi + i\omega\psi - \frac{i}{2}\left(\partial_u^2\psi + 2iq\partial_u\psi - q^2\psi\right) - si|\psi|^2\psi\right)e^{iqx+i\omega t} = \\ &= \left(\partial_u\psi\left(-c+q\right) + i\psi\left(\omega + \frac{1}{2}q^2\right)\right)e^{iqx+i\omega t}\end{aligned}$$

using that $\psi(t,u)$ satisfies NLS. Require

$$c = q \qquad \omega = -\frac{1}{2}q^2$$

Note:

- the boost velocity c or the background wavenumber q is a free parameter generating a continuous family of solutions

Thus:

The focussing Nonlinear Schrödinger equation has a *four-parameter family of solutions* of solitons

$$\psi(x,t) = \lambda\frac{1}{\cosh\left(\lambda(x - qt - x_0)\right)}e^{iqx+i\frac{1}{2}(\lambda^2-q^2)t+i\phi_0}$$

After a perturbation (change) in any of the four parameter q, λ, x_0, ϕ the solution does not relax back to the unperturbed solution but gets shifted along the corresponding family of solutions.

Note:

- Surprising feature of solitons:
  during collisions solutions become quite complicated, but after the collisions the solitons emerge unperturbed except for a shift in position x_0 and the phase ϕ_o. In particular, the other two parameters, λ and q, are unchanged, although there is also no 'restoring force' to push them back to the values before the collision.
- general solution can be described in terms of a nonlinear superposition of many interacting solitons and periodic waves (captured by inverse scattering theory).

13.3 Perturbed Solitons

Consider soliton-like solutions of the (focussing) NLS with small dissipative perturbations

$$\partial_t \Psi - \frac{i}{2}\partial_x^2 \Psi - i|\Psi|^2\Psi = \epsilon P(\Psi, \partial_x \Psi, \ldots)$$

For small perturbations expect slow evolution along the family of solutions:

$$T = \epsilon t \qquad \lambda = \lambda(T) \qquad q = q(T) \qquad x_0 = x_0(T) \qquad \phi_0 = \phi_0(T)$$

For simplicity: focus on perturbations for which soliton remains stationary: $c = 0$

Slow changes in the frequency $\omega = \frac{1}{2}(\lambda^2 - q^2)$: need to deal with the phase

$$\omega(T) = \frac{d}{dt}\phi$$

Ansatz for the expansion

$$\Psi = \psi(\Theta, T)\, e^{i\phi} = (\psi_0(\Theta, T) + \epsilon\psi_1(\Theta, T) + \ldots)\, e^{i\phi}$$

where

$$\psi_0(\Theta) = \lambda(T)\frac{1}{\cosh\Theta}, \qquad \Theta = \lambda(T)x$$

and

$$\lambda(T) = \lambda_0(T) + \epsilon\lambda_1(T) + \ldots, \qquad \omega(T) = \omega_0(T) + \epsilon\omega_1(T) + \ldots$$

with

$$\omega_0(T) = \frac{1}{2}\lambda_0(T)^2$$

Note:

- the phase ϕ evolves on the $\mathcal{O}(1)$ time scale, $\frac{d}{dt}\phi = \omega$, but the frequency changes on the slow time scale as the solution evolves along the family of solutions
- in the general case ($q \neq 0$) one would have to introduce a spatial phase $\theta(x, t)$ as well

$$\Theta = \lambda(T)\theta(x, t)$$

Rewrite NLS in terms of ψ rather than Ψ

$$i\omega\psi - \frac{i}{2}\lambda^2\partial_\Theta^2\psi - i|\psi|^2\psi = \epsilon\left(-\partial_T\psi + e^{-i\phi}P(\Psi, \partial_x\Psi, \ldots\right)$$

Insert expansion of ψ

$\mathcal{O}(\epsilon^0)$:

$$i\omega_0\psi_0 - \frac{i}{2}\lambda_0^2\partial_\Theta^2\psi_0 - i|\psi_0|^2\psi_0 = 0$$

confirms

$$\omega_0 = \frac{1}{2}\lambda_0^2$$

$\mathcal{O}(\epsilon)$:

$$\mathcal{L}\psi_1 \equiv i\omega_0\psi_1 - \frac{i}{2}\lambda_0^2\partial_\Theta^2\psi_1 - i\left(2|\psi_0|^2\psi_1 + \psi_0^2\psi_1^*\right) = -\partial_T\psi_0 - i\omega_1\psi_0 - i\lambda_0\lambda_1\partial_\Theta^2\psi_0 + e^{-i\phi}P$$

The unperturbed soliton ψ_0 is part of a four-parameter family of solutions,

$$\psi_0 = \psi_0(x, t; x_0, \phi_0, \lambda, q)$$

Taking derivatives of

$$\partial_t\psi_0 = \frac{i}{2}\partial_x^2\psi_0 + is|\psi_0|^2\psi_0$$

with respect to the four parameters shows that $\mathcal{L}$ has four vanishing eigenvalues:

the linear operator $\mathcal{L}$ is singular.

Note:

- these eigenvectors are analogous to translation modes (cf. homework). They arise from breaking of a continuous symmetry of the equation.

To get a solvability condition we need to project the $\mathcal{O}(\epsilon)$-equation onto the relevant left eigenvectors.

Projections need a scalar product. For functions the scalar product typically involves some integral over the domain. Here we can make $i\mathcal{L}$ self-adjoint[12] by a suitable choice of the scalar product. Choose

$$\langle\psi_1, \psi_2\rangle = \Re\left(\int_{-\infty}^{\infty}\psi_1^*\psi_2 d\Theta\right)$$

Then * **missing on** $\psi_1 in$**3rd row. check calculation again**

$$\begin{aligned}
\langle\psi_1, i\mathcal{L}\psi_2\rangle &= \Re\left(\int_{-\infty}^{\infty}\psi_1^* i\mathcal{L}\psi_2 d\Theta\right) = \\
&= \Re\left(\int\psi_1^*\left(-\omega_0\psi_2 + \frac{1}{2}\lambda_0^2\partial_\Theta^2\psi_2 + \left(2|\psi_0|^2\psi_2 + \psi_0^2\psi_2^*\right)\right)d\Theta\right) = \\
&= \Re\left(\int -\omega_0\psi_1^*\psi_2 + \frac{1}{2}\lambda_0^2\partial_\Theta^2\psi_1\,\psi_2 d\Theta\right) + \\
&\quad + \int\psi_{1r}\left(2|\psi_0|^2\psi_{2r} + \psi_0^2\psi_{2r}\right) + \psi_{1i}\left(2|\psi_0|^2\psi_{2i} + \psi_0^2\psi_{2i}\right)d\Theta = \\
&= \Re\left(\int\left(i\mathcal{L}\psi_1\right)\psi_2 d\Theta\right) = \langle i\mathcal{L}\psi_1, \psi_2\rangle
\end{aligned}$$

Thus, with this scalar product the left eigenvectors are identical to the right eigenvectors.

Note:

[12]Note that the terms $-\omega_0$ and $\frac{1}{2}\lambda_0^2\partial_\Theta^2$ are suggestive of $i\mathcal{L}$ being self-adjoint.

- it turns out that two of the 0-eigenvalues of $i\mathcal{L}$ are associated with *proper eigenvectors*, whereas the other two have *generalized eigenvectors*.

$$\Psi_{\phi_0}: \quad i\mathcal{L}i\psi_0 = 0 \qquad \Psi_{x_0}: \quad i\mathcal{L}\partial_\Theta\psi_0 = 0$$
$$\Psi_\lambda: \quad i\mathcal{L}\left(\Theta\partial_\Theta\psi_0 + \psi_0\right) = i\lambda_0^2\psi_0 \qquad \Psi_q: \quad i\mathcal{L}i\Theta\psi_0 = -\lambda_0^2\partial_\Theta\psi_0$$

- of course $\mathcal{L}i\psi_0 = 0$, but $i\psi_0$ is a left-eigenvector of $i\mathcal{L}$ but not of $\mathcal{L}$
- $\left(i\mathcal{L}\right)^2 i\Theta\psi_0 = 0$ and $\left(i\mathcal{L}\right)^2\left(\Theta\partial_\Theta\psi_0 + \psi_0\right) = 0$ as expected of generalized eigenvect

Focus on simple *dissipative* perturbation ($\mu, \alpha, \gamma \in \mathbb{R}$),

$$P(\Psi) = \mu\Psi + \alpha|\Psi|^2\Psi + \gamma|\psi\Psi|^4\Psi \tag{120}$$

Then we need only the eigenvector Ψ_ϕ associated with the phase invariance $\phi \to \phi + \Delta\phi$

$$\Psi_{\phi_0} = i\psi_0$$

Thus using $i\mathcal{L}$, i.e. after multiplying $\mathcal{O}(\epsilon)$-equation by i,

$$\begin{aligned} 0 &= \langle i\psi_0, i\mathcal{L}\psi_1\rangle = \Re\int -i\psi_0\, i\left(-\partial_T\psi_0 - i\omega_1\psi_0 - i\lambda_0\lambda_1\partial_\Theta^2\psi_0 + e^{-i\phi}P\right)d\Theta = \\ &= \int \psi_0\,\left(-\partial_T\psi_0 + \mu\psi_0 + \alpha\psi_0^3 + \gamma\psi_0^5\right)d\Theta \end{aligned}$$

Use $\psi_0 = \lambda\frac{1}{\cosh\Theta}$ and

$$\int\frac{1}{\cosh^2\Theta}d\Theta = 2 \qquad \int\frac{1}{\cosh^4\Theta}d\Theta = \frac{4}{3} \qquad \int\frac{1}{\cosh^6\Theta}d\Theta = \frac{16}{15}$$

to get

$$\frac{d}{dT}\lambda = \mu\lambda + \frac{2}{3}\alpha\lambda^3 + \frac{8}{15}\gamma\lambda^5 \tag{121}$$

Notes:

- the dissipative perturbations P lead to a slow evolution of the amplitude of the perturbed soliton close to the soliton family of solutions: slow manifold
- with increasing amplitude the perturbed soliton becomes narrower
- non-trivial fixed points
  - $\alpha < 0$:
    supercritical pitch-fork bifurcation

    $$\lambda^2 = \frac{3}{2}\frac{\mu}{\alpha} + h.o.t. \qquad \text{if } \mu > 0.$$

    Within the amplitude equation (121) the fixed point is stable.
    However: do not expect this localized soliton-like solution to be stable within the full NLS since $\Psi = 0$ is unstable for $\mu > 0$: perturbations will grow far away from the soliton

  - $\alpha > 0$:
    subcritical pitch-fork bifurcation

$$\lambda_{1,2}^2 = -\frac{5}{8}\frac{\alpha}{\gamma} \pm \frac{15}{16\gamma}\sqrt{\frac{4}{9}\alpha^2 - \frac{32}{15}\mu\gamma}$$

    two soliton-like solutions created in saddle-node bifurcation at $\alpha^2 = \frac{24}{5}\mu\gamma$
    within (121) the one with larger amplitude is stable, the other unstable.
    Background state $\Psi = 0$ is linearly stable for $\mu < 0$.

- full solution consists of four coupled evolution equations for λ, q, x_0, ϕ_0:
  - would have to check that for the perturbation (120) the equations for q, x_0, and ϕ_0 have *stable* fixed points with $q = 0$, $x_0 = const.$ and $\phi_0 = const.$
  - a general perturbation can make soliton travel, $q \neq 0$, $\frac{d}{dT}x_0 \neq 0$.

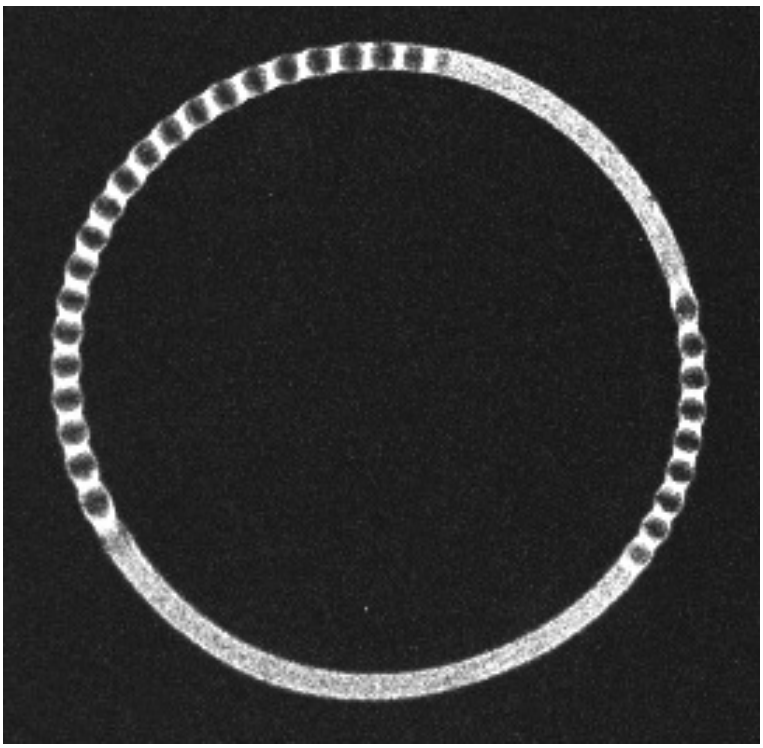
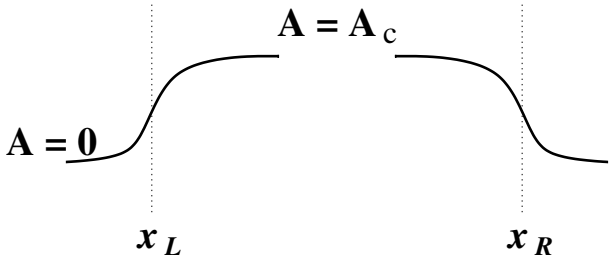


Figure 44: Localized wave trains in convection of water-alcohol mixtures. Top view on annular convection cell. Two localized wave trains are seen. (P. Kolodnder, D. Bensimon, and C.M. Surko, Phys. Rev. Lett. 60 (1988) 1723).

Notes:

- experiments in convection of water-alcohol mixtures: onset of convection via a subcritical Hopf bifurcation
- quintic complex Ginzburg-Landau equation

- for strong dispersion, i.e. large α and β, the complex Ginzburg-Landau equation can be considered as a perturbed NLS: expect localized solutions in the form of perturbed solitons (cf. O. Thual and S. Fauve, J. Phys. (Paris) 49 (1988) 1829).
- for weak dispersion perturbation approach via *interacting fronts* (B.A. Malomed and A.A. Nepomnyashchy, Phys. Rev. A (1990) 6009)
  subcritical bifurcation
  - bistability between conductive state ($\Psi = 0$) and convective state ($\Psi = \Psi_0 \neq 0$)
  - front solutions $\Psi_\pm(x,t) \to 0$ for $x \to \mp\infty$ and $\Psi_\pm(x,t) \to \Psi_0$ for $x \to \pm\infty$
  - fronts Ψ_+ and Ψ_- can interact and form a stable pair: wide localized wave train .



14 Appendix: Review of Some Aspects of 1-d Flows

14.1 Flow on the Line

Any first-order differential equation with constant coefficients,

$$\dot{x} = f(x), \tag{122}$$

can be solved exactly for any $f(x)$ by separation of variables

$$\int_{x_0}^{x} \frac{dx}{f(x)} = t - t_0 \tag{123}$$

Example: $\dot{x} = \sin x$

$$t = \int \frac{dx}{\sin x} = \int \csc x dx \tag{124}$$

$$t = -\ln|\csc x + \cot x| + C \tag{125}$$

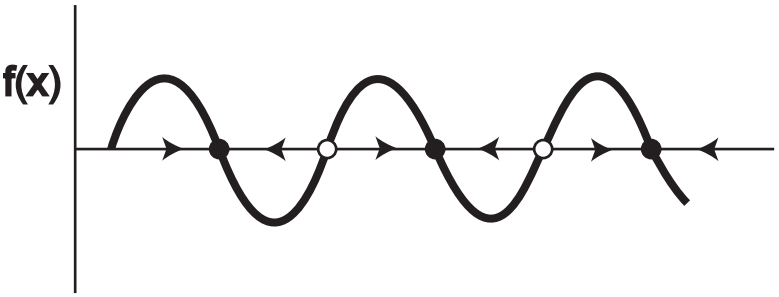
Now what? What have we learned?

Even if we could solve for x, would we have an overview of the behavior of system for arbitrary initial conditions?

Geometrical picture: **phase space** (or phase line in 1 dimension)

$\dot{x} = f(x)$ defines a *flow* in phase space or a *vector field*

For 1d: plot in addition $f(x)$



Conclude: for any initial condition system ends up in one of the **fixed points** at $x_n = (2n+1)\pi$.

Fixed points are **stagnation points** of the flow

Stability of Fixed Points:

Inspection of phase line:

- flow *into* $x_n = (2n+1)\pi$: *stable*

- flow *out of* $x_n = 2n\pi$: *unstable*

Compute linear stability by linearization around fixed point x_0

$$x = x_0 + \epsilon x_1(t) \qquad \epsilon \ll 1$$

insert

$$\frac{d}{dt}\left(x_o + \epsilon x_1\right) = f(x_0 + \epsilon x_1)$$

expand and collect like orders in ϵ

$$\mathcal{O}(\epsilon^0): \qquad 0 = f(x_0)$$

recovers equation for the fixed point x_0

$$\mathcal{O}(\epsilon^1): \qquad \frac{d}{dt}x_1 = \left.\frac{df}{dx}\right|_{x_0} x_1 \equiv \lambda x_1$$

Thus:

$$x_1 = x_1(0)\, e^{\lambda t}$$

$\lambda < 0:$ linearly stable
$\lambda = 0:$ marginally stable
$\lambda > 0:$ (linearly) unstable

In our example:

$$0 = f(x_0) = \sin x_0 \qquad x_0 = n\pi$$

$$\lambda = f'(x_0) = \cos x_0 = \cos n\pi$$

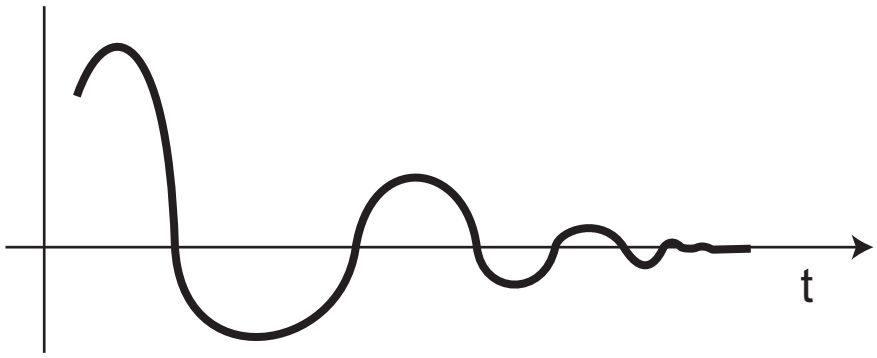
linearly stable: n odd linearly unstable: n even

Notes:

- for coupled systems $f'(x)$ is replaced by Jacobian matrix:
  linear stability determined by eigenvalues of the Jacobian.
- linear stability describes only infinitesimal perturbations
  a linearly stable fixed point can be unstable to perturbations of finite size
  e.g. in example perturbation could push system beyond the unstable fixed point separating the two linearly stable fixed points

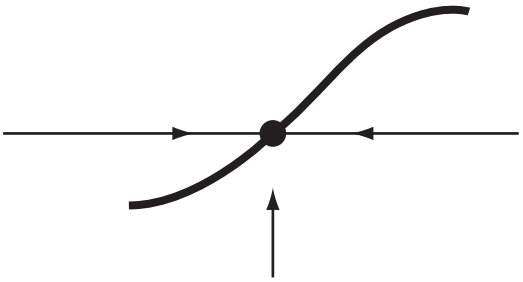
14.1.1 Impossibility of Oscillations:

Can the solution approach fixed point via **oscillations**?



No!

graphically:



to get to other side need to **cross** fixed point

$\rightarrow$ system evolves monotonically between fixed points

More general concept: Potential

In 1 dimension autonomous systems can always be written as

$$\dot{x} = f(x) = -\frac{dV}{dx} \qquad \text{with } V = -\int f(x)dx \tag{126}$$

Consider:

$$\frac{dV}{dt} = \frac{dV}{dx}\dot{x} = -(\frac{dV}{dx})^2 \leq 0 \tag{127}$$

V is non-increasing $\Rightarrow$ V cannot return to previous value

$$\frac{dV}{dt} = 0 \qquad \Rightarrow \qquad \frac{dV}{dx} = 0 \qquad \Rightarrow \qquad \dot{x} = 0 \qquad \text{fixed point} \tag{128}$$

Thus:

- if V is bounded from below x always goes to a fixed point
- the fixed point could be at infinity, e.g. for $V(x) = \frac{1}{1+x^2}$

Note:

- such a V that is bounded from below is also called a *Lyapunov functional* for the flow.

Compare: mechanical system in overdamped limit

$$m\ddot{x} = -\beta\dot{x} + F(x) \tag{129}$$

for very small mass (no inertia)

$$\dot{x} = \frac{1}{\beta}F(x) \tag{130}$$

Overshoot requires inertia, 2nd derivative, i.e. 2-dimensional flow.

Notes:

- 1-dimensional systems: there is always a potential in the autonomous case
- In higher dimensions even autonomous equations may or *may not have a potential*: persistent dynamics like oscillations are possible

14.2 Existence and Uniqueness

So far we assumed we **always** get a **unique** solution for all times:

- at any time 'we know where to go'
- we can continue this forever

Solutions to

$$\dot{x} = f(x) \tag{131}$$

1. do not have to exist for all times:
   for given initial condition solution may cease to exist beyond some time
2. do not have to be unique:
   same initial condition can lead to different states later.

1. Existence

solution can disappear by becoming infinite

if this happens in *finite* time then there is no solution beyond that time

Example:

$$\dot{x} = +x^{\alpha} \qquad \text{with} \qquad x(0) = x_0 > 0 \tag{132}$$

$$\int x^{-\alpha}\,dx = \int \frac{dx}{x^\alpha} = t + C \tag{133}$$

$$\frac{1}{1-\alpha}x^{1-\alpha} = t + C \tag{134}$$

initial conditions:

$$C = \frac{1}{1-\alpha}x_0^{1-\alpha} \tag{135}$$

$$x = \left((1-\alpha)t + x_0^{1-\alpha}\right)^{\frac{1}{1-\alpha}} \tag{136}$$

Solution diverges at

$$t^* = \frac{x_0^{1-\alpha}}{\alpha - 1} \qquad \text{if} \qquad \alpha > 1 \tag{137}$$

i.e. for $\alpha > 1$ divergence in **finite time**.

Note: divergence in infinite time no problem: $x(t) = e^t$

2. Uniqueness

Consider previous example for $0 < \alpha < 1$

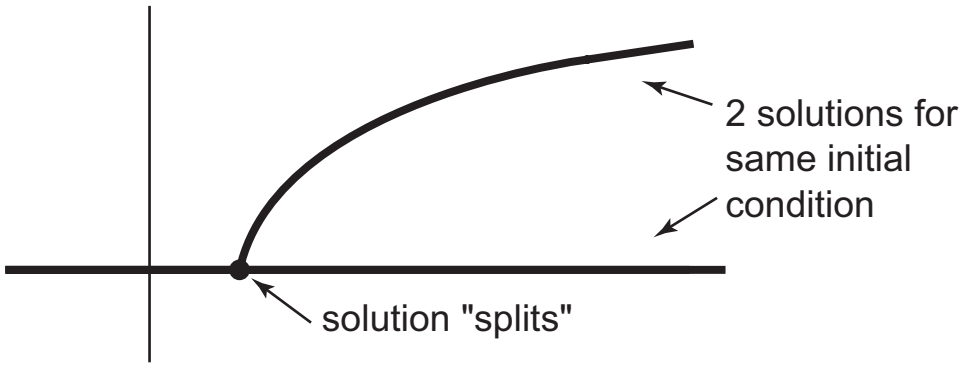
$$\Rightarrow x = 0 \qquad \text{for} \qquad t^* = \frac{x_0^{1-\alpha}}{\alpha - 1} < 0 \tag{138}$$

Solution can start at t^* with $x(t^*) = 0$ and grow from there.

But: $\tilde{x}(t) \equiv 0$ is a solution for all times

$\Rightarrow$ can start with $\tilde{x}(t) = 0$ for $t < t^*$ and 'switch' to $x(t) > 0$ beyond t^*. The combined solution is continuous and satisfies the differential equation.

Thus: two *different* solutions satisfy the *same* initial condition (at t^*).



Worse: t^* depends on x_0

$\Rightarrow$ can pick any t^* and patch the solutions at that t^*

$\Rightarrow$ infinitely many solutions with identical i.c. $x = 0$.

Note: in order to get "across the splitting" need to reach 0 in finite time
(splitting has to be crossed in finite time)

Theorem[13]**:**
If for

$$\dot{x} = f(x,t) \tag{139}$$

- $f(x,t)$ is continuous in $|t - t_0| < \Delta t$ in $|x - x_0| \leq \Delta x$ and has maximum M there, and
- $f(x,t)$ satisfies Lipschitz condition within Δx and Δt:

$$|f(x_1,t) - f(x_2,t)| \leq K\,|x_1 - x_2| \qquad \forall x_1, x_2 \in |x - x_0| \leq \Delta x \tag{140}$$

  with some constant K

then the solution exists for a finite time interval $|t - t_0| \leq \Delta T$ and is unique. The interval is given by

$$\Delta T = \min\left(\Delta t, \frac{\Delta x}{M}\right) \tag{141}$$

Discussion:

$f(x) = |x|^\alpha$ does not satisfy Lipschitz condition at $x = 0$ for $0 < \alpha < 1$:
would need

$$|x|^\alpha \leq Kx \quad *.2in \forall x \text{ near } x = 0 \tag{142}$$

i.e. $K \geq |x|^{\alpha-1} \to \infty$ for $x \to 0$ and $\alpha < 1$

Thus: uniqueness of solution is not guaranteed.

Note: If $f'(x)$ is continuous then $f(x)$ satisfies the Lipschitz condition and the solution is unique.

14.3 Unfolding of Degenerate Bifurcations

For a transcritical or for a pitch-fork bifurcation to occur *multiple conditions* need to be satisfied:

- bifurcation occurs: $\partial_x f|_{x_0,\mu_0} = 0$
- additional coefficients "happen to vanish"
  - because of some symmetry of the original system

[13] **see, e.g., Lin & Segel,** *Mathematics applied to deterministic problems in the natural sciences***, p.57**

– bifurcation is degenerate: the vanishing of the coefficients happens only for special choices of the parameters of the original system

Only the saddle-node bifurcation requires only 1 condition

Saddle-node bifurcation is a **codimension-1 bifurcation**:
in an n-dimensional parameter space the locations where a saddle-node bifurcation occurs form an $n-1$-dimensional hypersurface, since one of the parameters needs to be fixed. That hypersurface has co-dimension 1.

Question: What happens when the additional conditions are **weakly broken**?

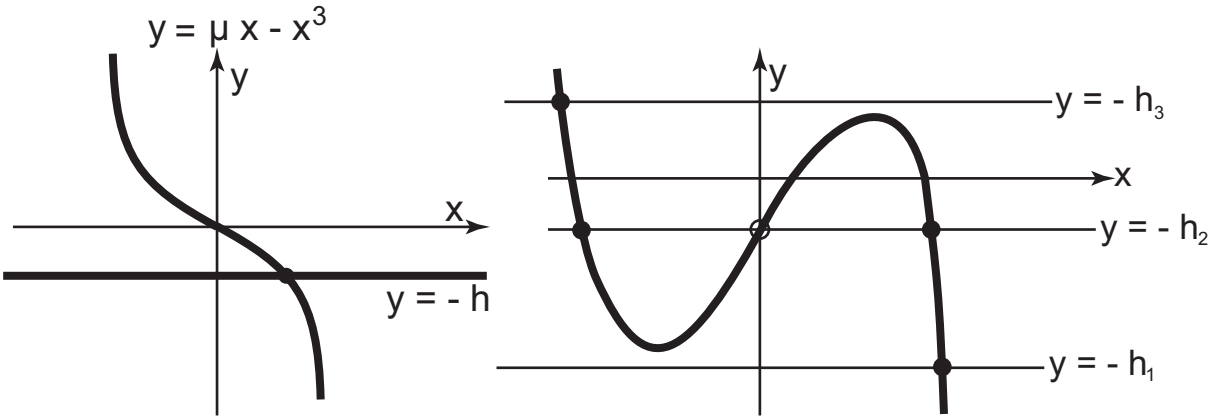
Consider perturbed pitchfork bifurcation

$$\dot{x} \;=\; \mu x - x^3 + h \tag{143}$$

Solving directly for fixed point is cumbersome (although possible).

Graphic solution:

$\mu < 0$ $\qquad\qquad$ $\mu > 0$



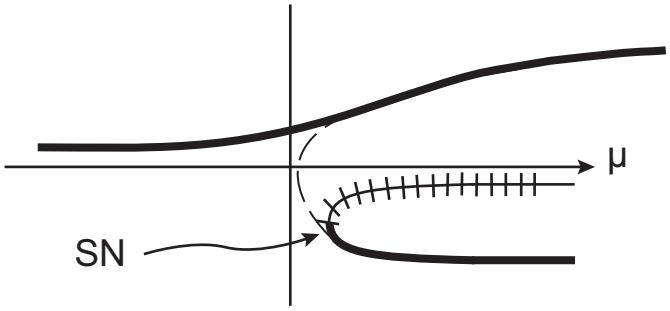
Creation/annihilation of 2 fixed points

Saddle-node bifurcation at extrema of $\mu x - x^3$ (if $\mu > 0$)

$$x_{SN} = \pm\sqrt{\frac{1}{3}\mu} \qquad h_{SN} = \mp\frac{2}{3}\mu\sqrt{\frac{1}{3}\mu} \tag{144}$$

Bifurcation diagrams depend on the parameter h

Vary μ:



for $h < 0$:

for $h > 0$ the lower branch is continuous and the saddle-node bifurcation occurs on the upper branch.

Note:

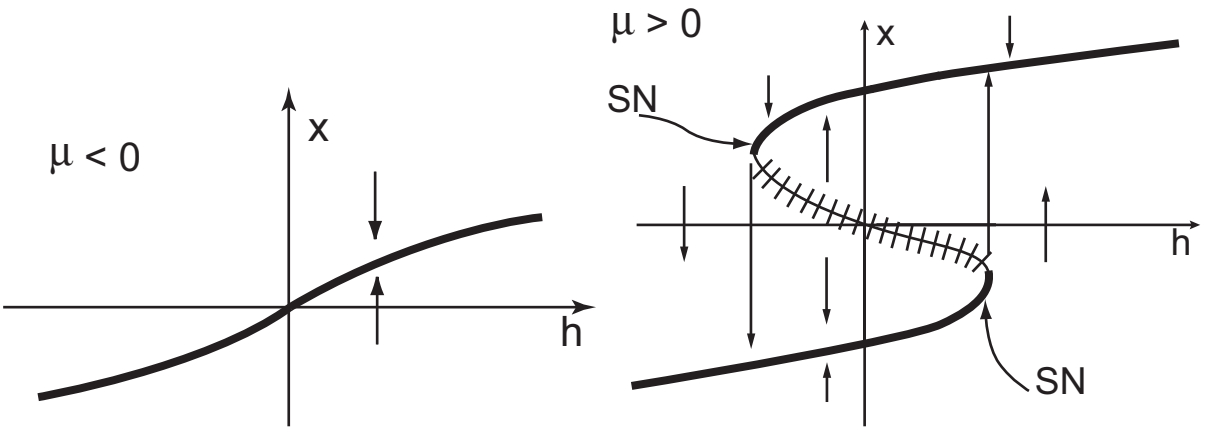
- even arbitrarily small perturbations $h \neq 0$ change the bifurcation diagram qualitatively: the pitch-fork bifurcation is not structurally stable
- with the perturbation $h \neq 0$ the only bifurcation that occurs is a saddle-node bifurcation: generic situation, structurally stable
- to get the original unperturbed pitch-fork bifurcation we have to tune 2 parameters

$$\mu = 0 \qquad \& \qquad h = 0$$

  pitch-fork bifurcation is a **codimension-2 bifurcation** in the setting of (143).
- symmetries of the system may render pitch-fork bifurcation a codimension-1 pheomenon (here reflection symmetry)

Could consider h as the bifurcation parameter:

Vary h:

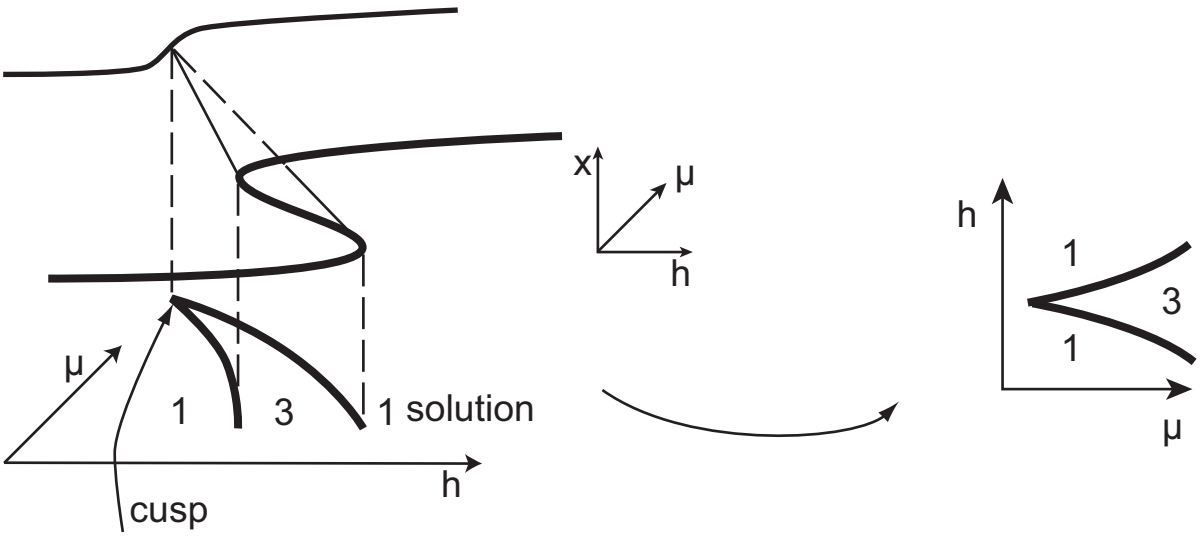


Note:

- vary h up and down beyond saddle-node bifurcations: hysteresis loop

Consider the full two-parameter bifurcation problem:

Solution surface



Degenerate pitch-fork bifurcation at $\mu = 0$, $h = 0$:

- different cuts through the solution surface yield different bifurcation diagrams
- degenerate pitch-fork bifurcation is the **organizing center** for the generation of two saddle-node bifurcations.

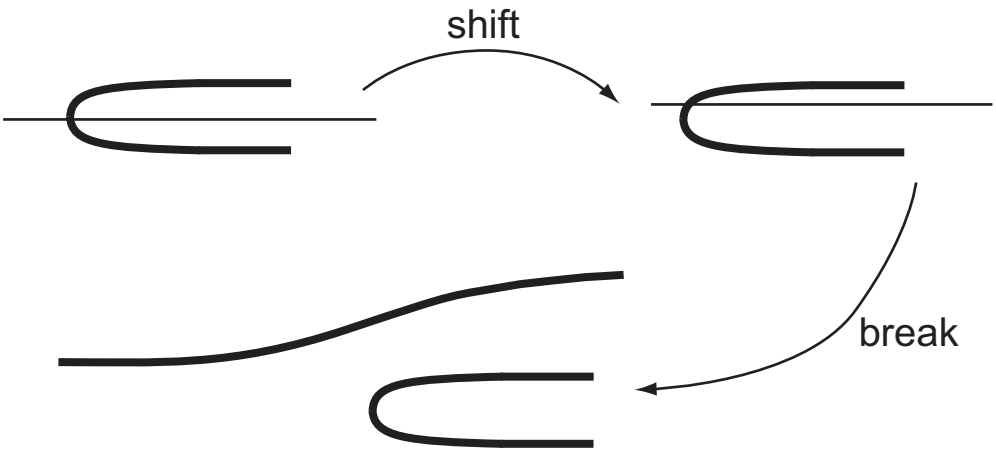
Notes:

- *Unfolding* of a degenerate bifurcation:
  introduce sufficiently many parameters so that no degeneracy is left.
- Often a complex bifurcation scenario containing multiple bifurcations can be understood by identifying and *unfolding an underlying degenerate bifurcation.*
- 'cusp catastrophe' is organizing center for two 'fold catastrophes' ($\rightarrow$ singularity theory: Thom, Zeeman).

Can consider more general unfolding of the pitch-fork bifurcation (see HW)

$$\dot{x} = \mu x - x^3 + h + gx^2.$$

Unfold perfect pitch-fork bifurcation now in two steps:
$(g \neq 0,\, h = 0)$: break symmetry $x \rightarrow -x \Rightarrow$ transcritical bifurcation
$(g \neq 0,\, h \neq 0)$: break transcritical $\Rightarrow$ only saddle-node bifurcation

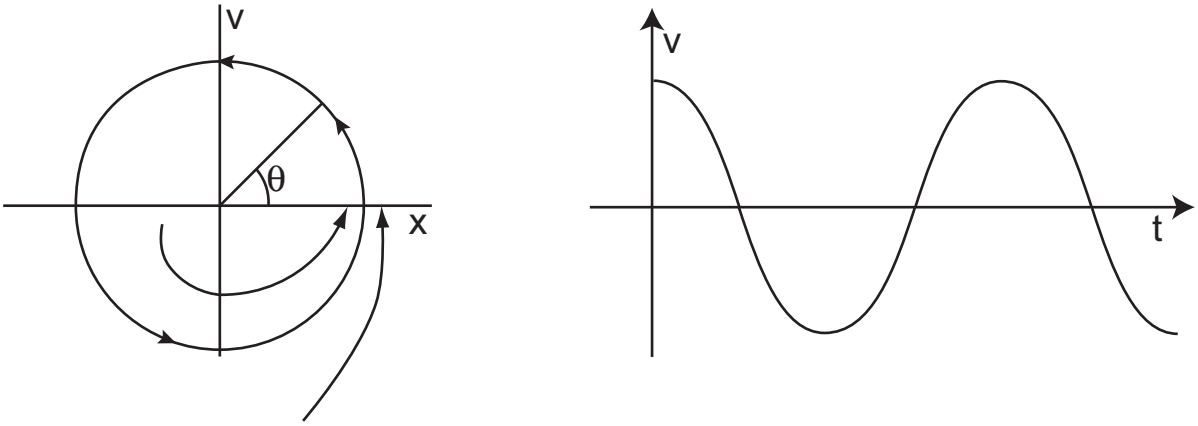


Note:

- in this unfolding additional bifurcation diagram possible (see HW).

14.4 Flow on a Circle

For oscillations need return: two dimensions needed



If oscillatory motion (circle) is sufficiently attractive consider only motion along closed orbit:

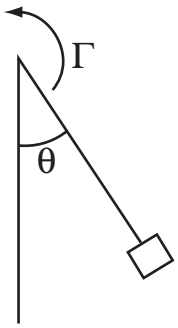
Flow on a circle

$$\dot{\theta} = f(\theta) \qquad \theta \in [0, 2\pi] \tag{145}$$

Notes:

- $f(\theta)$ cannot be arbitrary: has to be single-valued, i.e. 2π-periodic
- $f(\theta)$ gives the instantaneous frequency

Example: Overdamped Pendulum with Torque



$$m\ell^2\ddot{\theta} + \beta\dot{\theta} = -mg\ell\sin\theta + \tilde{\Gamma} \tag{146}$$

consider large damping

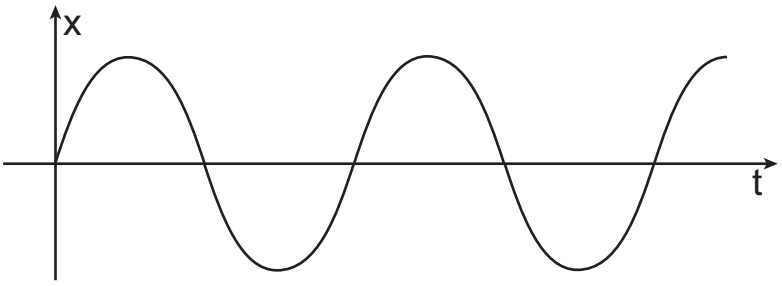
$$\dot{\theta} = \Gamma - a\sin\theta \tag{147}$$

i) $a = 0$ (no gravity)

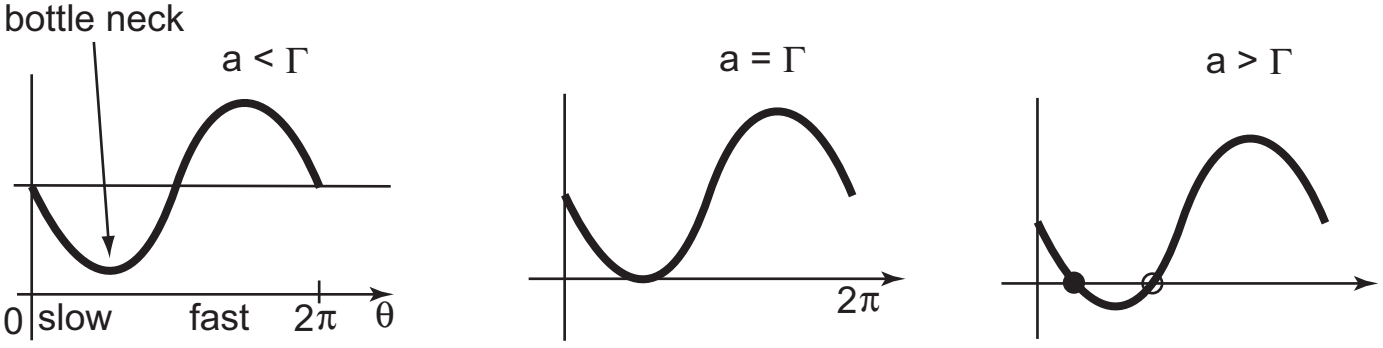
$$\theta \;=\; \theta_0 + \Gamma t \qquad \text{whirling motion} \tag{148}$$

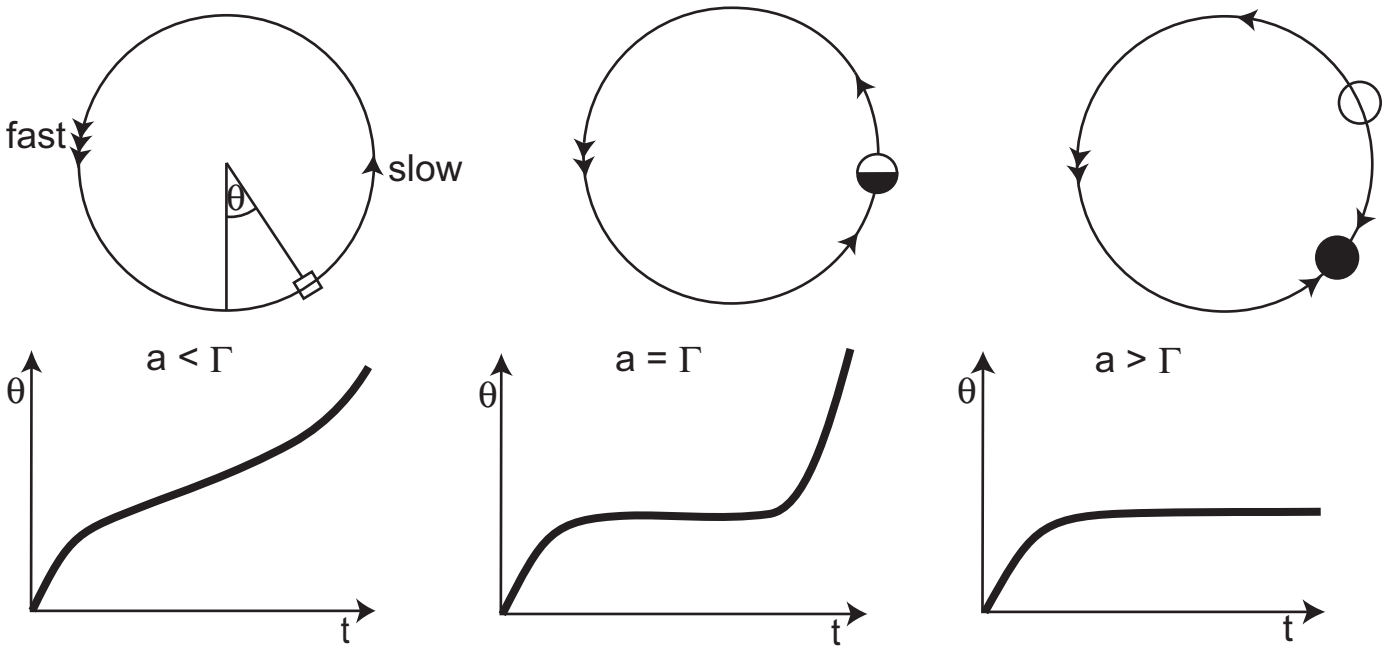
oscillation in horizontal coordinate:

$$x \;=\; \ell\sin\theta = \ell\sin(\theta_0 + \Gamma t) \tag{149}$$



ii) $a > 0$ (with gravity)





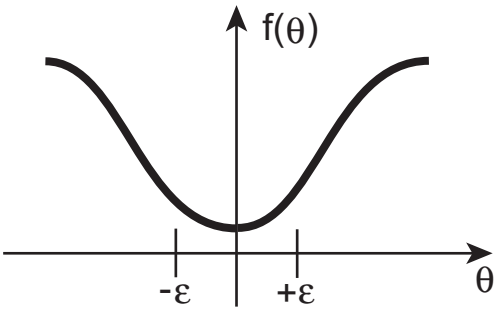
'ghost' of saddle-node bifurcation
$\Rightarrow$ extremely slow motion

Note:

- quite generally: near a steady bifurcation the dynamics become slow: growth/decay rates go to 0 ('critical slowing down').

Estimate period near bifurcation point:

$$T = \int dt = \int_0^{2\pi} \frac{d\theta}{\dot{\theta}} = \int_0^{2\pi} \frac{d\theta}{\omega - a\sin\theta} \tag{150}$$



Consider general case near saddle-node bifurcation

$$\dot{\theta} = f(\theta)$$

with

$$f(0) = \mu, \quad f'(0) = 0$$

$$\Rightarrow f(\theta) = \mu + \underbrace{\frac{1}{2} f''(0)}_{a} \theta^2 + \mathcal{O}(\theta^3)$$

$$T = \int_0^{2\pi} \frac{d\theta}{f(\theta)} = \int_{-\epsilon}^{+\epsilon} \underbrace{\frac{d\theta}{\mu + a\theta^2 + \mathcal{O}(\theta^3)}}_{\text{diverges as } \mu \to 0} + \underbrace{\int_{\epsilon}^{2\pi-\epsilon} \frac{d\theta}{f(\theta)}}_{\text{finite as } \mu \to 0}$$

$$\to \int_{-\epsilon}^{+\epsilon} \frac{d\theta}{\mu + a\theta^2} + T_0 \qquad \text{for } \mu \to 0$$

extract μ-dependence for $\mu \to 0$ (at fixed ϵ) using $\psi = \frac{\theta}{\sqrt{\mu}}$

$$\frac{1}{\mu} \int_{-\frac{\epsilon}{\mu^{1/2}}}^{\frac{\epsilon}{\mu^{1/2}}} \frac{\mu^{1/2}\, d\psi}{1 + a\psi^2} + T_0 \;\to\; \frac{1}{\mu^{1/2}} \int_{-\infty}^{\infty} \frac{d\psi}{1 + a\psi^2} + T_0 \;\propto\; \mu^{-1/2}$$

Notes:

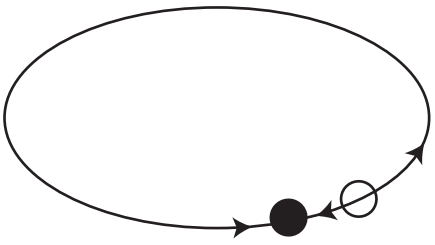
- Saddle-node bifurcation on a circle is **one** way to generate oscillations.
  Generically one has then

$$T \propto \mu^{-1/2} \tag{151}$$

- other types of bifurcations to oscillatory behavior lead to different $T(\mu)$,
  e.g. Hopf bifurcation

$$T(\mu = 0) = T_0 \text{ finite.} \tag{152}$$

- the fact that the saddle-node bifurcation leads to oscillations is a **global** feature of the system:
  need **global connection** between the generated fixed points



Examples:

i) Synchronization of fireflies

- light up periodically

- respond to neighboring fire flies

Consider extremely simple model for two periodically flashing fireflies

$$\begin{aligned} \dot{\phi_1} &= \omega_1 + a\sin(\phi_2 - \phi_1) \\ \phi_2 &= \omega_2 + a\sin(\phi_1 - \phi_2) \end{aligned}$$

$a > 0$: for $\phi_2 - \phi_1 > 0$ firefly ϕ_1 is pulled by firefly ϕ_2

introduce phase difference: $\theta = \phi_2 - \phi_1$

$$\dot{\theta} = \underbrace{\omega_2 - \omega_1}_{\Delta\omega} - 2a\sin(\theta)$$

$\Delta\omega$: frequency mismatch

- *Fixed point:*

$$\underbrace{|\Delta\omega|}_{\text{range of entrainment}} < 2a \qquad \text{and} \qquad \theta_0 = \arcsin\frac{\Delta\omega}{2a} \neq 0$$

  - flies flash with a fixed phase difference: **phase-locked** state
  - slower fly lags behind: phases are not synchronized
  - common oscillation frequency (frequency synchronization)

$$\Omega = \omega_1 + \frac{1}{2}\Delta\omega$$

  - phase-locked state disappears via a saddle-node bifurcation

- *"Whirling" motion:* $|\Delta\omega| > a$

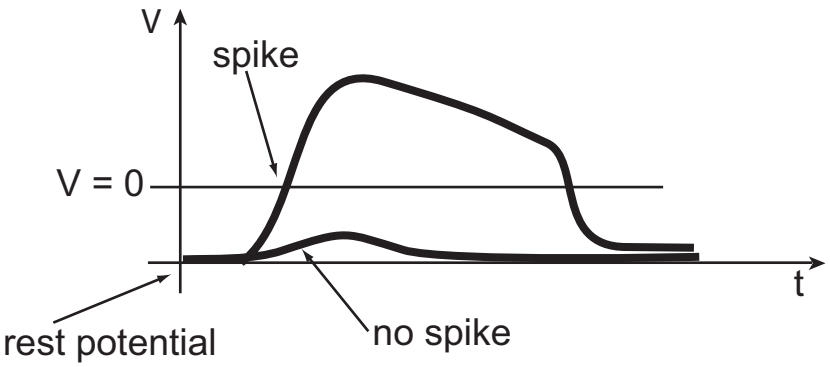
  - phase relationship between the flashes changes continuously in time

Notes:

- entrainment is a common feature of coupled oscillators
- in the case of fireflies the coupling is not present all the time but only during the flash ('pulsatile'), i.e. interaction consists of 'kicks'

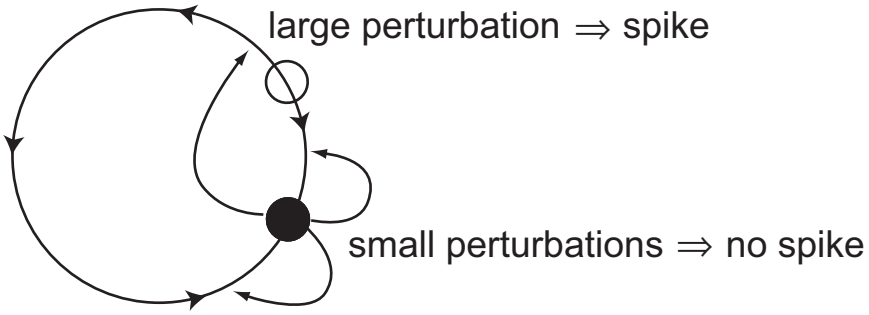
ii) Excitable systems:

Many systems have finite *threshold* for large response. For instance, neurons show little response to small inputs but give a large action potential ('spike') for inputs above a certain threshold



Very simple model

$$\dot{\theta} = \Gamma - a \sin\theta \qquad\qquad \Gamma \leq a$$



Note:

- The size of the spike (excursion) does not depend on the size of the perturbation once the threshold is reached.
- For Γ close to a even small perturbations can be sufficient to excite a spike.
  In a saddle-node bifurcation the distance between the unstable and stable fixed points is of $\mathcal{O}(\mu^{1/2})$:

$$\Delta\theta_{\text{threshold}} \approx \theta_{0,u} - \theta_{0,s} \quad \propto \quad |\Gamma - a|^{1/2}$$

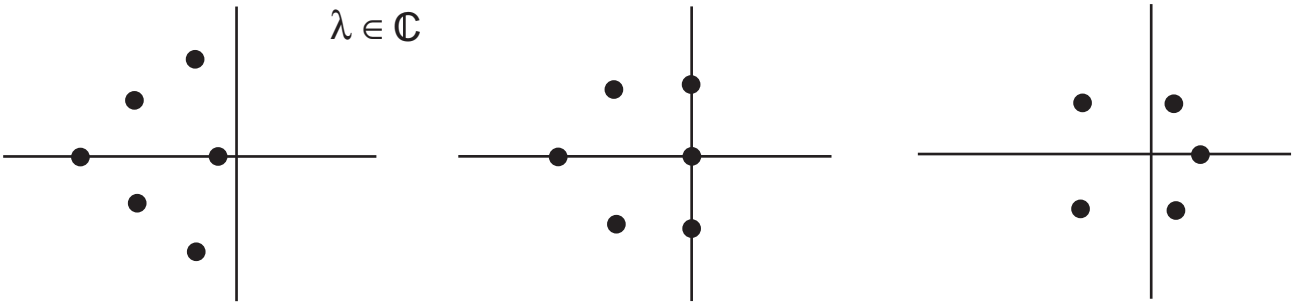
14.5 Stability

So far we had linear stability. In higher dimensions new aspects arise. Refine our notions of stability.

Linear Stability:

- with respect to *infinitesimal* perturbations
- determined by linearization

Eigenvalues in complex plane:

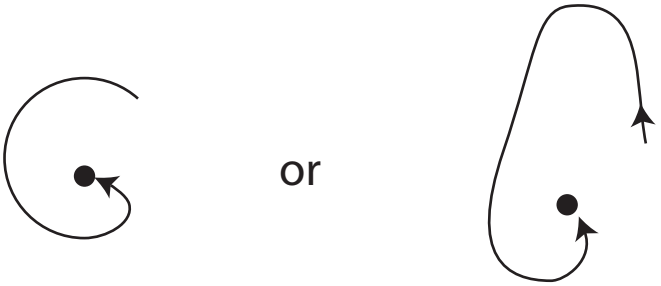


We are in particular interested in sets of points that are eventually approached by the system:

Definition:

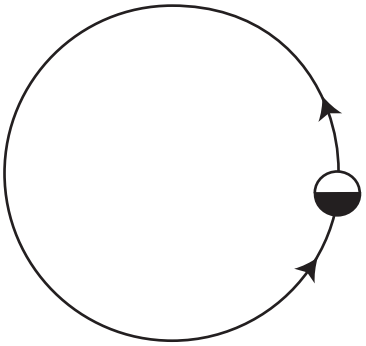
A set of points $\mathcal{A}$ (e.g. a fixed point) is **attracting** (is an attractor) if all trajectories that start close to it converge to it, i.e.

$$\text{for all } \underline{\mathbf{x}}(0) \text{ near } \mathcal{A}: \ \underline{\mathbf{x}}(t) \to \underline{x}_\infty \in \mathcal{A} \text{ for } t \to \infty$$



Notes:

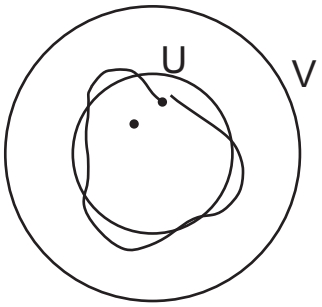
- The attractor need not be a fixed point. It could be a periodic or a chaotic orbit.
- The system need not approach the attractor right away. Even for arbitrarily small initial distances it could make a finite excursion:



Definition:

A set $\mathcal{A}$ is **Lyapunov stable** if all orbits that start close to it remain close to it for all times. More technically:

$\mathcal{A}$ is Lyapunov stable if for any neighborhood $\mathcal{V}$ of $\mathcal{A}$ there exists a neighborhood $\mathcal{U} \subseteq \mathcal{V}$ such that if $\mathbf{x}(0) \in \mathcal{U}$ then $\mathbf{x}(t) \in \mathcal{V}$ for all times.



Notes:

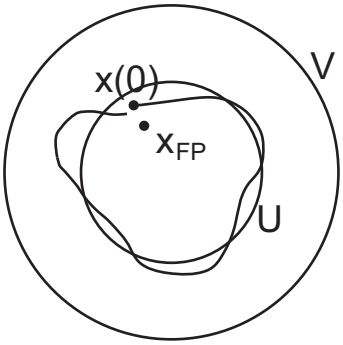
- Lyapunov stability of a set does not imply it is an attractor.
- An attractor does not have to be Lyapunov stable. E.g., the unstable fixed point at the SNIC bifurcation point is attracting but not Lyapunov stable (one cannot find neighborhood that limits excursion)

Definition:

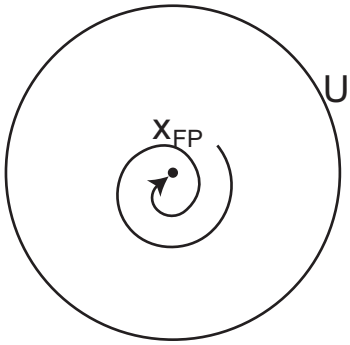
A set $\mathcal{A}$ is **asymptotically stable** if it is

- attracting and
- Lyapunov stable

i.e. if all orbits that start sufficiently close to $\mathcal{A}$ remain close to it for all times and converge to it as $t \to \infty$.



Lyapunov stable



assymptotically stable

Notes:

- linear stability $\Rightarrow$ asymptotic stability
- asymptotically stable $\Rightarrow$ Lyapunov stability
- asymptotically stable $\Rightarrow$ set is attracting, it is an attractor.
- linear instability $\Rightarrow$ instability

- **But:** asymptotic or Lyapunov stability **do not imply** linear stability

Lyapunov stability can be shown if the system has a Lyapunov functional, which is a generalization of the potential.

We had **Gradient Systems, Potential Systems:**

If

$$\dot{\underline{x}} = -\nabla V(\underline{x}) \qquad \text{i.e.} \qquad \dot{x}_i = -\frac{\partial V}{\partial x_i}$$

with $V \geq V_0$ for all $\underline{x}$ (bounded from below)
then

$$\frac{dV}{dt} = \sum_i \frac{\partial V}{\partial x_i}\dot{x}_i = -\sum_i (\frac{\partial V}{\partial x_i})^2 \leq 0$$

Thus V eventually reaches a (local) minimum.

Notes:

- The potential is constant in time *if and only* if the system is at a fixed point:

$$\frac{dV}{dt} = 0 \Leftrightarrow \frac{\partial V}{\partial x_i} = 0 \Leftrightarrow \dot{x}_i = 0 \qquad \text{for all } i$$

- Gradient systems show no persistent dynamics, cannot use a potential to show stability of a periodic orbit, say.

More general concept: **Lyapunov Functional** $\mathcal{F}$

Definition:

$\mathcal{F}$ is called a Lyapunov functional if $\mathcal{F}(\underline{x}) > \mathcal{F}(\underline{x}_0)$ for all $\underline{x} \neq \underline{x}_0 \in \mathcal{U}$, where $\underline{x}_0$ is a fixed point.

Notes:

- The dynamics need *not* be given by $\dot{\underline{x}} = -\nabla\mathcal{F}$.
- $\underline{x}_0$ could be replaced by a set of points (e.g. a periodic orbit)

If $\mathcal{F}$ is a Lyapunov functional for a system in a neighborhood $\mathcal{U}$ then we have:

- if $\frac{d\mathcal{F}}{dt} \leq 0$ for all $\underline{x} \neq \underline{x}_0$ in the neighborhood $\mathcal{U}$ then $\underline{x}_0$ is Lyapunov stable
- if $\frac{d\mathcal{F}}{dt} < 0$ for all $\underline{x} \neq \underline{x}_0$ in $\mathcal{U}$ then $\underline{x}_0$ asymptotically stable

Examples:

a) damped particle in bounded potential

$$\ddot{x} + \beta\dot{x} = -\frac{d\mathcal{U}}{dx}$$

i.e. written as a sytem we get

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= -\beta v - \frac{d\mathcal{U}}{dx} \end{aligned}$$

Try to use total energy as a Lyapunov function

$$m\mathcal{F} = \frac{1}{2}\dot{x}^2 + \mathcal{U} = \frac{1}{2}v^2 + \mathcal{U}$$

$$m\frac{d\mathcal{F}}{dt} = v\dot{v} + \frac{d\mathcal{U}}{dx}\dot{x} = v(-\beta v - \frac{d\mathcal{U}}{dx}) + \frac{d\mathcal{U}}{dx}v = -\beta v^2 < 0$$

$\Rightarrow$ all fixed points are asymptotically stable, the system has no periodic orbits.

b)

$$\begin{aligned} \dot{x} &= -x + 4y \\ \dot{y} &= -x - y^3 \end{aligned}$$

Simplest attempt: try quadratic function that is bounded from below:

$$\mathcal{F} = x^2 + ay^2 \quad \text{with } a > 0.$$

$$\frac{d\mathcal{F}}{dt} = 2x(-x+4y) + 2ay(-x-y^3) = \underbrace{-2x^2}_{\leq 0} + \underbrace{xy(8-2a)}_{\text{undetermined}} - \underbrace{2ay^4}_{\leq 0}$$

$\Rightarrow$ choose $a = 4 \Rightarrow \frac{d\mathcal{F}}{dt} < 0$ for $x \neq 0 \neq y$

$\Rightarrow (0,0)$ asymptotically stable and no periodic orbits

Note:

- finding a Lyapunov function is a matter of trial and error.

14.6 Poincaré-Bendixson Theorem

- How complex can the dynamics be in 2 dimensions?
- Can we guarantee a periodic orbit without explicitly calculating it?

Poincaré-Bendixson Theorem:

If

- R is a closed bounded subset of the **plane** ('trapping region')

- $\dot{\underline{x}} = \underline{f}(\underline{x})$ with $\underline{f}(\underline{x})$ continuously differentiable on an open set containing R

then any orbit that remains in R for all t either converges to a fixed point or to a periodic orbit.

Simple Illustration:

In one dimension we had: no periodic orbits

- fixed point *divides* phase line into *left* and *right*
  $\Rightarrow$ cannot go back and forth
  $\Rightarrow$ no oscillatory approach to fixed point and no persistent oscillations

In two dimensions:

What is more “complicated" than a periodic orbit?
A periodic orbit has single fundamental frequency ω

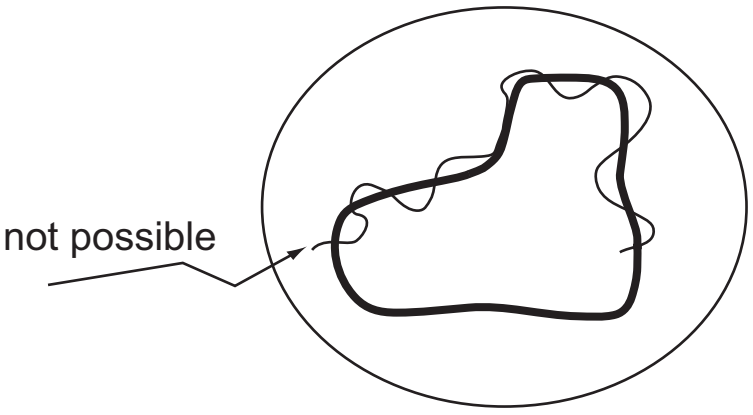
$$x(t) = A\cos\omega t + B\cos 2\omega t + C\cos 3\omega t + \ldots$$

Can we have 2 incommensurate frequencies,

$$\frac{\omega_1}{\omega_2} \neq \frac{m}{n} \qquad \text{irrational}$$

e.g. beating between two different frequencies?

- Cannot spiral in and then out again without the orbit intersecting itself.
- Consider approach to periodic orbit in two dimensions:
  Periodic orbit separates phase plane into *inside* and *outside*.
  Oscillatory approach to periodic orbit not possible $\Rightarrow$ No second frequency.
  System has to go to fixed point or periodic orbit.



Consequence of Poincaré-Bendixson:

- The only attractors of 2d-flows are fixed points or periodic orbits

- **No chaos in 2 dimensions.**

Application: Periodic orbit in a model for oscillations in glycolysis

Yeast cells break down sugar by glycolysis: simple model

ADP adenosine diphosphate

$$\dot{x} = -x + ay + x^2 y \equiv f(x, y)$$

F6P fructose-6-phosphate

$$\dot{y} = b - ay - x^2 y \equiv g(x, y)$$

For which parameter ranges can one guarantee the existence of a stable periodic orbit?

Obtain **phase portrait:**

Nullclines are those lines along which one of the variables are constant in time:

$$\dot{x} = 0 \quad \text{or} \quad \dot{y} = 0$$

i.e.

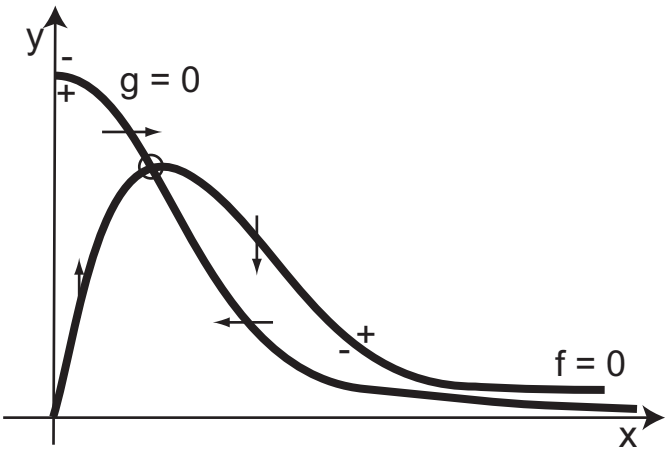
$$f = 0 \quad \Rightarrow \quad y = \frac{x}{a + x^2}$$
$$g = 0 \quad \Rightarrow \quad y = \frac{b}{a + x^2}$$

$\Rightarrow$ fixed point at the intersection of the nullclindes

$$y = \frac{x}{a + x^2} = \frac{b}{a + x^2}$$

$$\Rightarrow \qquad x = b \qquad \text{and} \qquad y = \frac{b}{a + b^2}$$

fixed point exists for all $b > 0$, $a > 0$ since $x \geq 0$ and $y \geq 0$ needed for concentrations x, y



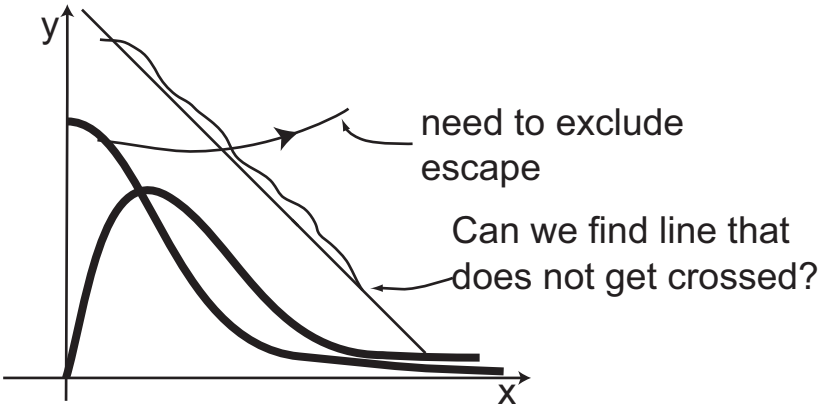
Indicate flow on null clines: spiraling motion around the fixed point:

- to fixed point?
- to periodic orbit? which?
- to infinity?

To use Poincaré-Bendixson:

1. need trapping region $\mathcal{R}$
2. exclude fixed points from trapping region

1) Trapping Region:



Consider first $x = 0$ and $y = 0$:

$$f(0, y) = ay > 0 \qquad \Rightarrow \dot{x} > 0$$

$$g(x, 0) = b > 0 \qquad \Rightarrow \dot{y} > 0$$

Consider large x and y (check possibility of escape)

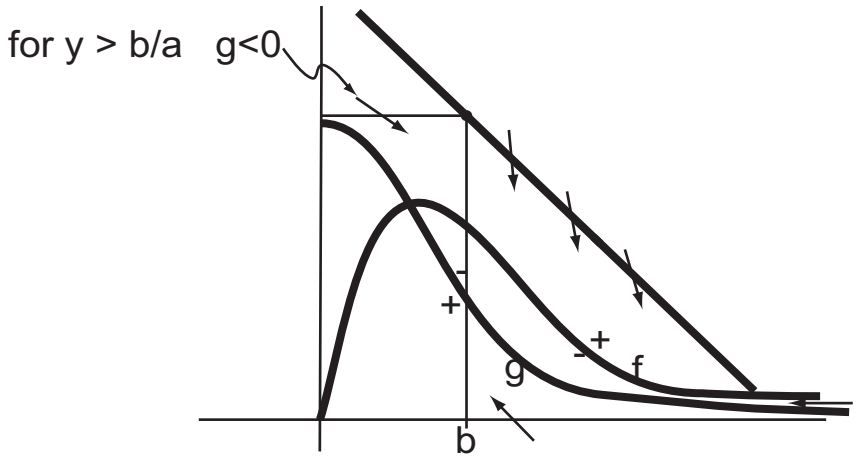
$$\left.\begin{array}{rcl} \dot{x} & \sim & x^2 y \\ \dot{y} & \sim & -x^2 y \end{array}\right\} \text{ on the orbit } x \text{ and } y \text{ satisfy} \qquad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \sim -1 \quad \text{for large} \quad x, y$$

Show that slope is steeper than -1
compare $|\dot{x}|$ with $|\dot{y}|$ more precisely

$$\begin{aligned} \dot{x} - (-\dot{y}) &= -x + ay + x^2\, y + b - ay - x^2\, y \\ &= b - x \end{aligned}$$

$\Rightarrow$ for $x > b \quad |\dot{x}| < |\dot{y}|$
$\Rightarrow$ flow inward along $y = -x + C$ for $x > b$ and C large enough to have nonlinearities dominate the flow

for $y > \frac{b}{a}$ we have $g < 0$
$\Rightarrow$ flow inward for $y > b/a$



2) Fixed Points:

only a single fixed point $\left(b, \frac{b}{a+b^2}\right)$

Stability analysis shows fixed point unstable for

$$1 - 2a - \sqrt{1-8a} < 2b^2 < 1 - 2a + \sqrt{1-8a}$$

$\Rightarrow$ limit cycle guaranteed for this range of b (if $a \leq \frac{1}{8}$)

outside this range of b expect convergence to fixed point.

Instability at $2(b_H^{(1,2)})^2 = 1 - 2a \pm \sqrt{1-8a}$ is a Hopf bifurcation (2 complex eigenvalues cross the imaginary axis).

Oscillations occur for $b_H^{(1)} < b < b_H^{(2)}$. No steady bifurcation possible.