Lecture Note Sketches

Perturbation Methods 420-2

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References

[1] W. Barten, M. Lücke, and M. Kamps. Localized traveling-wave convection in binary fluid mixtures. *Phys. Rev. Lett.*, 66:2621, 1991.

[2] W. Barten, M. Lücke, M. Kamps, and R. Schmitz. Convection in binary fluid mixtures. II: localized traveling waves. *Phys. Rev. E*, 51:5662, 1995.

[3] P. Coullet, C. Elphick, and D. Repaux. Nature of spatial chaos. *Phys. Rev. Lett.*, 58:431–434, 1987.

[4] T Dauxois, M Peyrard, and S Ruffo. The Fermi-Pasta-Ulam 'numerical experiment': history and pedagogical perspectives. *Eur. J. Physics*, 26(5):S3–S11, SEP 2005.

[5] P. Kolodner, D. Bensimon, and C. Surko. Traveling-wave convection in an annulus. *Phys. Rev. Lett.*, 60:1723, 1988.

[6] P. Kolodner, J.A. Glazier, and H. Williams. Dispersive chaos in one-dimensional traveling-wave convection. *Phys. Rev. Lett.*, 65:1579, 1990.

[7] P. Kolodner, S. Slimani, N. Aubry, and R. Lima. Characterization of dispersive chaos and related states of binary-fluid convection. *Physica D*, 85(1-2):165–224, July 1995.

[8] B.A. Malomed and A.A. Nepomnyashchy. Kinks and solitons in the generalized Ginzburg-Landau equation. *Phys. Rev. A*, 42:6009, 1990.

[9] U. Parlitz. Complex dynamics of nonlinear systems. In U. Kaatze T. Kurz, U. Parlitz, editor, *Oscillations, Waves and Interactions*, page 405. Universitaetsverlag Goettingen, 2007.

[10] O. Thual and S. Fauve. Localized structures generated by subcritical instabilities. *J. Phys. (Paris)*, 49:1829, 1988.

[11] Wim van Saarloos. Front propagation into unstable states: Marginal stability as a dynamical mechanism for velocity selection. *Phys. Rev. A 37, 211(1988)*, 1988.

[12] NJ Zabusky. Fermi-Pasta-Ulam, solitons and the fabric of nonlinear and computational science: History, synergetics, and visiometrics. *CHAOS*, 15(1), MAR 2005.

1 Asymptotic Evaluation of Integrals[1]

Motivation:

Many functions and solutions of differential equations are given or can be written in terms of definite integrals that depend on additional parameters.

E.g. modified Bessel function

$$K_0(x) = e^{-x} \int_0^\infty \left(t^2 + 2t\right)^{-1/2} e^{-xt} dt$$

Integrals with parameters also arise from Fourier and Laplace transforms

$$\tilde{f}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega t} f(t)\, dt.$$

It is often useful to extract the behavior of such a complicated solution in limiting cases, e.g. for large values of the argument,

- to get analytical insight into the behavior of the solution: is it decaying or diverging? if so, how? Any oscillatory behavior?
- for faster numerical evaluation
  - such a function could appear as a Green's function in a numerical code and could be called many times
  - the evaluation of the integral could be effected by round-off errors if the integrand is strongly oscillatory leading to many cancellations
  - divergences are difficult to treat numerically

Often one could analyse the differential equation leading to that special function in that limiting case. But then not all boundary conditions are exploited and one looses information (prefactors). If one has an integral representation of that function one can analyze that in the limit, which retains the full boundary information.

E.g.

$$y' = xy + 1 \qquad y(0) = 0$$

For large x one can get the approximate solution via

$$y' = xy$$

$$\frac{d}{dx} \ln y = x \qquad \Rightarrow \qquad y = y_0 e^{\frac{1}{2}x^2}$$

To obtain the prefactor y_0 one needs to use the initial condition. However, the approximate solution is only valid for $x \to \infty$ and not for $x = 0$.

[1]This chapter follows quite closely Bender & Orszag.

Here we can get an integral expression for the solution using an integrating factor

$$\frac{d}{dx}\left(e^{-\frac{1}{2}x^2}y\right) = e^{-\frac{1}{2}x^2}$$

Using the initial condition $y(0) = 0$ we get

$$y(x) = e^{\frac{1}{2}x^2}\int_0^x e^{-\frac{1}{2}t^2}\,dt$$

From this integral one can get the asymptotic behavior of $y(x)$ for $x \to \infty$ *including* the prefactor y_0.

Practical Note:

- For many of the expansions and perturbation methods **maple** and **mathematica** can be very useful. Both are available on the departmental computer network.

1.1 Elementary Examples

Consider

$$I(x) = \int_0^2 \cos\left[\frac{1}{x}\left(t^2+t^3\right)^{\frac{1}{4}}\right]dt$$

For large x one can easily approximate

$$\lim_{x\to\infty} I(x) = \int_0^2 \lim_{x\to\infty}\cos\left[\frac{1}{x}\left(t^2+t^3\right)^{\frac{1}{4}}\right]dt = \int_0^2 \cos\left[0\right]dt = 2$$

But: Not always one can interchange the limit with the integral!

Reminder:

If the series expansion $\sum_{n=0}^N f_n(t)$ converges to $f(t)$ *uniformly* for all t in the interval $[a,b]$, i.e. for any $\epsilon > 0$

$$\left|\sum_{n=0}^N f_n(t) - f(t)\right| < \epsilon \qquad \text{for } N > N_0(\epsilon)$$

with $N_0(\epsilon)$ independent of $t \in [a,b]$, then the series can be integrated term by term, i.e.

$$\int_a^b f(t)dt = \int_a^b \lim_{N\to\infty}\sum_{n=0}^N f_n(t)dt = \lim_{N\to\infty}\sum_{n=0}^N\int_a^b f_n(t)dt$$

Thus, to show the validity for our example $I(x)$ we would consider the series expansion of $\cos\left[\frac{1}{x}\left(t^2+t^3\right)^{\frac{1}{4}}\right]$ for $y \equiv x^{-1} \to 0$ and show uniform convergence within the interval $[0,2]$.

Example 1

Consider the small-x behavior of $I(x) = \int_0^1 \frac{1}{t}\sin xt\,dt$

$$\int_0^1 \frac{1}{t}\sin xt\,dt = \int_0^1 \frac{1}{t}\left\{xt - \frac{1}{3!}\left(xt\right)^3 + \ldots\right\}dt$$

The convergence of the series

$$x - \frac{1}{3!}x^3 t^2 + \frac{1}{5!}x^5 t^4 + \dots$$

to $t^{-1}\sin(xt)$ is uniform for all $t \in [0,1]$: use ratio test

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \frac{1}{(2(n+1)-1)\,(2(n+1)-2)}x^2t^2 \\ &= x^2t^2 \lim_{n\to\infty} \frac{1}{(2(n+1)-1)\,(2(n+1)-2)} = 0 \qquad \text{for all } t\in[0,1]\end{aligned}$$

In particular,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1 \qquad \text{bounded away from } 1$$

Therefore we integrate term by term to get

$$\int_0^1 \frac{1}{t}\sin xt\,dt \sim x - \frac{1}{3!}\frac{1}{3}x^3 + \frac{1}{5!}\frac{1}{5}x^5 \dots$$

Example 2 $I(x) = \int_x^\infty t^{\alpha-1}e^{-t}\,dt$

Determine the behavior of the incomplete Γ-function for small x

$$\Gamma(\alpha,x) = \int_x^\infty t^{\alpha-1}e^{-t}\,dt$$

We would like to expand the exponential and then integrate term by term. This would lead to terms

$$\int_x^\infty t^{\alpha-1}\frac{1}{n!}\left(-t\right)^n\,dt$$

which diverge at the upper limit for large enough n. To treat the upper limit we need to keep the exponential. Depending on a we need to do this in different ways:

i) $\alpha > 0$

Try to write the integral in terms of an integral that does not involve the limit $x \to \infty$.

Rewrite

$$\begin{aligned}\int_x^\infty t^{\alpha-1}e^{-t}\,dt &= \int_0^\infty t^{\alpha-1}e^{-t}\,dt - \int_0^x t^{\alpha-1}\sum_{n=0}^\infty \frac{(-t)^n}{n!}\,dt \\ &= \Gamma(\alpha) - \sum_{n=0}^\infty (-1)^n \frac{x^{n+\alpha}}{(n+\alpha)\,n!}\end{aligned} \tag{1}$$

since the second term can again be integrated term by term:

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{t^{n+1+\alpha-1}}{t^{n+\alpha-1}}\frac{n!}{(n+1)!} = \frac{t}{n+1} < \frac{x}{n+1} < 1 \qquad \text{for all } t\in[0,x] \text{ and } n+1>x. \tag{2}$$

Notes:

- Extending the integral to $x = 0$ worked because the definite integral $\int_0^\infty$ exists.
- The goal is to extract the x-dependence of the integral. It is therefore sufficient if the integral can be rewritten in terms of another x-dependent integral and an x-independent integral, even if the latter can only be evaluated numerically.
- What happens for $\alpha < 0$? Then the individual integrals in the series cannot be done.

ii) $\alpha \equiv -\bar{\alpha} < 0$ but not an integer

Now negative powers of t appear and the integral cannot be extended to $x = 0$ straightforwardly. Look at the low and high powers in t separately:

$$\int_x^\infty t^{-\bar{\alpha}-1} e^{-t} dt = \int_x^\infty \underbrace{\frac{1}{t^{\bar{\alpha}+1}} - \frac{1}{t^{\bar{\alpha}}} + \frac{1}{2}\frac{1}{t^{\bar{\alpha}-1}} \ldots (-1)^N \frac{1}{t^{\bar{\alpha}+1-N}}}_{\text{decay faster than } t^{-1} \text{ for } t\to\infty} + \underbrace{(-1)^{N+1} \frac{1}{t^{\bar{\alpha}+1-N-1}} + \ldots}_{\text{grows more slowly than } t^{-1} \text{ for } t\to 0} \quad dt$$

- low powers: up to a certain N the first integral can be done directly without using the exponential

$$\text{need} \quad \bar{\alpha} + 1 - N > 1 \qquad \text{i.e.} \quad N < \bar{\alpha}$$

  Let us choose the largest N that satisfies this condition.

- high powers: integral would diverge at $x \to \infty \Rightarrow$ the exponential has to be kept. For sufficiently large N the integral can be extended to $x = 0$. We need

$$\bar{\alpha} + 1 - N - 1 < 1 \qquad \text{i.e.} \qquad N + 1 > \bar{\alpha}$$

- Thus, we can satisfy both conditions if we choose N to satisfy

$$N < \bar{\alpha} < N + 1$$

  Note, that this condition can be met for *any non-integer* $\bar{\alpha}$.

Now, split the power series under the integral into two parts:

$$\int_x^\infty t^{\alpha-1} e^{-t}\, dt = \int_x^\infty t^{-\bar{\alpha}-1} \sum_{n=0}^{N} \frac{(-t)^n}{n!} dt + \int_x^\infty t^{-\bar{\alpha}-1} \sum_{n=N+1}^{\infty} \frac{(-t)^n}{n!} dt$$

Then, by construction the first integral, which does not involve a series, converges at the upper limit $t \to \infty$.

$$\int_x^\infty t^{-\bar{\alpha}-1} \sum_{n=0}^{N} \frac{(-t)^n}{n!} dt = -\sum_{n=0}^{N} (-1)^n \frac{x^{n-\bar{\alpha}}}{(n-\bar{\alpha})\, n!}$$

The second integral involving the series cannot be done term by term: because of the upper limit the convergence is not uniform (cf. (2)). But it poses no problem at the lower limit $t = x \to 0$: the term in the integrand with the lowest power in t is

$$t^{-\bar{\alpha}-1+N+1} = t^{N-\bar{\alpha}} \quad \text{with} \quad N - \bar{\alpha} > -1$$

Therefore exploit the integral over $[0,\infty)$

$$\int_x^\infty t^{-\bar{\alpha}-1}\sum_{n=N+1}^{\infty}\frac{(-t)^n}{n!}dt=\int_0^\infty t^{-\bar{\alpha}-1}\sum_{n=N+1}^{\infty}\frac{(-t)^n}{n!}dt-\int_0^x t^{-\bar{\alpha}-1}\sum_{n=N+1}^{\infty}\frac{(-t)^n}{n!}dt.$$

The second integral can be performed term by term

$$\int_0^x t^{-\bar{\alpha}-1}\sum_{n=N+1}^{\infty}\frac{(-t)^n}{n!}dt=\sum_{n=N+1}^{\infty}(-1)^n\,\frac{x^{n-\bar{\alpha}}}{(n-\bar{a})\,n!}. \tag{3}$$

Bring back the exponential function strategically for the first integral

$$\int_0^\infty t^{-\bar{\alpha}-1}\sum_{n=N+1}^{\infty}\frac{(-t)^n}{n!}dt=\int_0^\infty t^{-\bar{\alpha}-1}\left\{e^{-t}-\sum_{n=0}^{N}\frac{(-t)^n}{n!}\right\}dt.$$

This integral converges at the lower limit because all the low powers of t are eliminated due to $N-\bar{a}>-1$. It converges at the upper limit because the high powers are given by those of the exponential function and the highest exponent of the term without the exponential is $N-\bar{a}-1<-1$. We cannot evalute the 2 terms individually; evaluate the two terms together therefore by repeated integration by parts to eliminate the sum iteratively by taking derivatives

$$\begin{aligned}\int_0^\infty t^{-\bar{\alpha}-1}\left\{e^{-t}-\sum_{n=0}^{N}\frac{(-t)^n}{n!}\right\}dt &= -\frac{1}{\bar{\alpha}}t^{-\bar{\alpha}}\left\{e^{-t}-\sum_{n=0}^{N}\frac{(-t)^n}{n!}\right\}\Bigg|_0^\infty\\ &\quad-\int_0^\infty -\frac{1}{\bar{\alpha}}t^{-\bar{\alpha}}\left\{-e^{-t}-\sum_{n=0}^{N-1}\frac{(-t)^n}{n!}\right\}dt\end{aligned}$$

The boundary terms vanish:

- at $t\to\infty$ because $\bar{\alpha}>N$
- at $t=0$ because all powers inside the curly braces have exponents $N+1>\bar{\alpha}$ and higher

Thus after $N+1$ integrations by parts

$$\int_0^\infty t^{-\bar{\alpha}-1}\left\{e^{-t}-\sum_{n=0}^{N}\frac{(-t)^n}{n!}\right\}dt=(-1)^{N+1}\frac{1}{\bar{\alpha}\,(\bar{\alpha}-1)\ldots(\bar{\alpha}-N)}\underbrace{\int_0^\infty t^{-\bar{\alpha}+N}e^{-t}dt}_{\Gamma(N-\bar{\alpha}+1)}$$

Notes:

- The definition $\Gamma(a)=\int_0^\infty t^{\alpha-1}e^{-t}dt$ is valid only for $\alpha>0$.
- In rewriting the integral as $\Gamma\left(N-\bar{\alpha}+1\right)$ we used that $N+1>\bar{\alpha}$.

Now use

$$\begin{aligned}\Gamma\left(N-\bar{\alpha}+1\right) &= \left(N-\bar{\alpha}\right)\left(N-\bar{\alpha}-1\right)\ldots\left(N-\bar{\alpha}-N\right)\Gamma\left(-\bar{\alpha}\right)= \\ &= \left(-1\right)^{N+1}\left(\bar{\alpha}-N\right)\left(\bar{\alpha}-N+1\right)\ldots\left(\bar{\alpha}\right)\underbrace{\Gamma\left(-\bar{\alpha}\right)}_{\Gamma(\alpha)}\end{aligned}$$

Thus

$$\int_0^\infty t^{-\bar{\alpha}-1}\left\{e^{-t}-\sum_{n=0}^{N}\frac{\left(-t\right)^n}{n!}\right\}dt=\Gamma\left(\alpha\right)$$

Combining with (3) we get

$$\int_x^\infty t^{-\bar{\alpha}-1}e^{-t}dt=\Gamma\left(\alpha\right)-\sum_{n=0}^{\infty}\left(-1\right)^n\frac{x^{n-\bar{\alpha}}}{\left(n-\bar{\alpha}\right)n!}=\Gamma\left(\alpha\right)-\sum_{n=0}^{\infty}\left(-1\right)^n\frac{x^{n+\alpha}}{\left(n+\alpha\right)n!}$$

as in the case $\alpha>0$.

But: now the series contains also negative powers of x.

Summarizing again the strategy:

We separated the series into to parts

$$\int_x^\infty t^{-\bar{\alpha}-1}e^{-t}dt=\int_x^\infty\underbrace{\frac{1}{t^{\bar{\alpha}+1}}-\frac{1}{t^{\bar{\alpha}}}+\frac{1}{2}\frac{1}{t^{\bar{\alpha}-1}}\ldots\left(-1\right)^N\frac{1}{t^{\bar{\alpha}+1-N}}}_{I_{low}}dt+\int_x^\infty\underbrace{\left(-1\right)^{N+1}\frac{1}{t^{\bar{\alpha}+1-N-1}}+\ldots dt}_{I_{high}}$$

where by choosing N to satisfy $N<\bar{\alpha}<N+1$

- all the terms in I_{low} have an exponent smaller than -1 and decay faster than t^{-1} for $t\to\infty$; therefore the integrals can be done straightforwardly. We chose the largest value for N that satisfies that condition.
- the terms in I_{high} have an exponent larger than -1:
  - the terms decay more slowly than t^{-1} for $t\to\infty$ and cannot integrated individually. We need to retain the exponential.
  - these terms grow less than t^{-1} for $t\to 0$ and therefore allow the lower limit of the integral to be pushed to 0. This leads to the complete Γ-function and an integral involving only small values of t.

iii) $\alpha<0$ integer

using similar techniques one gets (cf. Bender&Orszag 6.2 example 4)

$$\Gamma\left(-N,x\right)=\frac{\left(-1\right)^{N+1}}{N!}\left(\gamma-\sum_{n=1}^{N}\frac{1}{n}\right)+\frac{\left(-1\right)^{N+1}}{N!}\ln x-\sum_{n=0,n\neq N}^{\infty}\left(-1\right)^n\frac{x^{n-N}}{\left(n-N\right)n!}$$

This expression differs qualitatively from the other ones by the appearance of the $\ln x$.

1.2 Integration by Parts

Consider integration by parts of an integral of the form $\int_a^x t^n f(t)dt$. There are two ways to integrate by parts

i)

$$\int_a^x t^n f(t)dt = \frac{1}{n+1} t^{n+1} f(t) \Big|_a^x - \int_a^x \frac{1}{n+1} t^{n+1} \frac{df}{dt} dt$$

ii)

$$\int_a^x t^n f(t)dt = t^n F(t)|_a^x - \int_a^x n t^{n-1} F(t)dt$$

with $F(t)$ being an arbitrary antiderivative of $f(t)$

$$\frac{dF(t)}{dt} = f(t)$$

We have made progress if we can neglect the integral term that arises.

In each step the power of the polynomial in the remaining integral

- increases in i): suggests using it for approximations for small x
- decreases in ii): suggests using it for approximations for large x

Example 1: Taylor series and remainder

$$\begin{aligned} f(x) &= f(0) + \int_0^x f'(t)dt \\ &= f(0) + (t-x)\, f'(t)|_{t=0}^{t=x} - \int_0^x (t-x)\, f''(t)\, dt \\ &= f(0) + x f'(0) + \int_0^x (x-t)\, f''(t)\, dt \end{aligned}$$

Note: in this case it is more useful to use $t - x$ rather than t as the antiderivative of 1

Repeated integration by parts yields

$$f(x) = \sum_{n=0}^{N} \frac{x^n}{n!} f^{(n)}(0) + \frac{1}{N!} \int_0^x (x-t)^N\, f^{(N+1)}(t)\, dt$$

No approximation has been made:

- the integral term gives exactly the remainder of the Taylor expansion; it can be used to obtain detailed error estimates.

For convergence we need that the integral grows more slowly than $N!$ for fixed x.

Example 2: $I(x) = \int_x^\infty e^{-t^4} dt$ for $x \to \infty$

Motivated by case ii) above we would like to integrate the exponential. Rewrite it therefore as a derivative with compensating terms and rewrite $I(x)$ as

$$\begin{aligned} I(x) &= -\frac{1}{4}\int_x^\infty \frac{1}{t^3}\frac{d}{dt}\left(e^{-t^4}\right) dt \\ &= -\frac{1}{4}\frac{1}{t^3}e^{-t^4}\Big|_x^\infty + \frac{1}{4}\int_x^\infty \frac{-3}{t^4}e^{-t^4} dt \end{aligned}$$

Estimate the integral term

$$\int_x^\infty \frac{1}{t^4}e^{-t^4} dt < \frac{1}{x^4} I(x) \ll I(x) \qquad \text{for } x \to \infty$$

Therefore we get

$$I(x) \sim \frac{1}{4}\frac{e^{-x^4}}{x^3} \qquad x \to \infty$$

Higher-order terms are obtained in the same way by repeated integration by parts.

Example 3: $I(x) = \int_0^x t^{-\frac{1}{2}} e^{-t} dt$ for $x \to \infty$

Try straight-forward integration by parts

$$I(x) = -t^{-\frac{1}{2}} e^{-t}\Big|_0^x - \frac{1}{2}\int_0^x t^{-\frac{3}{2}} e^{-t}\, dt$$

This will not work: both terms diverge at the lower limit, although the original integral does not have that problem.

In this case one can take care of the lower limit by noting

$$\int_0^\infty t^{-\frac{1}{2}} e^{-t} dt = \Gamma(\frac{1}{2}) = \sqrt{\pi}$$

$$\begin{aligned} I(x) &= \int_0^\infty t^{-\frac{1}{2}} e^{-t} dt - \int_x^\infty t^{-\frac{1}{2}} e^{-t} dt \\ &= \sqrt{\pi} + t^{-\frac{1}{2}} e^{-t}\Big|_x^\infty + \frac{1}{2}\underbrace{\int_x^\infty t^{-\frac{3}{2}} e^{-t}\, dt}_{<\frac{1}{x}\int_x^\infty t^{-\frac{1}{2}} e^{-t} dt} \end{aligned}$$

Thus

$$I(x) \sim \sqrt{\pi} - \frac{e^{-x}}{\sqrt{x}} \tag{4}$$

Note:

- if the integration by parts leads to a boundary contribution that diverges or is even only much larger than $I(x)$ itself the integration by parts will not work: that large contribution needs to be canceled by the integral in the remainder, i.e. that integral will also be large and cannot be neglected.

Example 4: Stieltjes integral $I(x) = \int_0^\infty \frac{e^{-t}}{1+xt}dt$ for small and large x

Small x:
we want to integrate by parts repeatedly to generate terms x^n, $n > 0$

Since

$$\frac{d}{dt}\frac{1}{1+xt} = -x\,\frac{1}{(1+xt)^2}$$

we write

$$\begin{aligned} I(x) \underbrace{=}_{i.b.p.} & -\frac{1}{1+xt}e^{-t}\Big|_{t=0}^{\infty} + \int_0^\infty -x\frac{1}{(1+xt)^2}e^{-t}dt \\ \underbrace{=}_{i.b.p.} & 1 + x\frac{1}{(1+xt)^2}e^{-t}\Big|_0^\infty + x^2\int_0^\infty \frac{2}{(1+xt)^3}e^{-t}dt \\ = & 1 - x + \ldots (-1)^{n-1}(n-1)!x^{n-1} + (-1)^n\, n!\, x^n \int_0^\infty \frac{1}{(1+xt)^{n+1}}e^{-t}dt \end{aligned}$$

The integrals generated by the integration by parts exist for all n. Therefore we get for small x the series

$$I(x) \sim \sum_{n=0}^{\infty} (-1)^n\, n!\, x^n \qquad x \to 0$$

Clearly, this series does not converge to $I(x)$ for any fixed x and $N \to \infty$: the individual terms even diverge for fixed x as $n \to \infty$.

What sense can we make of such a series? Can it be of any use?

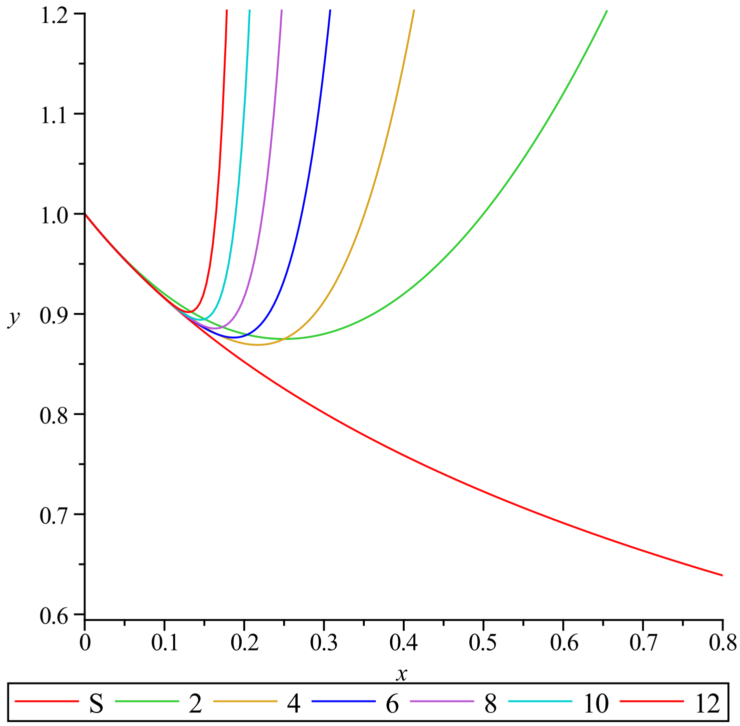

Figure 1: Stieltjes integral and its asymptotic approximations. Note that for only for small x the approximation with more terms is better than the approximation keeping fewer terms. For larger x the situations is reverse!

Estimate the error for fixed N:

$$
\begin{aligned}
E(x,N) &= \left| I(x) - \sum_{n=0}^{N} (-1)^n\, n!\, x^n \right| = \left| (-1)^n\ (N+1)!\, x^{N+1} \int_0^\infty \frac{1}{(1+xt)^{N+1+1}} e^{-t} dt \right| \\
&< (N+1)!\, x^{N+1} \int_0^\infty e^{-t} dt = (N+1)!\, x^{N+1} \qquad (5)
\end{aligned}
$$

Thus:

- For fixed N and $x \to 0$
  - the error goes to 0
  - the error is small compared to the last term in the series that is kept.

Such series appear quite often in asymptotic analysis and they can be very useful, even though they do not converge.

Definition

A series $\sum_n f_n (x - x_0)^{\alpha n}$ is called *asymptotic* to $f(x)$ at x_0,

$$
f(x) \sim \sum_{n=0} f_n (x - x_0)^{\alpha n} ,
$$

if for any *fixed* N

$$\left| f(x) - \sum_{n=0}^{N} f_n \, (x - x_0)^{\alpha n} \right| \ll |x - x_0|^{\alpha N} \quad \text{for} \quad x \to x_0.$$

Notes:

- In an asymptotic series, for $x \to x_0$ the error is small compared to the last term kept in the series.
- The series need not be an expansion in integer powers of $x - x_0$ ($\alpha \in \mathbb{R}^+$).
- If a series is asymptotic to $f(x)$ it does not have to converge to $f(x)$ for fixed x when $N \to \infty$.
- Any power series $\Sigma_{n=0}^{\infty} a_n x^n$ is asymptotic to some continuous function. One can construct such a function, e.g., in the form

  $$f(x) = \Sigma_{n=0}^{\infty} a_n \phi_n(x) \, x^n$$

  where $\phi_n(x)$ has only compact support within $|x| < \Delta_n$ with Δ_n shrinking with n and $\phi_n(x) = 1$ for $|x| < \delta_n$ with $\delta_n < \Delta_n$ (cf. Bender&Orszag Ch.3.8 Example 2).
  Therefore, it is not meaningful to say some series is asymptotic without specifying to which function it is asymptotic.
- A readable overview of asymptotic series and beyond (going beyond Bender & Orszag) are the first few chapters of J.P. Boyd's paper *The Devil's Invention: Asymptotic, Superasymptotic, and Hyperasymptotic Series*, Acta Applicandae Mathematicae **56** (1999) 1.

Thus, the series we obtained for the Stieltjes integral is asymptotic to the integral for small x.

Look at the error of the Stieltjes series in some more detail:
for increasing N the ratio ρ of successive error estimates is given by

$$\rho = \frac{(N+1)! x^{N+1}}{N! x^N} = (N+1) \, x$$

Thus,

- for $N + 1 < x^{-1}$ we have $\rho < 1$ and the error shrinks with increasing N
- for $N + 1 > x^{-1}$ we have $\rho > 1$ and the error increases with increasing N
- for each x there is an optimal N for which the error is minimal.

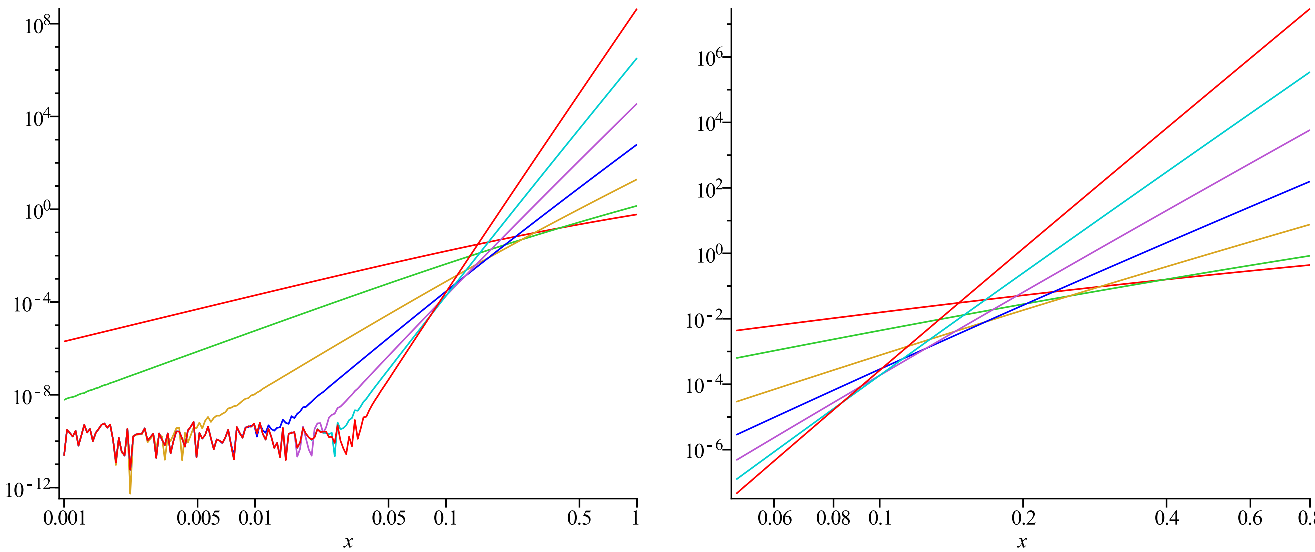

Figure 2: Error of the asymptotic series for the Stieltjes integral for $N = 1, 2, 4, 6, 8, 10$, 12 as a function of x. The error of the optimal choice N_{opt} is given by the envelope of all curves and decreases very rapidly with $x \to 0$.

Note:

- For the Stieltjes integral the error has the same sign and is smaller than the first omitted term. Therefore, for a given x the optimal N is that for which the $N + 1$-*th* term is minimal: *Optimal truncation rule.*
- The optimal truncation rule holds for Stieltjes-like integrals $\int_0^\infty \rho(t)/(1 + xt)dt$.

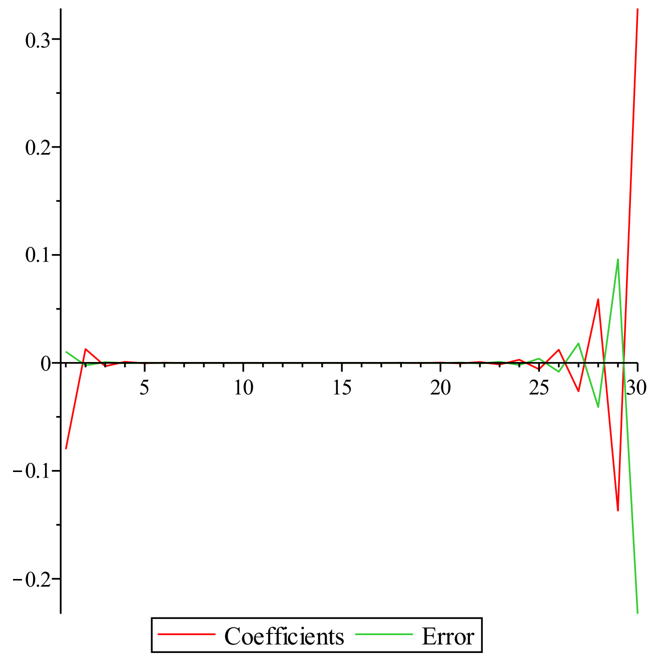

Figure 3: Terms a_n of the Stieltjes series and the error of the series $\Sigma_{i=0}^{n} a_i$ for $x = 0.08$. Consistent with (5) the error is smaller than the first omitted term.

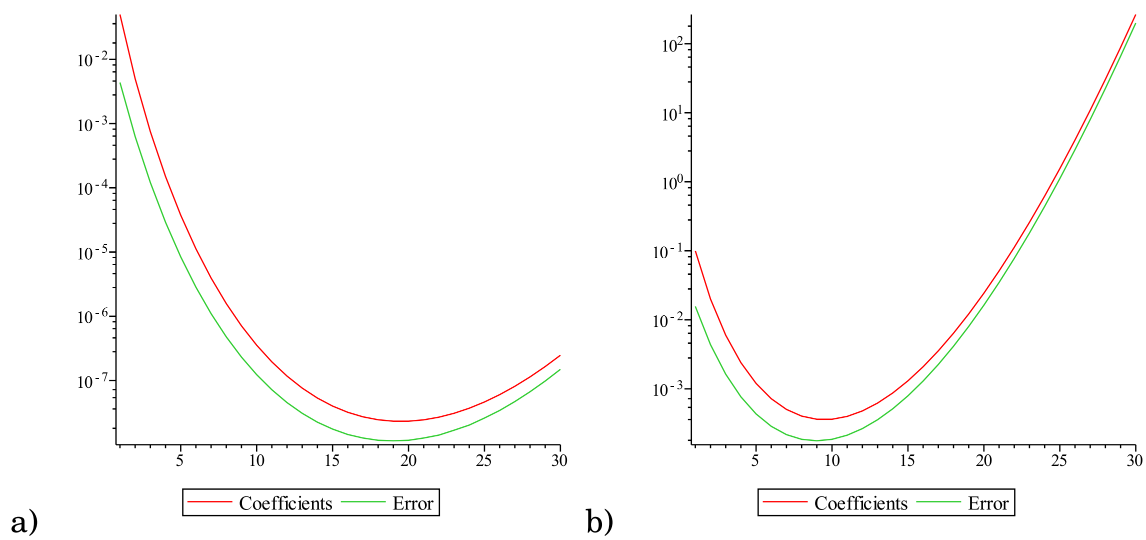

Figure 4: Absolute values of the terms a_n of the Stieltjes series and the absolute value of the error of the series $\Sigma_{i=0}^{n} a_i$ on a log scale for $x = 0.05$ (a) and $x = 0.1$ (b).

Note:

- For an asymptotic series the error cannot be made smaller than some *minimal value* and that value *increases* with increasing x, i.e. with increasing distance from the expansion point.
- In a converging series the error can always be made to go to 0 for any x within the radius of convergence.

Large x:
we would like to generate terms x^{-n}, $n > 0$. Try therefore integration by parts the other way around.

$$\begin{aligned}
I(x) &= \int_0^\infty e^{-t}\,\frac{1}{x}\frac{d}{dt}\ln(1+xt)\,dt \\
&\underbrace{=}_{\text{i.b.p.}} \underbrace{\frac{1}{x}\ln(1+xt)\,e^{-t}\Big|_{t=0}^{\infty}}_{0} + \frac{1}{x}\int_0^\infty \ln(1+xt)\,e^{-t}dt \\
&\underbrace{=}_{\text{i.b.p.}} \frac{1}{x^2}\left[(1+xt)\ln(1+xt)-(1+xt)\right]e^{-t}\Big|_0^\infty + \frac{1}{x^2}\int_0^\infty \left[(1+xt)\ln(1+xt)-(1+xt)\right]e^{-t}\,dt \\
&= \frac{1}{x^2} + \frac{1}{x^2}\int_0^\infty \left[(1+xt)\ln(1+xt)-(1+xt)\right]e^{-t}\,dt
\end{aligned}$$

So it seems, that $I(x) \sim x^{-2}$. But the integral still depends on x and therefore this scaling holds only if the integral does not grow with x.
But: the integral in the remainder diverges for $x \to \infty$:

$$\begin{aligned}
\int_0^\infty \left[(1+xt)\ln(1+xt)-(1+xt)\right]e^{-t}\,dt &> \int_0^\infty \left[xt\,\ln(xt)-(1+xt)\right]e^{-t}\,dt \\
&= x\ln x\int_0^\infty te^{-t}dt + x\left\{\int_0^\infty t\ln te^{-t} - te^{-t}dt\right\} - \int_0^\infty e^{-t}dt
\end{aligned}$$

Thus, the remainder goes like $x^{-1}\ln x \gg x^{-2}$.

One can show by other methods that

$$I(x) \sim \frac{\ln x}{x} \qquad x\to\infty$$

Note:

- Integration by parts can only generate series in integer powers of x.
  Whenever the asymptotic series is not a power series *integration by parts must fail.*

Therefore, no kind of integration by parts can work for the Stieltjes integral for large x: we need to learn more advanced methods yet.

Important:

- Always check whether the integral in the remainder can really be neglected.

1.3 Laplace Integrals[2]

Laplace integrals have the form

$$I(x) = \int_a^b f(t) e^{x\phi(t)} dt$$

Central Idea: For large x the contributions to the integral for t near the maximal value(s) of $\phi(t)$ dominate exponentially. Therefore it is sufficient to restrict the integral to a vicinity of the maximum (maxima).

If $\phi(t)$ has a maximum at $c \in (a,b)$ consider the approximation

$$I(x;\epsilon) = \int_{c-\epsilon}^{c+\epsilon} f(t) e^{x\phi(t)} dt$$

If the maximum of $\phi(t)$ occurs at a consider the approximation

$$I(x;\epsilon) = \int_{a}^{a+\epsilon} f(t) e^{x\phi(t)} dt$$

and analogously if the maximum is at b.

To evaluate $I(x;\epsilon)$ it is then sufficient to use an approximation for $f(t)$ that is valid in the vicinity of the maximum.

For this to work one needs that

- the final approximation does not depend on ϵ
- the asymptotic expansions of $I(x)$ and $I(x;\epsilon)$ are identical
$$I(x) \sim I(x;\epsilon)$$

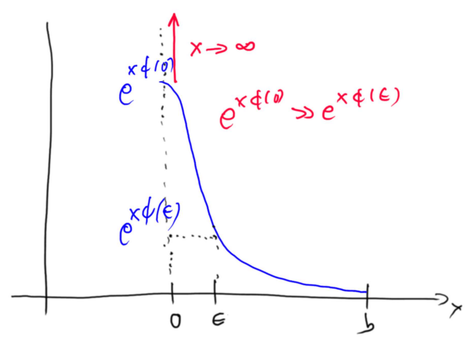

Figure 5: The maximum becomes much larger than all other points for $x \to \infty$ with ϵ fixed.

This is actually the case:
Typically $I(x)$ and $I(x;\epsilon)$ differ only by terms that are *exponentially small in* x, because for $a < t < c - \epsilon$ and $c + \epsilon < t < b$ the term $e^{x\phi(t)}$ is exponentially small compared to $e^{x\phi(c)}$ when $x \to \infty$ with ϵ fixed.

To show explicitly $I(x) \sim I(x;\epsilon)$ estimate

$$\frac{I(x) - I(x;\epsilon)}{I(x;\epsilon)}$$

For simplicity assume the maximum occurs at $c = 0$, which could be inside the interval or at its boundary.

First consider maximum at the lower boundary, $c = 0 = a$, i.e. $\phi(t)$ is decreasing from a to b

[2]This method was first presented by Laplace (1774).

We need an upper bound for the numerator

$$|I(x) - I(x;\epsilon)| = \left|\int_\epsilon^b f(t)e^{x\phi(t)}dt\right| < e^{x\phi(\epsilon)}\left|\int_\epsilon^b f(t)dt\right|$$

and a lower bound for the denominator

$$I(x;\epsilon) = \int_0^\epsilon f(t)e^{x\phi(t)}dt$$

We are assuming that $f(t)$ is sufficiently smooth. If $f(0) \neq 0$ we can therefore assume that $f(t)$ does not change sign in $[0,\epsilon]$ and has a minimal value $f_{min} > 0$ in $[0,\epsilon]$.

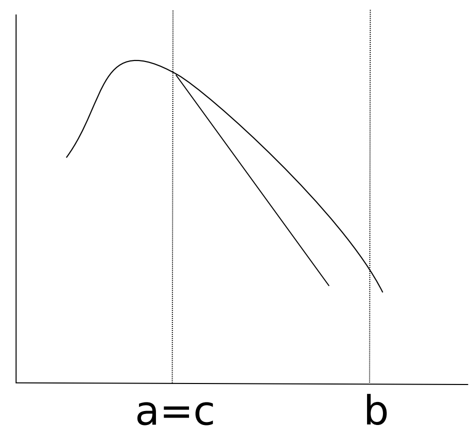

Figure 6: Sketch for the estimates for the exponential.

We can find $\gamma > 0$ such that

$$\phi(t) \geq \phi(0) - \gamma t \qquad \text{for } t \in [0,\epsilon]$$

Then

$$\begin{aligned} I(x;\epsilon) &> e^{x\phi(0)}\int_0^\epsilon f_{min}e^{-x\gamma t}dt \\ &= e^{x\phi(0)}\frac{f_{min}}{\gamma x}\left(1 - e^{-\gamma x\epsilon}\right) \end{aligned}$$

and one has

$$I(x,;\epsilon) > C\,x^{-1}e^{x\phi(0)} \qquad x \to \infty$$

with some x-independent constant C.

Thus

$$\frac{|I(x) - I(x;\epsilon)|}{|I(x;\epsilon|} < \frac{\left|\int_\epsilon^b f(t)dt\right|}{C}x^\rho e^{-x(\phi(0)-\phi(\epsilon))} \underbrace{\rightarrow}_{\text{exponentially}} 0 \qquad \text{for } x \to \infty \tag{6}$$

Note:

- since $I(x)$ and $I(x;\epsilon)$ differ only by terms that are exponentially small in x for $x \to \infty$ their expansions in powers of x are equal to all orders:

$$I(x;\epsilon) \sim I(x)$$

Example 1: $I(x) = \int_0^a (1-t)^{-1}e^{-xt}dt$ for $a < 1$.

Here $\phi(t) = -t$. It is maximal at $t = 0$. Expand therefore $f(t)$ around $t = 0$.

$$I(x;\epsilon) \sim \int_0^\epsilon (1 + \ldots)\,e^{-xt}dt = \frac{1}{x}\left(1 - e^{-\epsilon x}\right)$$

The term containing ϵ is exponentially smaller than the other term (it is *subdominant*):

$$I(x) \sim \frac{1}{x} \qquad x \to \infty$$

Go to higher orders in the expansion und use that the series converges uniformly for $t \in [0,\epsilon]$,

$$I(x;\epsilon) \sim \int_0^\epsilon \sum_{n=0}^\infty t^n e^{-xt} dt = \sum_{n=0}^\infty \int_0^\epsilon t^n e^{-xt} dt = \sum_{n=0}^\infty \int_0^{\epsilon x} \left(\frac{s}{x}\right)^n \frac{1}{x} e^{-s} ds.$$

We are keeping ϵ fixed as $x \to \infty$. The upper limit of that integral goes therefore to ∞ for $x \to \infty$. Although all these integrals can be done easily directly, it is yet easier to replace ϵx by ∞,

$$I(x) \sim \sum_{n=0}^\infty \int_0^\infty t^n e^{-xt} dt = \sum_{n=0}^\infty (-1)^n \frac{d^n}{dx^n} \int_0^\infty e^{-xt} dt = \sum_{n=0}^\infty (-1)^n \frac{d^n}{dx^n} \frac{1}{x} \qquad x \to \infty.$$

Why can we replace ϵx by ∞? The approximation for $(1+t)^{-1}$ is only valid for small t, $0 \le t < 1$, but the difference is subdominant relative to $I(x)$ since $x \to \infty$ for ϵ fixed,

$$\sum_{n=0}^\infty \int_\epsilon^\infty t^n e^{-xt} dt = \sum_{n=0}^\infty (-1)^n \frac{d^n}{dx^n} \left\{ \frac{1}{x} e^{-x\epsilon} \right\}.$$

Note:

- The contributions to $I(x)$ all come from inside the interval $[c-\epsilon, c+\epsilon]$. There we need to approximate $f(t)$ systematically to make the integrals doable.
- The contributions from outside of $[c-\epsilon, c+\epsilon]$ are only subdominant with respect to $I(x)$; therefore it does not matter how we approximate $f(t)$ outside that interval. In particular, we can replace ϵx by ∞, even if the approximation of $f(t)$ is not valid there.

1.3.1 Watson's Lemma

For the less general integral

$$I(x) = \int_0^b f(t) e^{-xt} dt$$

one can give a general expression if $f(t)$ is given by an asymptotic series

$$f(t) \sim t^\alpha \sum_{n=0}^\infty a_n t^{\beta n} \qquad t \to 0$$

For the integral to converge we need $\alpha > -1$ and $\beta > 0$.

The asymptotic series for $f(t)$ satisfies

$$\left| f(t) - t^\alpha \sum_{n=0}^N a_n t^{\beta n} \right| \ll K t^{\alpha+\beta N}, \qquad t \to 0 \qquad \text{for all N}$$

For given N we can therefore choose $\epsilon > 0$ such that

$$\left| f(t) - t^\alpha \sum_{n=0}^N a_n t^{\beta n} \right| \le K t^{\alpha+\beta N}\, t^\beta = K t^{\alpha+\beta(N+1)}, \qquad t \in [0,\epsilon]$$

Now consider

$$I(x;\epsilon) = \int_0^{\epsilon} f(t)e^{-xt}dt$$

Compared to $I(x)$ we make only an exponentially small error by replacing $I(x)$ by $I(x;\epsilon)$.

$$\begin{aligned}
\left| I(x;\epsilon) - \int_0^{\epsilon} t^{\alpha}\sum_{n=0}^{N} a_n t^{\beta n} e^{-xt}dt \right| &\leq \int_0^{\epsilon} \left| f(t)\, e^{-xt} - t^{\alpha}\sum_{n=0}^{N} a_n t^{\beta n} e^{-xt}\right| dt \\
&\leq K\int_0^{\epsilon} t^{\alpha+\beta(N+1)}e^{-xt}dt \\
&< K\int_0^{\infty} t^{\alpha+\beta(N+1)}e^{-xt}dt \\
&= K\frac{\Gamma\left(\alpha+\beta(N+1)+1\right)}{x^{\alpha+\beta(N+1)+1}}
\end{aligned}$$

Now replace again ϵ by ∞ in the integral on the left hand side and obtain

$$\left| I(x;\epsilon) - \sum_{n=0}^{N} a_n \frac{\Gamma\left(\alpha+\beta N+1\right)}{x^{\alpha+\beta N+1}} \right| \leq K\frac{\Gamma\left(\alpha+\beta(N+1)+1\right)}{x^{\alpha+\beta(N+1)+1}} \ll \frac{\Gamma\left(\alpha+\beta N+1\right)}{x^{\alpha+\beta N+1}} \qquad x\to\infty.$$

Thus for all N the error is small compared to the last term in the series and we have established the asymptotic series

$$I(x) \sim \sum_{n=0}^{\infty} a_n \frac{\Gamma\left(\alpha+\beta n+1\right)}{x^{\alpha+\beta n+1}} \qquad x\to\infty$$

Note:

- Watson's lemma will always generate a power series in an algebraic power of x because the maximum of $\phi(t) \equiv -xt$ is at $t=0$.

Example 2: Bessel function $K_0(x) = \int_1^{\infty}\left(s^2-1\right)^{-1/2} e^{-xs}ds$

To bring $K_0(x)$ in the correct form shift the limit $t = s-1$

$$K_0(x) = e^{-x}\int_0^{\infty}\left(t^2+2t\right)^{-\frac{1}{2}} e^{-xt}dt$$

To expand around $t=0$ rewrite

$$\left(t^2+2t\right)^{-\frac{1}{2}} = (2t)^{-\frac{1}{2}}\left(1+\frac{1}{2}t\right)^{-\frac{1}{2}}$$

For small u Taylor series leads to the binomial theorem in the form

$$\frac{1}{(1-u)^s} = 1 + \left.\frac{s}{(1-u)^{s+1}}\right|_{u=0} u + \left.\frac{s\,(s+1)}{(1-u)^{s+2}}\right|_{u=0}\frac{1}{2!}u^2 + \ldots = \sum_{n=0}^{\infty}\frac{1}{n!}u^n\frac{\Gamma\left(s+n\right)}{\Gamma\left(s\right)} \qquad \textbf{(7)}$$

Insert to get

$$\begin{aligned}
K_0(x) &= e^{-x}\int_0^\infty \left(t^2+2t\right)^{-\frac{1}{2}} e^{-xt}dt \sim e^{-x}\int_0^\epsilon \left(t^2+2t\right)^{-\frac{1}{2}} e^{-xt}dt \\
&= e^{-x}\int_0^\epsilon (2t)^{-\frac{1}{2}}\sum_{n=0}^\infty \left(-\frac{1}{2}t\right)^n \frac{\Gamma\left(\frac{1}{2}+n\right)}{n!\Gamma\left(\frac{1}{2}\right)}e^{-xt}dt \\
&\underbrace{=}_{\text{uniform convergence in }[0,\epsilon]} e^{-x}\sum_{n=0}^\infty \int_0^\epsilon (2t)^{-\frac{1}{2}} \left(-\frac{1}{2}t\right)^n \frac{\Gamma\left(\frac{1}{2}+n\right)}{n!\Gamma\left(\frac{1}{2}\right)}e^{-xt}dt \\
&\sim e^{-x}\sum_{n=0}^\infty \int_0^\infty (2t)^{-\frac{1}{2}} \left(-\frac{1}{2}t\right)^n \frac{\Gamma\left(\frac{1}{2}+n\right)}{n!\Gamma\left(\frac{1}{2}\right)}e^{-xt}dt
\end{aligned}$$

Note:

- Although the series coming from the binomial theorem only converges for $|\frac{1}{2}t| < 1$, it can be used in the integral over $[0,\infty)$ because all dominant contributions to the integral come from $t \in [0,\epsilon]$.

We have rederived Watson's lemma with $\alpha = -\frac{1}{2}$, $\beta = 1$, and

$$a_n = \frac{(-1)^n}{2^{n+1/2}}\frac{\Gamma\left(\frac{1}{2}+n\right)}{n!\Gamma\left(\frac{1}{2}\right)}$$

and get

$$\begin{aligned}
K_0(x) &= e^{-x}\sum_{n=0}^\infty \frac{(-1)^n}{2^{n+1/2}}\frac{\Gamma\left(\frac{1}{2}+n\right)\Gamma\left(-\frac{1}{2}+n+1\right)}{n!\Gamma\left(\frac{1}{2}\right)x^{-\frac{1}{2}+n+1}} \\
&= e^{-x}\sum_{n=0}^\infty \frac{(-1)^n}{2^{n+1/2}}\frac{\left[\Gamma\left(\frac{1}{2}+n\right)\right]^2}{n!\Gamma\left(\frac{1}{2}\right)}\frac{1}{x^{n+\frac{1}{2}}} \qquad \text{for} \qquad x\to\infty
\end{aligned}$$

Note:

- In this case the series for $f(t)$ is not only asymptotic but converges.

1.3.2 General Laplace Integrals

Obtain the leading-order behavior of the general Laplace integral

$$I(x) = \int_a^b f(t)e^{x\phi(t)}dt$$

As indicated before one has to distinguish the cases when $\phi(t)$ has local maxima and when the maxima are attained at the boundaries of the interval. In each case $\phi(t)$ and $f(t)$ are expanded around the maximum $t = c$.

i) Maximum at endpoint, $c = a$

Expand

$$\phi(t) = \phi(a) + (t-a)\phi'(a) \qquad \text{with} \quad \phi'(a) < 0$$

$$\begin{aligned}
I(x;\epsilon) &\approx \int_a^{a+\epsilon} \left[f(a) + f'(a)\,(t-a)\right] e^{x\phi(a)+x\phi'(a)(t-a)}\,dt \\
&= f(a)e^{x\phi(a)} \int_a^{a+\epsilon} e^{x\phi'(a)(t-a)}dt + f'(a)e^{x\phi(a)} \int_a^{a+\epsilon} (t-a)\, e^{x\phi'(a)(t-a)}dt \\
&= f(a)e^{x\phi(a)} \frac{-1}{x\phi'(a)} \left\{1 - e^{x\phi'(a)\epsilon}\right\} + f'(a)e^{x\phi(a)} \frac{1}{\phi'(a)} \frac{d}{dx} \left[\frac{-1}{x\phi'(a)} \left\{1 - e^{x\phi'(a)\epsilon}\right\}\right] \\
&= -f(a)\frac{e^{x\phi(a)}}{x\phi'(a)} + f'(a)e^{x\phi(a)} \frac{1}{x^2\phi'(a)^2} + \text{exponentially small terms}
\end{aligned}$$

Thus

$$I(x) \sim -f(a)\frac{e^{x\phi(a)}}{x\phi'(a)} \qquad x \to \infty \tag{8}$$

Note:

- the leading-order term depends on the value of f at $x = a$, but on the value *and* the derivative of ϕ at $x = a$: changes in ϕ are amplified by ϕ being in the exponential and $x \to \infty$
- since we kept only the first non-trivial term in the expansion of $\phi(t)$ only the leading-order term in the result can be trusted. For higher-order terms see Sec.1.3.3.
- obviously, this result is not valid at a local maximum where $\phi'(a) = 0$.

ii) Maximum inside the interval

Now the maximum is a local maximum

$$\phi(t) = \phi(c) + \frac{1}{p!}\phi^{(p)}(c)\,(t-c)^p \qquad p \text{ even and } \phi^{(p)}(c) < 0 \tag{9}$$

$$I(x;\epsilon) = \int_{c-\epsilon}^{c+\epsilon} \left[f(c) + \ldots\right] e^{x\phi(c)} e^{x\frac{1}{p!}\phi^{(p)}(c)(t-c)^p}\,dt$$

Use

$$s = \left(\frac{x}{p!}\phi^{(p)}(c)\right)(t-c)$$

to get

$$\sim e^{x\phi(c)} f(c) \left[\frac{-x\phi^{(p)}(c)}{p!}\right]^{-\frac{1}{p}} \int_{-\infty}^{\infty} e^{-s^p} ds$$

Note:

- the limits of the integral go to $\pm\infty$ for $x \to \infty$ and fixed ϵ.

Using

$$\int_{-\infty}^{\infty} e^{-s^p} ds = 2\frac{\Gamma\left(\frac{1}{p}\right)}{p}$$

we get

$$I(x) \sim 2\frac{\Gamma\left(\frac{1}{p}\right)}{p}\left[-\frac{p!}{x\phi^{(p)}}\right]^{\frac{1}{p}} f(c)\, e^{x\phi(c)} \tag{10}$$

In the generic case of a quadratic maximum, $p = 2$, this reduces (using $\Gamma(\frac{1}{2}) = \sqrt{\pi}$) to

$$I(x) \sim \sqrt{\frac{2\pi}{-x\phi''(c)}}\, f(c)e^{x\phi(c)} \tag{11}$$

Note:

- this derivation is a bit careless in the treatment of the expansion of $\phi(t)$. A more careful analysis is done in the next example.

Example $I(x) = \int_0^{\pi/2} e^{-x\sin^2 t} dt$

Here $\phi(t)$ has its maximum at $t = 0$. Expect that $\sin^2 t$ can be replaced by t^2 since only small t contribute. If that is the case one gets

$$I(x;\epsilon) \sim \int_0^{\epsilon} e^{-xt^2} dt$$

Can again extend the integral to $+\infty$

$$I(x;\epsilon) \sim \int_0^{\infty} e^{-xt^2} dt = \frac{1}{2}\sqrt{\frac{\pi}{x}}$$

Is the replacement $\sin^2 t \to t^2$ justified? We need to show for small ϵ that

$$\int_0^{\epsilon} e^{-x\sin^2 t} dt \sim \int_0^{\epsilon} e^{-xt^2} dt$$

Clearly, $e^{-x\sin^2 t} \approx e^{-xt^2}$ for sufficiently small t at fixed x.
But we need to look at $x \to \infty$ for fixed ϵ: for $x \to \infty$ the absolute (not the relative) error that we make with this approximation is magnified. Because the error is in the exponential, this error leads to a vastly wrong magnitude: we have at small, but fixed t

$$\frac{e^{-x\sin^2 t}}{e^{-xt^2}} \to \infty \quad \text{exponentially large for} \quad x \to \infty$$

Thus, for the two exponentials to be close to each other, the acceptable values of t go to 0 as $x \to \infty$. How small does t need to be?

Expand $\sin t$ to higher order

$$x\sin^2 t = x\left\{t - \frac{1}{6}t^3 + \mathcal{O}(t^5)\right\}^2 = xt^2 - \frac{1}{3}xt^4 + \mathcal{O}(xt^5)$$

Then we get for the integral

$$\int_0^\epsilon e^{-x\sin^2 t}dt = \int_0^\epsilon e^{-xt^2+\frac{1}{3}xt^4+\mathcal{O}(xt^5)}dt$$

Divide the integration domain into two parts

$$\int_0^\epsilon e^{-xt^2-\frac{1}{3}xt^4+\mathcal{O}(xt^5)}dt = \int_0^{t_{max}} e^{-xt^2-\frac{1}{3}xt^4+\mathcal{O}(xt^5)}dt + \int_{t_{max}}^\epsilon e^{-x\sin^2 t}dt$$

such that

- in $[0, t_{max}]$ the replacement of $x\sin^2 t$ is valid, we want to expand $e^{-\frac{1}{3}xt^4}$ need $xt^4_{max} \to 0$ for $x \to \infty$
- the integral over $[t_{max}, \epsilon]$ is negligible: need $xt^2_{max} \to \infty$ for $x \to \infty$

Therefore, for $t_{max} = x^{-\alpha}$ we need

$$1 - 4\alpha < 0 \qquad 1 - 2\alpha > 0$$

i.e.

$$\frac{1}{4} < \alpha < \frac{1}{2}$$

Then in the first integral the integrand can be expanded[3]

$$\begin{aligned}\int_0^{x^{-\alpha}} e^{-x\sin^2 t}dt &= \int_0^{x^{-\alpha}} e^{-xt^2}e^{+\frac{1}{3}xt^4+\dots}dt \to \\ &\int_0^{x^{-\alpha}} e^{-xt^2}\left\{1 + \frac{1}{3}xt^4 + \dots\right\}dt \qquad \text{for } x \to \infty\end{aligned}$$

With $s = x^{\frac{1}{2}}t$ we get

$$\begin{aligned}\int_0^{x^{-\alpha}} e^{-x\sin^2 t}dt &\to x^{-\frac{1}{2}}\int_0^{x^{\frac{1}{2}-\alpha}} e^{-s^2}ds + \frac{1}{3}x^{-\frac{1}{2}}\int_0^{x^{\frac{1}{2}-\alpha}} x\left(\frac{s}{x^{\frac{1}{2}}}\right)^4 e^{-s^2}ds \\ &\to \frac{1}{x^{\frac{1}{2}}}\frac{1}{2}\sqrt{\pi} + \mathcal{O}\left(x^{-\frac{3}{2}}\right)\end{aligned}$$

Note, that again the limit of the integral in terms of s goes to ∞.

[3] An error in the exponential that cannot be expanded, i.e. a small relative but large absolute error, leads to a multiplicative rather than an additive error in the result.

What about the integral from $x^{-\alpha}$ to ϵ? It is exponentially small compared to $I(x)$:

$$\int_{x^{-\alpha}}^{\epsilon} e^{-x\sin^2 t}dt < e^{-x\sin^2 x^{-\alpha}}\epsilon$$

since $\sin^2 t$ is increasing for small t.

For $\alpha < \frac{1}{2}$ we have $x\sin^2 x^{-\alpha} \to x^{1-2\alpha} \to \infty$ with $x \to \infty$.

Thus, the expansion of the function in the exponent is justified.

Example: Modified Bessel function $I_n(x) = \frac{1}{\pi}\int_0^{\pi} e^{x\cos t}\cos nt\, dt$

$\phi(t)$ has its maximum at $t = 0$. Try to approximate both cosines by lowest order terms and consider

$$\int_0^{\epsilon} e^x\, 1 dt = \epsilon e^x \tag{12}$$

Why does this result depend on the artificial parameter ϵ?

We need to keep the first *non*-constant term in the expansion for ϕ because limiting the integration domain to $[0, \epsilon]$ is only allowed if ϕ and its approximation truly decrease and damp contributions from $x > \epsilon$ exponentially fast (cf. (6)).
The maximum at the boundary is at the same time a local maximum: we need to keep the quadratic terms of $\phi(t)$, otherwise there is no damping of the contributions away from the maximum (cf. (9))[4].

$$\begin{aligned} I_n(x;\epsilon) &= \frac{1}{\pi}\int_0^{\epsilon} e^{x-\frac{1}{2}xt^2}dt \\ &\sim \frac{1}{\pi}e^x\frac{1}{2}\sqrt{\frac{2\pi}{x}} = \frac{1}{\sqrt{2\pi x}}e^x \end{aligned}$$

Now there is no ϵ-dependence any more, as required, since the rapid decay with increasing t allowed us to extend the upper limit of the integral to $+\infty$.

Example: $I(x) = \int_0^{\infty} e^{-\frac{1}{t}}e^{-xt}dt$

It looks like a case for Watson's lemma with $f(t) = e^{-\frac{1}{t}}$.

However: $e^{-\frac{1}{t}}$ and **all** its derivatives vanish at $t = 0 \Rightarrow$ Watson's lemma would yield

$$I(x) = 0$$

The dominant contribution to the integral comes from the maximum of the integrand. This maximum is not at $t = 0$ as Watson's lemma assumes. In fact, in this case the maximum depends on x,

$$t_{max} = \frac{1}{\sqrt{x}}$$

Note:

[4]Note, the expression (8) has $\phi'(c)$ in the denominator.

- a maximum that depends on x is called a *moveable maximum.*
- to apply Laplace's method to a moveable maximum it is best to transform t into a coordinate in which the maximum is fixed, independent of x.

We could introduce a shift of the coordinate

$$s = t - t_{max}.$$

This would lead to complicated expressions in $e^{-\frac{1}{t}}$.

Instead, the maximum can be fixed by a rescaling

$$s = \sqrt{x}t \qquad s_{max} = 1$$

$$I(x) = \frac{1}{\sqrt{x}} \int_0^\infty e^{-\sqrt{x}\left(\frac{1}{s}+s\right)} ds$$

Expand now $\phi(s) \equiv \frac{1}{s} + s$ around its maximum at $s = 1$ using $s = 1 + u$

$$\phi(u) = \frac{1}{1+u} + 1 + u = 1 - u + u^2 + \ldots + 1 + u = 2 + u^2 + \mathcal{O}(u^3)$$

Thus, we get

$$\begin{aligned} I(x) &\sim \frac{1}{\sqrt{x}} \int_{-\epsilon}^{\epsilon} e^{-\sqrt{x}\left(2+u^2+\mathcal{O}(u^3)\right)} du \\ &\underbrace{=}_{v=x^{\frac{1}{4}}u} \frac{1}{\sqrt{x}} e^{-2\sqrt{x}} \frac{1}{x^{\frac{1}{4}}} \int_{-\epsilon x^{\frac{1}{4}}}^{\epsilon x^{\frac{1}{4}}} e^{-v^2} dv \\ &\sim \frac{1}{\sqrt{x}} e^{-2\sqrt{x}} \frac{1}{x^{\frac{1}{4}}} \int_{-\infty}^{\infty} e^{-v^2} dv \\ &= \frac{1}{\sqrt{x}} e^{-2\sqrt{x}} \frac{1}{x^{\frac{1}{4}}} \sqrt{\pi} \end{aligned}$$

This is consistent with our previous derivation of (11), noting that $\sqrt{x}$ plays the role of x in (11) and $\phi''(1) = -2$.

Notes:

- $I(x)$ decreases faster than any power. Watson's lemma can only generate results that depend on x as a power series and can therefore not be applicable.
- This approach does not work for all integrals in which the location of the maximum scales with x. E.g., it does not work for $\int_0^a t^\alpha e^{-xt^\beta} dt$. Why not?
  When the ϕ is expanded around the maximum the quadratic term must have a prefactor that goes to ∞ as x goes to ∞, i.e. the width of the maximum must go to 0 for $x \to \infty$ and it must do so faster than the moveable maximum goes to 0. In the calculation above this implied that the limits in v go to $\pm\infty$. Otherwise the integral would depend on ϵ. In this example this is the case because $e^{-\frac{1}{t}}$ goes to 0 very fast for $t \to 0$.

1.3.3 Higher-order Terms with Laplace's Method

Consider the case of a maximum of $\phi(t)$ at $c \in (a,b)$ with $\phi'(c) = 0$, $\phi''(c) < 0$, and $f(c) \neq 0$. To get higher-order terms expand $\phi(t)$ and $f(t)$ to higher orders. How far do we need to go? Without loss of generality assume $c = 0$ to simplify the algebra.

$$I(x) \sim \int_{-\epsilon}^{+\epsilon} \left[f(0) + f'(0)t + \frac{1}{2} f''(0) t^2 + \ldots \right] e^{x\left[\phi(0) + \frac{1}{2}\phi''(0)t^2 + \frac{1}{6}\phi'''(0)t^3 + \frac{1}{4!}\phi^{(4)}t^4 + \ldots\right]} dt$$

The integral extends only over small values of $|t|$: expand the exponential in t:

- In order to be able to extend the integral to $(-\infty, +\infty)$ we need to **keep the quadratic term** up in the exponential and cannot include it in the expansion of the exponential
- If we did not extend the domain of integration we may be tempted to include the quadratic term in the expansion of the exponential. As illustrated in the example above (cf. (12)), the resulting integrals would, however, depend on ϵ.
- The **cubic and quartic terms must not be kept** in the exponential:
  - without expanding the exponential in the cubic and quartic term the extension to $(-\infty, +\infty)$ would not be possible if $\phi'''(c) \neq 0$ or $\phi^{(iv)}(c) > 0$: the integral would diverge.
  - the polynomial expansion in the exponent is likely to have maxima outside the interval $(-\epsilon, \epsilon)$. These maxima are only a result of the approximation of $\phi(t)$ $\Rightarrow$ if we were not to expand the exponential these spurious maxima may contribute to the integral once the domain of integration is extended to $(-\infty, +\infty)$ although we know that the contributions from outside $(-\epsilon, \epsilon)$ are exponentially smaller for the full, non-expanded ϕ.

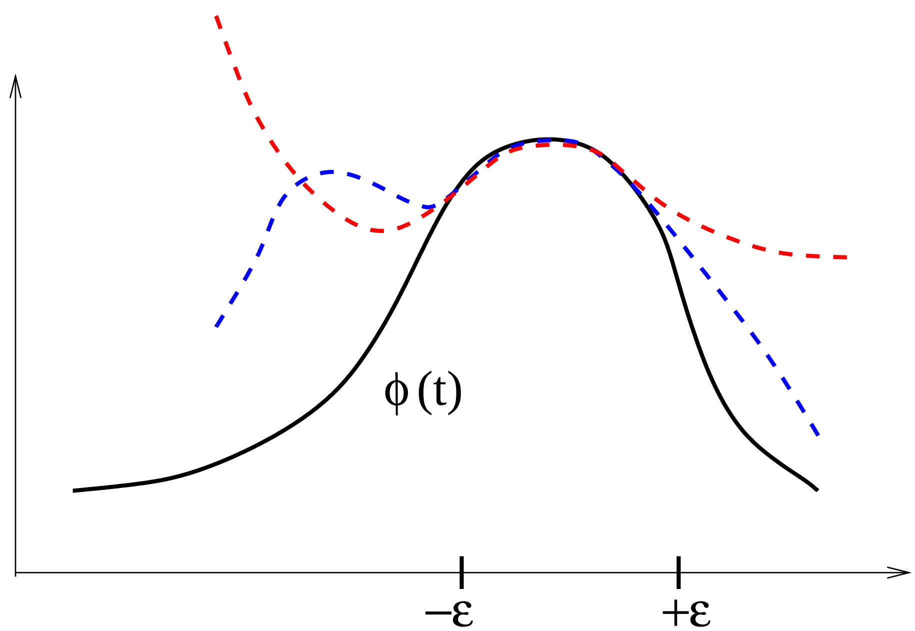

Figure 7: Possible divergence and spurious maxima of Taylor expansion of $\phi(t)$.

Rescale integration variable to extract the x-dependence from the integrals, $s=\sqrt{x}t$, and extend the integration from $[-\epsilon\sqrt{x},\epsilon\sqrt{x}]$ to $(-\infty,+\infty)$ and

$$I(x)\sim\frac{1}{\sqrt{x}}\int_{-\infty}^{+\infty}\left[f(0)+f'\frac{s}{\sqrt{x}}+\frac{1}{2}f''\frac{s^2}{x}+\frac{1}{6}f'''\frac{s^3}{x^{\frac{3}{2}}}+\ldots\right]e^{\left[x\phi(0)+\frac{1}{2}\phi''s^2+\frac{1}{6}\phi'''\frac{s^3}{x^{\frac{1}{2}}}+\frac{1}{4!}\phi^{(4)}\frac{s^4}{x}+\frac{1}{5!}\phi^{(5)}\frac{s^5}{x^{\frac{3}{2}}}\ldots\right]}ds$$

where all derivatives are evaluated at $s=c=0$.

In the integration odd terms in s cancel
$\Rightarrow$ the first non-trivial term beyond the leading-order term will involve the terms of $\mathcal{O}(\frac{1}{x})$
$\Rightarrow$ we need to keep two additional terms in the expansion of $\phi(t)$ and $f(t)$ to get one additional order in the expansion for $I(x)$

$$I(x)\sim\frac{1}{\sqrt{x}}e^{x\phi(0)}\int_{-\infty}^{+\infty}e^{\frac{1}{2}\phi''s^2}\left[f(0)+\frac{1}{x}\left\{f(0)\left(\frac{1}{4!}\phi^{(4)}s^4+\frac{1}{2}\left(\frac{1}{6}\phi'''\right)^2s^6\right)+\frac{1}{6}f'\phi'''s^4+\frac{1}{2}f''s^2\right\}+\mathcal{O}\left(\frac{1}{x^2}\right)\right]$$

Using

$$\int_{-\infty}^{+\infty}s^{2n}e^{-\alpha s^2}ds=(-1)^n\frac{d^n}{d\alpha^n}\int_{-\infty}^{+\infty}e^{-\alpha s^2}ds=(-1)^n\frac{d^n}{d\alpha^n}\sqrt{\frac{\pi}{\alpha}}$$

we get (reinstating again c)

$$I(x)\sim\sqrt{\frac{2\pi}{-x\phi''(c)}}e^{x\phi(c)}\left\{f(c)+\frac{1}{x}\left[-\frac{f''(c)}{2\phi''(c)}+\frac{f(c)\phi^{(iv)}(c)}{8\phi''^2(c)}+\frac{f'(c)\phi'''(c)}{2\phi''^2(c)}-\frac{5\phi'''^2(c)f(c)}{24\phi''^3(c)}\right]+\mathcal{O}\left(\frac{1}{x^2}\right)\right\}$$

Note:

- since odd powers in s cancel the higher-order corrections are an expansion in $\frac{1}{x}$ rather than in $\frac{1}{\sqrt{x}}$.

Note:

- **Important:** It is better to think in terms of an expansion in $1/x$ than in $t-c$ and to rewrite the integral extracting the quadratic term explicitly using $t-c=\sqrt{-\frac{2}{x\,\phi''(c)}}s$

$$\begin{aligned}I(x)&=\int f(t)e^{\frac{1}{2}x\phi''(c)(t-c)^2}e^{x\phi(t)-\frac{1}{2}x\phi''(c)(t-c)^2}dt=\\&=\sqrt{-\frac{2}{x\,\phi''(c)}}\int\underbrace{f\left(c+\sqrt{-\frac{2}{x\,\phi''(c)}}s\right)e^{x\phi\left(c+\sqrt{-\frac{2}{x\,\phi''(c)}}s\right)+s^2}}_{\text{expand in }1/x}e^{-s^2}ds\end{aligned}$$

Example: Again $I(x)=\int_0^{\frac{\pi}{2}}e^{-x\sin^2t}dt$

To get a higher-order approximation expand

$$\begin{aligned} I(x) &\sim \int_0^{\epsilon} e^{-x\left(t-\frac{1}{6}t^3+\ldots\right)^2} dt = \int_0^{\epsilon} e^{-x\left(t^2-\frac{1}{3}t^4+\ldots\right)} dt \\ &= \int_0^{\infty} e^{-xt^2}\left\{1+\frac{1}{3}xt^4+\ldots\right\} dt \\ &= \frac{1}{2}\sqrt{\frac{\pi}{x}}\left(1+\frac{1}{4x}+\ldots\right) \qquad x\to\infty \end{aligned}$$

Note:

- If the exponential had not been expanded in the quartic term the integral over $(0,\infty)$ would have (erroneously) diverged.

1.4 Generalized Fourier Integral

In general the Laplace integral could involve a complex $\phi(t)$.

Consider here the case of purely imaginary ϕ

$$\phi(t) = i\psi(t) \qquad I(x) = \int_a^b f(t)e^{ix\psi(t)}dt$$

Note:

- $I(x)$ is a *generalized Fourier integral*

Often one can use integration by parts, which results in

$$I(x) = \left.\frac{f(t)}{ix\psi'(t)}e^{ix\psi(t)}\right|_a^b - \frac{1}{ix}\int_a^b \frac{d}{dt}\left(\frac{f(t)}{\psi'(t)}\right)e^{ix\psi(t)}dt \qquad (13)$$

For $\psi(t)=t$ the integral term can be neglected under quite general conditions:

Riemann-Lebesgue Lemma:

$$\int_a^b f(t)e^{ixt}dt \to 0 \qquad \text{for} \quad x\to\infty \qquad \text{if} \int_a^b |f(t)|\,dt \quad \text{exists}$$

Notes:

- in the context of Fourier transformations the Riemann-Lebesgue lemma states that under very general conditions the amplitudes of the highest Fourier modes of a function go to 0.
- the integral vanishes since the high-frequency oscillations lead to a cancellation of the smooth parts of $f(t)$

- if $\psi'(t) \neq 0$ in the whole integration interval the Riemann-Lebesgue lemma can also be applied to $\int f(t)e^{ix\psi(t)}dt$ using a variable transformation $u = \psi(t)$, which is invertible and leads to
$$\int f(t)e^{ix\psi(t)}dt = \int \frac{1}{\psi'(t(u))} f(t(u))e^{ixu}du$$
- if $\psi'(t_0) = 0$ the point t_0 is a point of stationary phase (see Sec.1.4.1)

Example: $I(x) = \int_0^1 \frac{e^{ixt}}{1+t}dt$

Using integration by parts we get

$$I(x) = \underbrace{\frac{1}{1+t}\frac{1}{ix}e^{ixt}\Big|_0^1}_{\frac{i}{x}-\frac{i}{2x}e^{ix}} - \frac{1}{ix} \underbrace{\int_0^1 \frac{-1}{(1+t)^2}e^{ixt}dt}_{\text{Riemann-Lebesgue } \to 0}$$

According to Riemann-Lebesgue the integral term is small, but it does not tell us what order it is. To confirm the Riemann-Lebesgue explicitly and to get an estimate for the order of the error perform an additional integration by parts

$$\int_0^1 \frac{-1}{(1+t)^2}e^{ixt}dt = \underbrace{\frac{-1}{ix\,(1+t)^2}e^{ixt}\Big|_0^1}_{\to 0 \quad x\to\infty} - \frac{1}{ix}\int_0^1 \frac{2}{(1+t)^3}e^{ixt}dt$$

Bound the integral term

$$\left|\frac{1}{ix}\int_0^1 \frac{2}{(1+t)^3}e^{ixt}dt\right| \leq \frac{1}{x}\frac{2}{1}1 \to 0 \qquad x \to \infty$$

To get higher-order approximations one could repeat the integration by parts.

Example: $I(x) = \int_0^1 \sqrt{t}e^{ixt}$

Again integration by parts

$$I(x) = \underbrace{\frac{\sqrt{t}}{ix}e^{ixt}\Big|_0^1}_{-\frac{i}{x}e^{ix}} - \frac{1}{ix}\int_0^1 \frac{1}{2\sqrt{t}}e^{ixt}dt$$

By Riemann-Lebesgue the integral remainder term can be neglected.

If we want to get the orderof that error term we *could not* use an additional integration by parts:
the boundary term

$$\frac{1}{(ix)^2}\frac{1}{2\sqrt{t}}e^{ixt}\Big|_0^1$$

does not exist at $t = 0$, nor does the integral

$$\int_0^1 \frac{1}{t^{\frac{3}{2}}} e^{ixt} dt$$

because of the divergence at $t = 0$. What is going on at that next order?

To extract the x-dependence rescale using $s = xt$

$$\int_0^1 \frac{1}{\sqrt{t}} e^{ixt} dt = \frac{1}{\sqrt{x}} \int_0^x \frac{1}{\sqrt{s}} e^{is} ds$$

Deform the integration contour and separate it into two contours:
$\mathcal{C}_1$, $s \equiv iu$, $0 \le u < x$, and $\mathcal{C}_R$, $s = Re^{i\theta}$, $0 \le \theta \le \frac{\pi}{2}$, with $R = x$,

$$\frac{1}{\sqrt{x}} \int_0^x \frac{1}{\sqrt{s}} e^{is} ds = \frac{1}{\sqrt{x}} \int_{\mathcal{C}_1} \frac{1}{\sqrt{s}} e^{is} ds + \frac{1}{\sqrt{x}} \int_{\mathcal{C}_R} \frac{1}{\sqrt{s}} e^{is} ds \tag{14}$$

Estimate the $\mathcal{C}_1$-integral

$$\begin{aligned} \int_{\mathcal{C}_1} \frac{1}{\sqrt{s}} e^{is} ds &= \int_0^x \frac{1}{(iu)^{\frac{1}{2}}} e^{-u} i du = e^{-i\frac{\pi}{4}} e^{+i\frac{\pi}{2}} \int_0^x u^{-\frac{1}{2}} e^{-u} du \\ &\to \Gamma(\frac{1}{2}) - \frac{e^{-x}}{\sqrt{x}} \text{ for x} \to \infty \qquad \text{according to (4)} \end{aligned}$$

The $\mathcal{C}_R$-integral vanishes by Jordan's lemma:

$$\begin{aligned} \left| \int_{\mathcal{C}_R} \frac{1}{\sqrt{s}} e^{is} ds \right| = \left| \int_{\mathcal{C}_R} \frac{1}{\sqrt{R} e^{\frac{i}{2}\theta}} e^{iR(\cos\theta + i \sin\theta)} R i e^{i\theta} d\theta \right| &\le \sqrt{R} \int_0^{\frac{\pi}{2}} e^{-R\sin\theta} d\theta \\ &\le \sqrt{R} \int_0^{\delta} e^{-R\theta} d\theta + \sqrt{R} \int_\delta^{\frac{\pi}{2}} e^{-R\sin\delta} d\theta \\ &\le \sqrt{R} \frac{1}{R} \left(1 - e^{-R\delta}\right) + \sqrt{R} e^{-R\sin\delta} \frac{\pi}{2} \\ &= \mathcal{O}\left(\frac{1}{\sqrt{x}}\right) \end{aligned}$$

Thus,

$$\frac{1}{\sqrt{x}} \int_0^x \frac{1}{\sqrt{s}} e^{is} ds \sim e^{\frac{i\pi}{4}} \sqrt{\frac{\pi}{x}}$$

and

$$I(x) + \frac{i}{x} e^{ix} \sim \frac{e^{i\frac{3\pi}{4}}}{2} \sqrt{\pi} \frac{1}{x^{\frac{3}{2}}}$$

Note:

- since the remainder is $O(x^{-\frac{3}{2}})$ it is clear that it could not have been obtained by integration by parts.

1.4.1 Method of Stationary Phase

As mentioned above, if $\psi'(t) = 0$ somewhere in the interval $[a,b]$, integration by parts may not work (cf. (13))
$\Rightarrow$ use the method of **stationary phase:**

- Near the stationary point the oscillations of $e^{ix\psi(t)}$ are less rapid leading to less cancellation
  $\Rightarrow$ the dominant contributions to the integral come from the stationary point.
  $\Rightarrow$ we can restrict the integral to the vicinity of the stationary points

Show the dominance of the contributions from the stationary point:
One can always divide the integration domain into sections such that the stationary point $x = c$ ends up at an endpoint a of a subintegral. It is therefore sufficient to consider $c \equiv a$, $\psi'(c) = 0$, $\psi'(t) \neq 0$ for $t \in (a,b]$

$$I(x) = \int_a^{a+\epsilon} f(t)e^{ix\psi(t)}dt + \int_{a+\epsilon}^{b} f(t)e^{ix\psi(t)}dt$$

Note:

- the second integral is of $\mathcal{O}(x^{-1})$ since $\psi'(t) \neq 0$ (cf. (13))

To get the leading behavior of the first integral expand $f(t) = f(a) + \ldots$, and

$$\psi(t) = \psi(a) + \frac{1}{p!}\psi^{(p)}(a)\,(t-a)^p$$

where $\psi^{(p)}(a)$ is the first non-vanishing derivative of $\psi(t)$ at $t = a$ and $p > 1$,

$$I(x) \sim \int_a^{a+\epsilon} \left[f(a) + \ldots\right] e^{ix\left[\psi(a)+\frac{1}{p!}\psi^{(p)}(a)(t-a)^p\right]}dt$$

Extending the integral to ∞ introduces errors of $O(\frac{1}{x})$ since there are no stationary points in the added interval (integration by parts and Riemann-Lebesgue)

$$I(x) \sim f(a)e^{ix\psi(a)} \int_0^{\infty} e^{ix\frac{1}{p!}\psi^{(p)}(a)\,(t-a)^p}dt$$

As before, the integral can be obtained by deforming the contour to go out to infinity along a ray $e^{\pm i\frac{\pi}{2p}}$ (cf. (14)):
We want

$$i\psi^{(p)}(a)\,\frac{x}{p!}\,(t-a)^p = -u \qquad u \geq 0$$

to ensure decay of the exponential along that ray.
Need to choose $t - a \in e^{i\frac{\pi}{2p}}\mathbb{R}$ for $\psi^{(p)} > 0$ and $t - a \in e^{i\frac{\pi}{-2p}}\mathbb{R}$ for $\psi^{(p)} < 0$

$$\psi^{(p)}(a) > 0: \qquad t-a = e^{i\frac{\pi}{2p}}\left(\frac{p!}{x\psi^{(p)}(a)}\right)^{\frac{1}{p}}u^{\frac{1}{p}}$$

$$\psi^{(p)}(a) < 0: \qquad t-a = e^{-i\frac{\pi}{2p}}\left(-\frac{p!}{x\psi^{(p)}(a)}\right)^{\frac{1}{p}}u^{\frac{1}{p}}$$

$$I(x) \sim f(a)e^{ix\psi(a)}e^{\pm i\frac{\pi}{2p}}\left(\pm\frac{p!}{x\psi^{(p)}(a)}\right)^{\frac{1}{p}}\int_0^\infty \frac{1}{p}u^{\frac{1}{p}-1}e^{-u}du$$

Thus: the **method of stationary phase** yields the leading-order term of $I(x)$ as

$$I(x) \sim f(a)e^{ix\psi(a)}e^{\pm i\frac{\pi}{2p}}\left(\pm\frac{p!}{x\psi^{(p)}(a)}\right)^{\frac{1}{p}}\frac{\Gamma\left(\frac{1}{p}\right)}{p} \qquad x\to\infty \tag{15}$$

where the sign in $\pm$ has to agree with that of $\psi^{(p)}(a)$ and the endpoint a is the point of stationary phase.

Note:

- $p > 1$, therefore $I(x) \gg \mathcal{O}\left(\frac{1}{x}\right)$ which is the size of the contributions away from the stationary point.

Example: $I(x) = \int_0^\infty \cos\left(xt^2 - t\right)dt$

To use the method of stationary phase rewrite the integral[5]

$$I(x) = \Re\left[\int_0^\infty e^{i\left(xt^2-t\right)}dt\right]$$

and identify

$$f(t) = e^{-it} \qquad \psi(t) = t^2$$

Stationary point is at $t = 0$ with $\psi''(0) = 2 > 0$

$$I(x) \sim \Re\left[e^{i\frac{\pi}{4}}\sqrt{\frac{\pi}{x}}\frac{1}{2}\right] = \frac{1}{2}\sqrt{\frac{\pi}{2x}} \qquad x\to\infty$$

Notes:

- (15) gives the leading-order behavior $x^{-\frac{1}{p}}$
- Terms omitted include terms $\mathcal{O}(\frac{1}{x})$ arising from the integration *away* from the stationary point $\Rightarrow$ to obtain higher-order approximations is not so easy, since contributions may arise from the whole integration interval $[a, b]$.
  This is to be compared with the Laplace's method, where the contributions from the domain away from the maximum are exponentially rather than algebraically small $\Rightarrow$ use method of steepest descend (Sec.1.5).
- If $f(a) = 0$ it is not clear whether the integral is still dominated by the point of stationary phase

[5] In the derivation we used the exponential decay in the imaginary direction $t = iu$.

1.5 Method of Steepest Descent[6]

Consider now Laplace integrals with fully complex exponent

$$I(x) = \int_a^b h(t)e^{x\rho(t)}dt$$

with $h(t)$ and $\rho(t)$ analytic functions.

Approach: deform the integration contour such that the imaginary part ψ of $\rho \equiv \phi + i\psi$ is constant

$$I(x) = e^{ix\psi}\int_{\mathcal{C}} h(t)e^{x\phi(t)}dt$$

For the resulting integral one can then use Laplace's method.

Note:

- one could use also contours of constant real part ϕ and then use the method of stationary phase for the resulting integral. However, only the leading-order contribution is given by the vicinity of the point of stationary phase, while with Laplace's method the whole asymptotic expansion is determined by the neighborhood of the maximum.

Example: Compute the full asymptotic series for $I(x) = \int_0^1 \ln t\, e^{ixt}dt$

Integration by parts does not work since $\ln t$ diverges at $t = 0$

Method of stationary phase does not work because there is no point of stationary phase.

Deform the integration path to make the resulting integrals suitable for Laplace's method: need to keep the imaginary part of ixt constant:

$$\Im\,(ixt) = const \Leftrightarrow \Re\,(t) = const$$

Use three contours

$\mathcal{C}_1 : t = is$, $0 \le s < T$ $\mathcal{C}_2$: $t = iT + s$, $0 \le s \le 1$,
$\mathcal{C}_3$: $t = 1 + is$, $T \ge s \ge 0$ and let $T \to \infty$

$$\begin{aligned} I(x) &= i\int_{\mathcal{C}_1} \ln(is)\, e^{-xs}ds + \int_{\mathcal{C}_2} \ln(iT+s)\, e^{-xT+ixs}ds + \\ & \quad i\int_{\mathcal{C}_3} \ln(1+is)\, e^{ix-xs}ds \end{aligned}$$

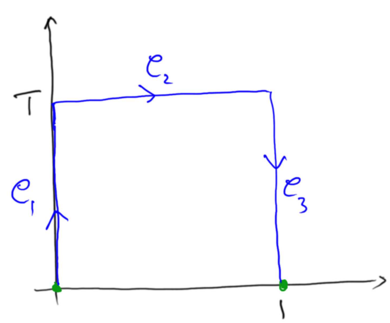

Figure 8: Contours for $I(x) = \int_0^1 \ln t\, e^{ixt}dt$

i) Integral $\int_{\mathcal{C}_2}$:
This integral vanishes for $T \to \infty$ because of the factor e^{-xT}.

[6] First published by Debye (1909) who pointed to an unpublished note by Riemann (1863).

ii) Integral $\int_{\mathcal{C}_1}$:
Exponential has maximum at $s=0$, but the $\ln(is)$ cannot be expanded at $s=0$
try to extract the x-dependence:

$$\begin{aligned} i\int_0^\infty \ln(is)\, e^{-xs}ds &= \frac{i}{x}\int_0^\infty (\ln i + \ln u - \ln x)\, e^{-u}du \\ &= -\frac{1}{x}\frac{\pi}{2} - i\frac{\ln x}{x} + \frac{i}{x}\int_0^\infty \ln u\, e^{-u}du \\ &= -\frac{1}{x}\left\{\frac{\pi}{2}+i\gamma\right\} - i\frac{\ln x}{x} \end{aligned}$$

using $\int_0^\infty e^{-u}\ln u\, du = -\gamma \approx 0.572\ldots$ (Euler's constant).

iii) Integral $\int_{\mathcal{C}_3}$:
Expand

$$\ln(1+is) = -\sum_{n=1}^{\infty}\frac{(-is)^n}{n}$$

now use Watson's lemma with global maximum at $s=0$, $\alpha=0$, $\beta=1$,

$$ie^{ix}\int_{\mathcal{C}_3}\ln(1+is)\, e^{-xs}ds = ie^{ix}\sum_{n=1}^{\infty}(-1)(-i)^n\,\frac{\Gamma(n+1)}{x^{n+1}n} \qquad x\to\infty$$

Combined we get

$$I(x) = -\frac{1}{x}\left(\frac{\pi}{2}+i\gamma\right) - i\frac{\ln x}{x} + ie^{ix}\sum_{n=1}^{\infty}\frac{(-i)^n\,(n-1)!}{x^{n+1}} \qquad x\to\infty$$

Notes:

- in this example $\rho = it$
  $\Rightarrow \Im(\rho(t=0))$ and $\Im(\rho(t=1))$ differ from each other
  $\Rightarrow$ there is no constant-phase contour that connects $\rho(t=0)$ and $\rho(t=1)$
  $\Rightarrow$ we needed three contours, to be chosen such that two lead to Laplace integrals and one gives a vanishing contribution.

Example: Determine the full asymptotic behavior of $I(x)=\int_0^1 e^{ixt^2}dt$

To get the leading-order behavior the method of stationary phase is sufficient:
$\psi = t^2$ with stationary point $t=0$ and $\psi''(a)=2>0$.

$$I(x)\sim \frac{1}{2}e^{i\frac{\pi}{4}}\sqrt{\frac{2\pi}{x\,2}}$$

To get the full asymptotic behavior is difficult with this method. Use steepest descent instead.

Deform contour into contours along which the imaginary part of $\rho \equiv it^2$ is constant.

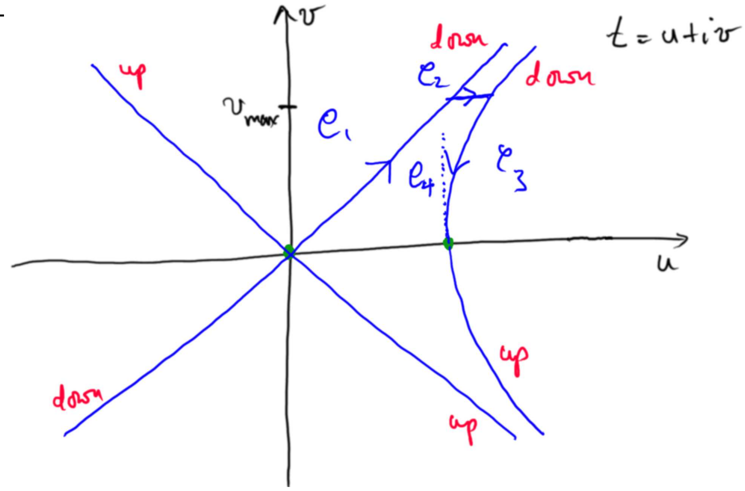

Figure 9: Contours for $I(x)=\int_0^1 e^{ixt^2}dt$

Note:

- t is to be considered a complex variable $t\equiv u+iv$: $\Im(\rho)=u^2-v^2$, $\Re(\rho)=-2uv$

Identify suitable contours:

- At $t=0$: $\Im(\rho)=0 \Rightarrow u=\pm v$

$$\Re(\rho)=-2uv=\mp 2v^2$$

  $u=-v$:

$$e^{x\rho}=e^{x(\Re(\rho)+i\Im(\rho))}=e^{x(-2uv+i0)}=e^{+2xv^2} \qquad \text{ascent, diverges for } u\to\infty$$

  $u=+v$:

$$e^{x\rho}=e^{-2xv^2} \qquad \text{descent}$$

  $\Rightarrow$ to get steepest descent path choose $\mathcal{C}_1: t=v+iv$, $0\le v\le v_{max}$ with $v_{max}\to\infty$ eventually

- At $t=1: \Im(\rho)=1 \Rightarrow u=\pm\sqrt{1+v^2}$ and $t=\pm\sqrt{1+v^2}+iv$
  $\Re(\rho)=-2uv=\mp v\sqrt{1+v^2}$. To get descent for $v\to+\infty$ choose $\mathcal{C}_3$ using $u=+\sqrt{1+v^2}$, $0\le v\le v_{max}$

$$\rho=\Re(\rho)+i\Im(\rho)=-2v\sqrt{1+v^2}+i$$

- Connecting path $\mathcal{C}_2$: $t=v_{max}(1+i)+u$, $0\le u\le u_{max}\equiv\sqrt{1+v_{max}^2}-v_{max}$

Evaluate integrals

$\mathcal{C}_2: t^2=(v_{max}+u+iv_{max})^2=(v_{max}+u)^2-v_{max}^2+2iv_{max}(v_{max}+u)$

$$\int_{\mathcal{C}_2} e^{ixt^2}dt=\int_0^{u_{max}} e^{ix\left\{(v_{max}+u)^2-v_{max}^2\right\}}e^{-2xv_{max}(v_{max}+u)}du\to 0 \qquad x\to\infty$$

$\mathcal{C}_1$:

$$\int_{\mathcal{C}_1} e^{ixt^2} dt = (1+i) \int_0^\infty e^{-2xv^2} dv = (1+i) \frac{1}{2}\sqrt{\frac{\pi}{2x}} = \frac{1}{2}\sqrt{\frac{\pi}{x}} e^{i\frac{\pi}{4}}$$

This is the same result as the leading-order result obtained with method of stationary phase

$\mathcal{C}_3$:

$$\int_{\mathcal{C}_3} e^{ixt^2} dt = -\int_0^\infty e^{ix-2xv\sqrt{1+v^2}} \left(\frac{v}{\sqrt{1+v^2}} + i \right) dv$$

where the minus sign reflects that $\mathcal{C}_3$ starts at infinity rather than 0.

If we only wanted to get the leading-order term we could expand in small v. But to get the full asymptotic expansion we need to avoid the higher derivatives of ϕ that generate all those additional terms in the expansion (see Sec.1.3.3). Instead we want to use Watson's lemma to get the full asymptotic expansion. Therefore we want to have

$$2v\sqrt{1+v^2} = s$$

Instead of doing a second variable tansformation from v to s make transformation directly from t to s. Along contour $\mathcal{C}_3$

$$it^2 = i - 2v\sqrt{1+v^2} \overbrace{=}^{!} i - s$$

therefore

$$t = (1+is)^{\frac{1}{2}} \qquad dt = \frac{i}{2\,(1+is)^{\frac{1}{2}}} ds$$

$$\begin{aligned}
\int_{\mathcal{C}_3} e^{ixt^2} dt &= -e^{ix} \int_0^\infty e^{-xs} \frac{i}{2\,(1+is)^{\frac{1}{2}}} ds \\
&= -e^{ix} \frac{i}{2} \sum_{n=0}^\infty \int_0^\infty e^{-xs} (-is)^n \frac{\Gamma\left(n+\frac{1}{2}\right)}{n!\Gamma\left(\frac{1}{2}\right)} ds \\
&= -e^{ix} \frac{i}{2} \sum_{n=0}^\infty (-i)^n \frac{\Gamma\left(n+\frac{1}{2}\right)}{n!\Gamma\left(\frac{1}{2}\right)} \frac{\Gamma(n+1)}{x^{n+1}}
\end{aligned}$$

using the series expansion from the binomial theorem (7).

Thus, we have

$$\int_0^\infty e^{ixt^2} dt = \frac{1}{2}\sqrt{\frac{\pi}{x}} e^{i\frac{\pi}{4}} - e^{ix} \frac{i}{2} \sum_{n=0}^\infty (-i)^n \frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} \frac{1}{x^{n+1}}$$

Note:

- Since the integrand is analytic everywhere the return contour can also be deformed away from the steepest-descent contout $\mathcal{C}_3$ without affecting the value of the integral. This can be exploited to make the integration simpler.

- The integral over the deformed contour has to be amenable to an asymptotic method like Laplace's method.

Deform the steepest-descent contour $\mathcal{C}_3$ into a contour $\mathcal{C}_4$ that simplifies the calculation

$\mathcal{C}_4$: choose the contour $t = 1 + iv$, $0 \le v \le v_m$ from $t_m = 1 + iv_m$ to contour $\mathcal{C}_3$, which is tangential to $\mathcal{C}_3$ at $t = 1$.

Since the contour $\mathcal{C}_4$ is tangential to $\mathcal{C}_3$ at $t = 1$ the contributions to the integral will be exponentially small for $v > \epsilon$. It is therefore sufficient to integrate only over the interval $[0, \epsilon]$. In this interval $v^2 \ll v$ and e^{ixv^2} can be expanded,

$$\begin{aligned}\int_{\mathcal{C}_3} e^{ixt^2} dt &= \int_{\mathcal{C}_4} e^{ixt^2} dt \sim i \int_0^\epsilon e^{ix(1+iv)^2} dv = i \int_0^\epsilon e^{ix\left(1-v^2\right)-2xv} dv \\ &= ie^{ix} \int_0^\epsilon \sum_{n=0}^{\infty} \frac{1}{n!} \left(-ix\right)^n v^{2n} e^{-2xv} dv\end{aligned}$$

Using Laplace's method and extending the integration again to ∞ one gets (cf. treatment of higher order-terms in ϕ in Sec. 1.3.3)

$$\begin{aligned}\int_{\mathcal{C}_4} e^{ixt^2} dt &= ie^{ix} \sum_{n=0}^{\infty} \int_0^\epsilon \frac{1}{n!} \left(-ix\right)^n v^{2n} e^{-2xv} dv \\ &= ie^{ix} \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\left(-ix\right)^n}{\left(2x\right)^{2n+1}} \int_0^\infty u^{2n} e^{-u} du \\ &= ie^{ix} \sum_{n=0}^{\infty} \frac{\left(-i\right)^n (2n)!}{n!\, 2^{2n+1}} \frac{1}{x^{n+1}} \qquad x \to \infty\end{aligned}$$

Compare with the result using $\mathcal{C}_3$:

$$\begin{aligned}\frac{(2n)!}{n! 2^{2n}} &= \frac{1}{n!} \frac{2n}{2} \frac{2n-1}{2} \frac{2n-2}{2} \cdots \frac{2}{2} \frac{1}{2} = \\ &= \frac{2n-1}{2} \frac{2n-3}{2} \cdots \frac{3}{2} \frac{1}{2} = \left(n - \frac{1}{2}\right) \left(n - \frac{3}{2}\right) \cdots \left(\frac{1}{2}\right) = \frac{\Gamma\left(n + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}\end{aligned}$$

since $\Gamma\left(x\right) = \left(x - 1\right) \Gamma(x - 1)$.

Thus, both contours give exactly the same result.

Note:

- the path of steepest descend avoids the oscillations in the integral arising from the non-constant imaginary part ψ and therefore allows Laplace's method; but any other path that allows Laplace's method works as well. A path that is tangential to the contour with constant ψ will have only slow oscillations near the saddle, which can be captured as higher-order corrections.

1.5.1 Steepest Descent and Saddle Points

Why is this method called 'Steepest Descent'?

$\rho(t) = \phi(t) + i\psi(t)$ is an analytic function of $t \equiv u + iv$. It therefore satisfies the Cauchy-Riemann conditions

$$\frac{\partial \phi}{\partial u} = \frac{\partial \psi}{\partial v} \qquad \frac{\partial \phi}{\partial v} = -\frac{\partial \psi}{\partial u}$$

Consider (u, v) as a two-dimensional vector. One can then write

$$\left(\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}\right) = \nabla f$$

One then gets

$$\nabla\phi \cdot \nabla\psi = \frac{\partial \phi}{\partial u}\frac{\partial \psi}{\partial u} + \frac{\partial \phi}{\partial v}\frac{\partial \psi}{\partial v} = \frac{\partial \phi}{\partial u}\frac{\partial \psi}{\partial u} + \left(-\frac{\partial \psi}{\partial u}\right)\frac{\partial \phi}{\partial u} = 0$$

$$\left.\begin{array}{c}\text{Lines of constant } \psi \text{ are orthogonal to } \nabla\psi \\ \\ \nabla\phi \text{ is orthogonal to } \nabla\psi\end{array}\right\} \quad \Rightarrow \quad \psi = const. \quad \| \quad \nabla\phi$$

Since $\nabla\phi$ gives the direction of steepest ascent, going away from a maximum along lines $\psi = const.$ amounts to going in the direction of steepest descent.

So far the maxima were at the end of the contours. Consider now local maxima of ϕ in the interior of the integration interval.

For analytic $\rho = \phi + i\psi$ the functions ϕ and ψ are harmonic

$$\Delta\phi = 0 \qquad \Delta\psi = 0$$

They cannot take on maxima or minima in the interior of a bounded domain.

At the maxima of ϕ along the lines of constant ψ one has for the directional derivatives

$$\frac{d\psi}{ds} = 0 \quad \text{and} \quad \frac{d\phi}{ds} = 0 \qquad \Rightarrow \qquad \frac{d\rho}{ds} = 0$$

By Cauchy-Riemann the directional derivatives of ψ and ϕ vanish in *all* directions:

$$\rho' = 0$$

- the maxima of ϕ along the lines of constant ψ are *saddles* in the complex plane: at least two orientations for steepest ascent/descent
- each line of steepest ascent/descent corresponds to a line of constant ψ:
  at a saddle multiple lines of constant ψ intersect.

Example: Saddle point of e^{xt^2}

$\rho(t) = t^2 = (u + iv)^2 = u^2 - v^2 + 2iuv$ and $\rho'(t) = 2t = 2u + i2v$

The only saddle point is

$$\rho' = 0 \qquad u = 0 = v$$

The steepest paths are given by $uv = 0$:

$$\begin{aligned} u &= 0 \qquad \rho = -v^2 \qquad \text{steepest descent} \\ v &= 0 \qquad \rho = +u^2 \qquad \text{steepest ascent} \end{aligned}$$

Example: Behavior near the saddle point of $e^{x(\sinh t - t)}$ near $t = 0$

At $t = 0$ we have

$$\rho = \sinh t - t = 0 \qquad \rho' = \cosh t - 1 = 0 \qquad \rho'' = \sinh t = 0 \qquad \rho''' = \cosh t = 1$$

Thus, also the second derivative vanishes, but not third one (third-order saddle point).

Steepest paths

$$\rho = \sinh(u + iv) - u - iv = \sinh u \cos v + i \cosh u \sin v - u - iv$$

$$\cosh u \sin v - v = 0 \qquad \Leftrightarrow \qquad v = 0 \quad \text{or} \quad u = \cosh^{-1}\left(\frac{v}{\sin v}\right)$$

The second condition defines two lines because $\cosh^{-1}$ is double-valued.

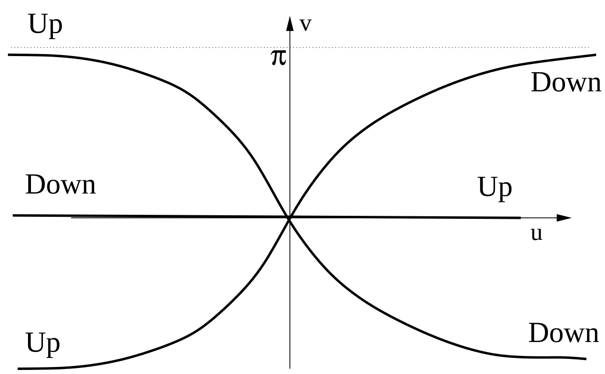

Figure 10: Saddle of $e^{x(\sinh t - t)}$.

Example: $I(x) = \int_0^1 e^{-4xt^2} \cos(5xt - xt^3)\, dt$ for large x

This is not a Laplace integral because x appears also in the cosine.

i) Try nevertheless to argue that integral dominated by small values of t because of the exponential.

One would get

$$I(x) \sim \int_0^\infty e^{-4xt^2} dt = \frac{1}{2}\sqrt{\frac{\pi}{4x}} \qquad \text{wrong}$$

But: exponential has decayed only for $t > x^{-\frac{1}{2}} \Rightarrow$ for $t = \mathcal{O}(x^{-\frac{1}{2}})$ we have $xt \sim x^{\frac{1}{2}} \to \infty$ and the argument of $\cos$ is large and the rapid oscillations of the $\cos$ lead to substantial cancellation.

ii) For $t \le \mathcal{O}(x^{-\frac{1}{2}})$ the second term in the $\cos$ is small: $xt^3 \le \mathcal{O}(x^{-\frac{1}{2}}) \Rightarrow$ tempting to ignore that term.
Then

$$\begin{aligned}
I(x) &\sim \int_0^1 e^{-4xt^2}\cos(5xt)\,dt = \frac{1}{2}\int_{-1}^{1} e^{-4xt^2}\cos(5xt)\,dt \\
&\underbrace{\sim}_{\text{'as usually'}} \frac{1}{2}\int_{-\infty}^{\infty} e^{-4xt^2}\cos(5xt)\,dt = \\
&= \frac{1}{2}\Re\left\{\int_{-\infty}^{+\infty} e^{-4xt^2+5ixt}dt\right\} = \frac{1}{2}\Re\left\{\int_{-\infty}^{+\infty} e^{-4x\left(t-\frac{5}{8}i\right)^2-\frac{25}{16}x}dt\right\} = \\
&\underbrace{=}_{\text{can translate contour}} \frac{1}{2}\sqrt{\frac{\pi}{4x}}e^{-\frac{25}{16}x} \qquad \text{wrong}
\end{aligned}$$

Integral now exponentially small, but still wrong (can be checked by expanding the $\cos$ and comparing the order of the omitted term with the retained term (see below)).

iii) Use method of steepest descent. Rewrite

$$I(x) = \frac{1}{2}\int_{-1}^{+1} e^{-4xt^2+5ixt-ixt^3}dt = \frac{1}{2}\int_{-1}^{+1} e^{x\rho(t)}dt$$

with

$$\rho(t) = -it^3 - 4t^2 + 5it$$

Phases at the end points differ from each other

$$\Im\left(\rho(t=\pm 1)\right) = \pm 4$$

Identify contours $\mathcal{C}_{1,2}$ of constant $\Im(\rho)$ using $t = u + iv$

$$\begin{aligned}
\rho &= -i\left(u^3 + 3iu^2v - 3uv^2 - iv^3\right) - 4\left(u^2 - v^2 + 2iuv\right) + 5iu - 5v \\
&= \underbrace{-v^3 + 3u^2v - 4u^2 + 4v^2 - 5v}_{\phi} + i\underbrace{\left\{-u^3 + 3uv^2 - 8uv + 5u\right\}}_{\psi}
\end{aligned}$$

Thus

$$\phi = -v^3 + 3u^2v - 4u^2 + 4v^2 - 5v \qquad \psi = -u^3 + 3uv^2 - 8uv + 5u$$

Note:

- ϕ is even in u and ψ is odd in u.

$\psi = 4\sigma$ at $t = \sigma$ with $\sigma = \pm 1$:

$$3uv^2 - 8uv + 5u - u^3 - 4\sigma = 0$$

$$v = \frac{1}{6u}\left(8u \pm \sqrt{64u^2 - 12u\left(5u - u^3 - 4\sigma\right)}\right) = \frac{1}{3u}\left(4u \pm \sqrt{u^2 + 3u^4 + 12\sigma u}\right)$$

We need the contours passing through $t = \sigma$, i.e. $u = \sigma$ and $v = 0$

$$\sigma = +1: \qquad v_2 = \frac{1}{3u}\left(4u - \sqrt{u^2 + 3u^4 + 12u}\right)$$
$$\sigma = -1: \qquad v_1 = \frac{1}{3u}\left(4u + \sqrt{u^2 + 3u^4 - 12u}\right)$$

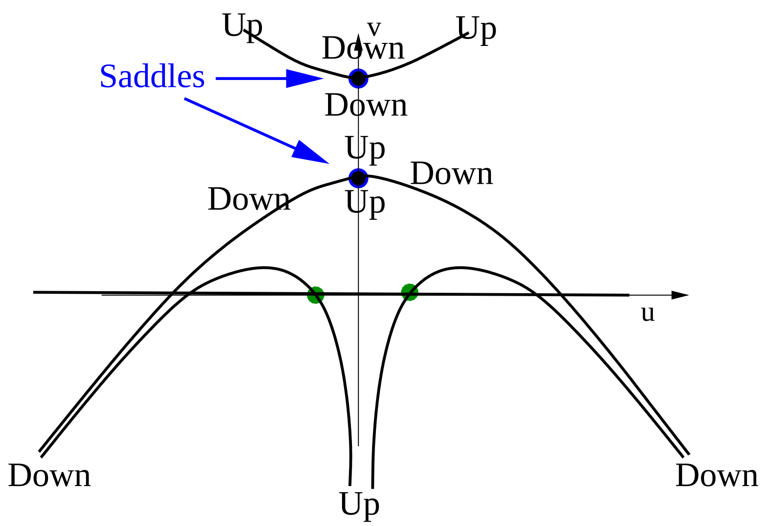

Figure 11: Contours with $\psi = const$ and saddle points for $e^{-4xt^2+5ixt-ixt^3}$.

Limiting behavior of the contours $v^{\pm}_{1,2}$

$$\sigma = +1: \qquad v_2 \to -\frac{u}{\sqrt{3}} \quad \text{for} \quad u \to +\infty \qquad v_2 \to -\frac{2}{\sqrt{3}}\frac{1}{\sqrt{u}} \quad \text{for} \quad u \to 0^+$$
$$\sigma = -1: \qquad v_1 \to +\frac{u}{\sqrt{3}} \quad \text{for} \quad u \to -\infty \qquad v_1 \to -\frac{2}{\sqrt{3}}\frac{1}{\sqrt{-u}} \quad \text{for} \quad u \to 0^-$$

Limiting behavior of real part ϕ along the contours $v^{\pm}_{1,2}$ for $u \to \pm\infty$ and $v \to -\infty$. Using $u^2 \to 3v^2$

$$\sigma = \pm 1: \qquad \phi \to -v^3 + 3u^2 v \to -v^3 + 9v^3 = 8v^3 \to -\infty \qquad \text{for} \qquad v \to -\infty$$

Identify all saddle points of $\rho(t)$

$$\frac{d\rho}{dt} = -3it^2 - 8t + 5i = 0$$

$$t^{(s)}_{1,2} = -\frac{8 \pm \sqrt{64 - 60}}{6i} \qquad \begin{cases} t^{(s)}_1 = \frac{5}{3}i \\ t^{(s)}_2 = i \end{cases}$$

Along the contours with $\psi = const.$ the real part ϕ can only have a local maximum at one of the two saddle points

$\Rightarrow \phi$ does not have a local maximum along the contours $v_{1,2}$

$\Rightarrow \phi$ is going up along $\mathcal{C}_{1,2}$ for $u \to 0$.

Thus

- If we deformed the integration contour to include the singularity at $u = 0$ the main contribution to the integral would arise at the singularity. Poor choice
- Integrate from $t = \pm 1$ outward to $u \to \pm\infty$
  - the main contribution to this contour integral arises near $t = \pm 1$
  - we need to connect the end points with an additional contour $\mathcal{C}_3$

Identify contours with $\psi = const.$ through the saddle point $t = i$

$$\psi(t = i) = 0 \qquad \Rightarrow \qquad u = 0 \quad \text{or} \quad 3v^2 - 8v + 5 - u^2 = 0 \quad \Rightarrow \quad v_{3,4} = \frac{4 \pm \sqrt{1 + 3u^2}}{3}$$

The contour v_4 is asymptotic to the contours $v_{1,2}$.

The integral along $\mathcal{C}_4$ does not vanish: ϕ reaches a maximum at the saddle at $t = i$.

Therefore the dominant contribution to that integral arises near that saddle.

$$\begin{aligned} 2I(x) &\sim \int_{\mathcal{C}_1} e^{x\rho} dt + \int_{\mathcal{C}_4} e^{x\rho} dt + \int_{\mathcal{C}_2} e^{x\rho} dt + \underbrace{\int_{\mathcal{C}_{connect}} e^{x\rho} dt}_{\to 0 \text{ for } u_{max} \to -\infty} \\ &\sim \int_{t=-1}^{t=-1-\mathcal{O}(\epsilon)} e^{x\rho} dt + \int_{t=i-\mathcal{O}(\epsilon)}^{t=i+\mathcal{O}(\epsilon)} e^{x\rho} dt + \int_{t=1+\mathcal{O}(\epsilon)}^{t=1} e^{x\rho} dt \end{aligned}$$

$\mathcal{C}_4$: use contour that is tangential to the steepest descent contour

$$t = i + s \qquad -\epsilon \le s \le \epsilon \qquad v = 1 \quad u = s$$

$$\begin{aligned} \int_{\mathcal{C}_4} &\sim \int_{-\epsilon}^{+\epsilon} e^{x\left\{-s^2-2\right\}} e^{ix\left\{-s^3\right\}} ds \\ &\sim \frac{1}{\sqrt{x}} e^{-2x} \int_{-\sqrt{x}\epsilon}^{+\sqrt{x}\epsilon} e^{-u^2} e^{-\frac{i}{\sqrt{x}} u^3} du \\ &\sim \frac{1}{\sqrt{x}} e^{-2x} \int_{-\infty}^{+\infty} e^{-u^2} \left(1 - \frac{i}{\sqrt{x}} u^3\right) du \\ &\sim \sqrt{\frac{\pi}{x}}\, e^{-2x} \end{aligned}$$

$\mathcal{C}_1$:

$$t = -1 - (1 + i\alpha)\, s \qquad 0 \le s \le \epsilon$$

with $\alpha \in \mathbb{R}$ chosen so that the contour is tangential to the contour $\psi = const.$

Thus

$$u = -1 - s \qquad v = -\alpha s$$

and

$$\phi = -4 + \mathcal{O}(s) \qquad \psi = -4 + \mathcal{O}(s)$$

Therefore we can estimate

$$\begin{aligned} \left|\int_{\mathcal{C}_1}\right| &\sim \left|\int_0^{+\epsilon} e^{-4x+\mathcal{O}(\epsilon)} e^{-4ix+\mathcal{O}(s)} ds\right| \\ &\leq e^{-4x}\int_0^{\epsilon} 1 ds \\ &= \mathcal{O}(e^{-4x}) \end{aligned}$$

analogously for $\mathcal{C}_2$.

Thus the integrals over contours $\mathcal{C}_{1,2}$ are exponentially smaller than that over contour $\mathcal{C}_3$ and we have

$$I(x) \sim \frac{1}{2}\sqrt{\frac{\pi}{x}}\, e^{-2x} \tag{16}$$

Note:

- the leading contribution to the integral is given by a saddle that is not even close to the original integration contour.

How do we know that ignoring ixt^3 was incorrect? Reconsider the expansion carefully:

$$I(x) = \frac{1}{2}\int_{-1}^{1} e^{-4xt^2}\cos\left(5xt - xt^3\right) dt \sim \frac{1}{2}\int_{-\infty}^{\infty} e^{-4xt^2}\cos\left(5xt - xt^3\right) dt$$

Can the integration limits be pushed to $\pm\infty$?

$$\begin{aligned} \left|\frac{1}{2}\int_1^{\infty} e^{-4xt^2}\cos\left(5xt - xt^3\right) dt\right| &< \frac{1}{2}\int_1^{\infty} e^{-4xt^2} dt = \frac{1}{2}e^{-4x}\int_0^{\infty} e^{-4xu^2-8xu} du \\ &< \frac{1}{2}e^{-4x}\frac{1}{2}\sqrt{\frac{\pi}{4x}} \ll |I(x)| \end{aligned}$$

The error term is exponentially small. But if the integral $I(x)$ itself is also exponentially small (as is the case here), we need to compare them explicitly: in this case the integral can be extended to $\pm\infty$ (cf. (16)).

We therefore consider again

$$I(x) = \frac{1}{2}\int_{t=-1}^{1} e^{-4x\left(t-\frac{5}{8}i\right)^2-\frac{25}{16}x-ixt^3} dt \sim \frac{1}{2}\int_{t=-\infty}^{+\infty} e^{-4x\left(t-\frac{5}{8}i\right)^2-\frac{25}{16}x-ixt^3} dt \tag{17}$$

To evaluate the Gaussian integral we need to shift the contour to $\frac{5}{8}i-\infty < t < \frac{5}{8}i+\infty$. We had not done this carefully enough before.

The contours $t = \pm R + is$, $0 \leq s \leq \frac{5}{8}i$, do not contribute for $R \to \infty$.

Therefore we go ahead and use

$$s = 2\sqrt{x}\left(t - \frac{5}{8}i\right) \qquad t = \frac{s}{2\sqrt{x}} + \frac{5}{8}i$$

and

$$t^3 = \frac{1}{8}\frac{s^3}{x^{\frac{3}{2}}} + i\frac{15}{32}\frac{s^2}{x} + \frac{75}{128}\frac{s}{x^{\frac{1}{2}}} - \frac{125}{8^3}i$$

Thus

$$e^{-ixt^3} = 1 - ixt^3 + \ldots = 1 + x\frac{15}{32}\frac{s^2}{x} - x\frac{125}{8^3} - i\left\{\frac{1}{8}\frac{s^3}{x^{\frac{1}{2}}} + \frac{75}{128}sx^{\frac{1}{2}}\right\} + \ldots$$

Note:

- The 'correction' from expanding e^{-ixt^3} contributes terms that are large ($\mathcal{O}(x, ix^{\frac{1}{2}})$) compared to the term we kept: we cannot expand the exponential!

What happens if we do not expand the exponential?

The term ixt^3 contributes to the real Gaussian term $\sim s^2$ and also a term $\sim ix^{\frac{1}{2}}s$. The latter term needs to be absorbed again into the completed square, which induces another shift in the integration variable, which induces further corrections through the $-ixt^3$ term
$\rightarrow$ we need to determine the correct shift of the contour simultaneously with completing the square. Make the ansatz

$$s = (t - i\alpha)\, 2\sqrt{x} \qquad t = \frac{1}{2\sqrt{x}}s + i\alpha$$

with α to be determined.

Then

$$t^3 = \frac{1}{8x^{\frac{3}{2}}}s^3 + \frac{3}{4}i\alpha s^2 - \frac{3}{2}\frac{\alpha^2}{x^{\frac{1}{2}}}s - i\alpha^3$$

Thus

$$-4t^2 + 5it - it^3 = \frac{s^2}{x}\left[-1 + \frac{3}{4}\alpha\right] + i\frac{s}{x^{\frac{1}{2}}}\left[-4\alpha + \frac{3}{2}\alpha^2 + \frac{5}{2}\right] + 4\alpha^2 - 5\alpha - \alpha^3 - \frac{is^3}{8x^{\frac{3}{2}}}$$

We need to shift the contour such that the leading term is a pure Gaussian (no term linear in s) and decaying

$$-4\alpha + \frac{3}{2}\alpha^2 + \frac{5}{2} = 0 \qquad \rightarrow \qquad \alpha_{1,2} = \frac{4 \pm 1}{3}$$

Consider the quadratic term

$$-1 + \frac{3}{4}\alpha = \begin{cases} \frac{1}{4} & \text{for } \alpha = \frac{5}{3} \\ -\frac{1}{4} & \text{for } \alpha = 1 \end{cases}$$

Since the quadratic term has to lead to exponential decay away from $s = 0$, only $\alpha = 1$ is acceptable. Then we get

$$\begin{aligned} I(x) &\sim \frac{1}{2}\int_{s=-\infty}^{+\infty} e^{-\frac{1}{4}s^2 - 2x - i\frac{s^3}{8x^{\frac{1}{2}}}}\frac{1}{2\sqrt{x}}ds \\ &= \frac{1}{4\sqrt{x}}2\sqrt{\pi}e^{-2x} + \frac{1}{4\sqrt{x}}e^{-2x}\int_{-\infty}^{\infty} e^{-\frac{1}{4}s^2}\left\{-i\frac{s^3}{8x^{\frac{1}{2}}} + \frac{1}{2}\left(i\frac{s^3}{8x^{\frac{1}{2}}}\right)^2 + \ldots\right\}ds \end{aligned}$$

in agreement with the result from the saddle-point calculation.

Note:

- the correct shift $\alpha = 1$ moved the contour exactly through the saddle point.

Notes:

- when shifting contours omitted terms can become significant
- if the final result is exponentially small, one needs to check whether the integration limits could indeed be extended to ∞.

1.5.2 Complex x and the Stokes Phenomenon

Generalize the integral further to allow x to be complex

- now the asymptotic expansion can depend on the argument of x
  depending on the argument of x different terms can be dominant and subdominant
- the interchange of dominance and subdominance when the argument of the expansion variable x is varied is called *Stokes Phenomenon*

Example: Again $I(x) = \int_0^1 e^{-4xt^2} \cos(5xt - xt^3)\, dt$ for large complex $x = X + iY$

Saddle points do not depend on x
$\Rightarrow$ dominant contributions will still come from the endpoints at $t = \pm 1$ or from the saddle point, $t = i$.

We need to consider steepest descent contours near the end points. For $x \in \mathbb{R}$ we found that those contributions are subdominant (negligible compared to the contribution from the saddle), but as the argument of x varies these contributions can interchange their roles.

$t = -1 + U + iV$ with $|U|, |V| \ll 1$:

$$\begin{aligned}
\rho &= -V^3 + 3(-1+U)^2 V - 4(-1+U)^2 + 4V^2 - 5V + \\
&\quad + i\left\{-(-1+U)^3 + 3(-1+U)V^2 - 8(-1+U)V + 5(-1+U)\right\} \\
&= -4 + 3V + 8U - 5V + h.o.t. + i\{1 - 5 - 3U + 5U + 8V\} \\
&= -4 + 8U - 2V + i\{-4 + 2U + 8V\} + h.o.t.
\end{aligned}$$

Write

$$x\rho = (X + iY)\rho \equiv \Phi + i\Psi$$

We need contour with $\Psi = const.$ to leading non-trivial order in U and V,

$$\begin{aligned}
\Psi &= X\{-4 + 2U + 8V\} + Y\{-4 + 8U - 2V\} \\
&= \underbrace{-4(X+Y)}_{\text{value of } \psi \text{ at endpoint}} \qquad + U(2X + 8Y) + V(8X - 2Y)
\end{aligned}$$

Thus,

$$V = -\frac{X+4Y}{4X-Y}U + h.o.t.$$

Then to leading non-trivial order in U and V

$$\begin{aligned}
\Phi &= X\left(-4+8U-2V\right) - Y\left(-4+2U+8V\right) \\
&= 4\left(Y-X\right) + U\left(8X-2Y\right) + V\left(-2X-8Y\right) \\
&= 4\left(Y-X\right) + U\left\{8X-2Y+2\frac{\left(X+4Y\right)^2}{4X-Y}\right\} \\
&= 4\left(Y-X\right) + U\frac{32X^2-8XY-8XY+2Y^2+2X^2+16XY+32Y^2}{4X-Y} \\
\Phi &= 4\left(Y-X\right) + U\frac{34\left(X^2+Y^2\right)}{4X-Y}
\end{aligned}$$

and

$$dt = \left(1 - i\frac{X+4Y}{4X-Y}\right)dU$$

Φ is decreasing with decreasing U for $4X > Y$; it is increasing otherwise.
Thus we need to integrate

$$\begin{aligned}
U &\in [0,-\epsilon] \qquad \text{for} \quad 4X-Y>0 \\
U &\in [0,\epsilon] \qquad \text{for} \quad 4X-Y<0
\end{aligned}$$

At $4X-Y=0$ the contour becomes vertical (cf. Fig.12).

For $|x| \to \infty$ the prefactor of U in Φ goes to ∞ as well. Thus, we get for the contribution from the end point $t=-1$

$$\begin{aligned}
\int_{\mathcal{C}_1} &= \frac{1}{2}\left(1-i\frac{X+4Y}{4X-Y}\right)\int_0^{-\epsilon} e^{4(Y-X)}e^{-4i(X+Y)+i\mathcal{O}(U^2)}e^{\frac{34\left(X^2+Y^2\right)}{4X-Y}U}dU \\
&\sim -\frac{1}{2}\left(1-i\frac{X+4Y}{4X-Y}\right)e^{4(Y-X)}e^{-4i(X+Y)}\frac{4X-Y}{34\left(X^2+Y^2\right)} \\
&= -\frac{1}{2}e^{-4x(1+i)}\frac{4X-Y-iX-4iY}{34\left(X+iY\right)\left(X-iY\right)} \\
&= -\frac{1}{2}e^{-4x(1+i)}\frac{1}{34x}\left(4-\frac{Y+iX}{X-iY}\right) \\
&= \frac{1}{2}\frac{i-4}{34x}\,e^{-4x(1+i)}
\end{aligned}$$

Analogously, one obtains

$$\int_{\mathcal{C}_2} = -\frac{1}{2}\frac{i+4}{34x}\,e^{-4x(1-i)}$$

For the contribution from the saddle point we had previously for $x \in \mathbb{R}$

$$\int_{\mathcal{C}_3} \sim \sqrt{\frac{\pi}{x}}\,e^{-2x}$$

Since the integral $I(x)$ is analytic in x, this expression must also be valid for $x \in \mathbb{C}$.

Check this:

Contour tangent to $\mathcal{C}_4$ at $t = i$

$$t = i + U\,(1 + i\alpha) \qquad |U| \ll 1$$

For x complex the tangent to contour $\mathcal{C}_4$ is not necessarily horizontal any more: α not known.

Insert into $x\rho$

$$x\rho = -2X + \underbrace{\left(X\alpha^2 + 2Y\alpha - X\right)}_{-\beta} U^2 + i\left\{-2Y + \left(Y\alpha^2 - 2X\alpha - Y\right)U^2\right\} + \mathcal{O}(U^3)$$

Need $\Psi = const$ to $\mathcal{O}(U^2)$

$$\alpha_{1,2} = \frac{X \pm \sqrt{X^2 + Y^2}}{Y}$$

To go in the descending direction of Φ,

$$\Phi_{1,2} = -2X + \frac{2}{Y^2}\left(X^2 + Y^2\right)\left(X \pm \sqrt{X^2 + Y^2}\right)U^2$$

need to choose $\alpha = \alpha_2$.

$$\int_{\mathcal{C}_4} \sim \sqrt{\frac{\pi}{\beta}} e^{-2(X+iY)}\,(1 + i\alpha_2)$$

Consider

$$\frac{\beta}{(1 + i\alpha_2)^2} = \ldots = -\frac{2\left(X^2 + Y^2\right)\left(X - \sqrt{X^2 + Y^2}\right)}{Y + i\left(X - \sqrt{X^2 + Y^2}\right)} = \ldots = X + iY = x.$$

Thus, as before for $x \in \mathbb{R}$

$$\int_{\mathcal{C}_4} \sim \sqrt{\frac{\pi}{x}} e^{-2x}$$

Compare the exponential growth/decay of the three integrals

$\int_{\mathcal{C}_1}$ dominates $\int_{\mathcal{C}_4}$ for

$$4\Re\left((X + iY)(1 + i)\right) = 4(X - Y) < 2X \qquad \Leftrightarrow \qquad Y > \frac{1}{2}X$$

$\int_{\mathcal{C}_2}$ dominates $\int_{\mathcal{C}_4}$ for

$$4\Re\left((X + iY)(1 - i)\right) = 4(X + Y) < 2X \qquad \Leftrightarrow \qquad Y < -\frac{1}{2}X$$

$\int_{\mathcal{C}_2}$ dominates $\int_{\mathcal{C}_1}$ for

$$4\Re\left((X + iY)(1 - i)\right) = 4(X + Y) < 4(X - Y) = 4\Re\left((X + iY)(1 + i)\right) \qquad \Leftrightarrow \qquad Y < 0$$

Thus

- for $|\arg(x)| < \arctan\frac{1}{2}$: $X > 0$ and $-\frac{1}{2}X < Y < \frac{1}{2}X$

$$I(x) \sim \int_{\mathcal{C}_4} = \frac{1}{2}\sqrt{\frac{\pi}{x}}\, e^{-2x}$$

- for $\arctan\frac{1}{2} < \arg(x) < \pi$: $Y > 0$

$$I(x) \sim \int_{\mathcal{C}_1} = \frac{1}{2}\frac{i-4}{34x}\, e^{-4x(1+i)}$$

- for $-\arctan\frac{1}{2} > \arg(x) > -\pi$> $Y < 0$

$$I(x) \sim \int_{\mathcal{C}_2} = -\frac{1}{2}\frac{i+4}{34x}\, e^{-4x(1-i)}$$

Note:

- At the *Stokes lines*

$$\arg(x) = \frac{1}{2} \qquad \text{and} \qquad \arg(x) = -\frac{1}{2} \qquad \text{and} \qquad \arg(x) = \pm\pi$$

  dominant and subdominant terms interchange their roles

Compare the imaginary parts of the exponents at the Stokes lines

$\int_{\mathcal{C}_4}$:

$$-2Y = \begin{cases} -X & \text{for} \quad Y = \frac{1}{2}X \\ +X & \text{for} \quad Y = -\frac{1}{2}X \end{cases}$$

$\int_{\mathcal{C}_1}$:

$$-4\,(Y+X) = -6X \quad \text{for } Y = \frac{1}{2}X$$

$\int_{\mathcal{C}_2}$:

$$-4\,(Y-X) = +6X \quad \text{for } Y = -\frac{1}{2}X$$

Notes:

- The dominant character of the integral changes across the Stokes lines:
  - while the exponential decay of the two integrals is the same, their oscillatory character is different (different 'frequencies')
  - the prefactors in front of the exponential is also different
- With varying $\arg x$ the contours of steepest descent can also switch suddenly and omit a saddle point
- If we had tried the 'naive method' (17):

- we would not know when the contributions from the endpoints are dominant
- we would have to deform the contour through the saddle to avoid oscillations

$\Rightarrow$ to take care of these issues we would end up doing the same work as in the systematic approach.

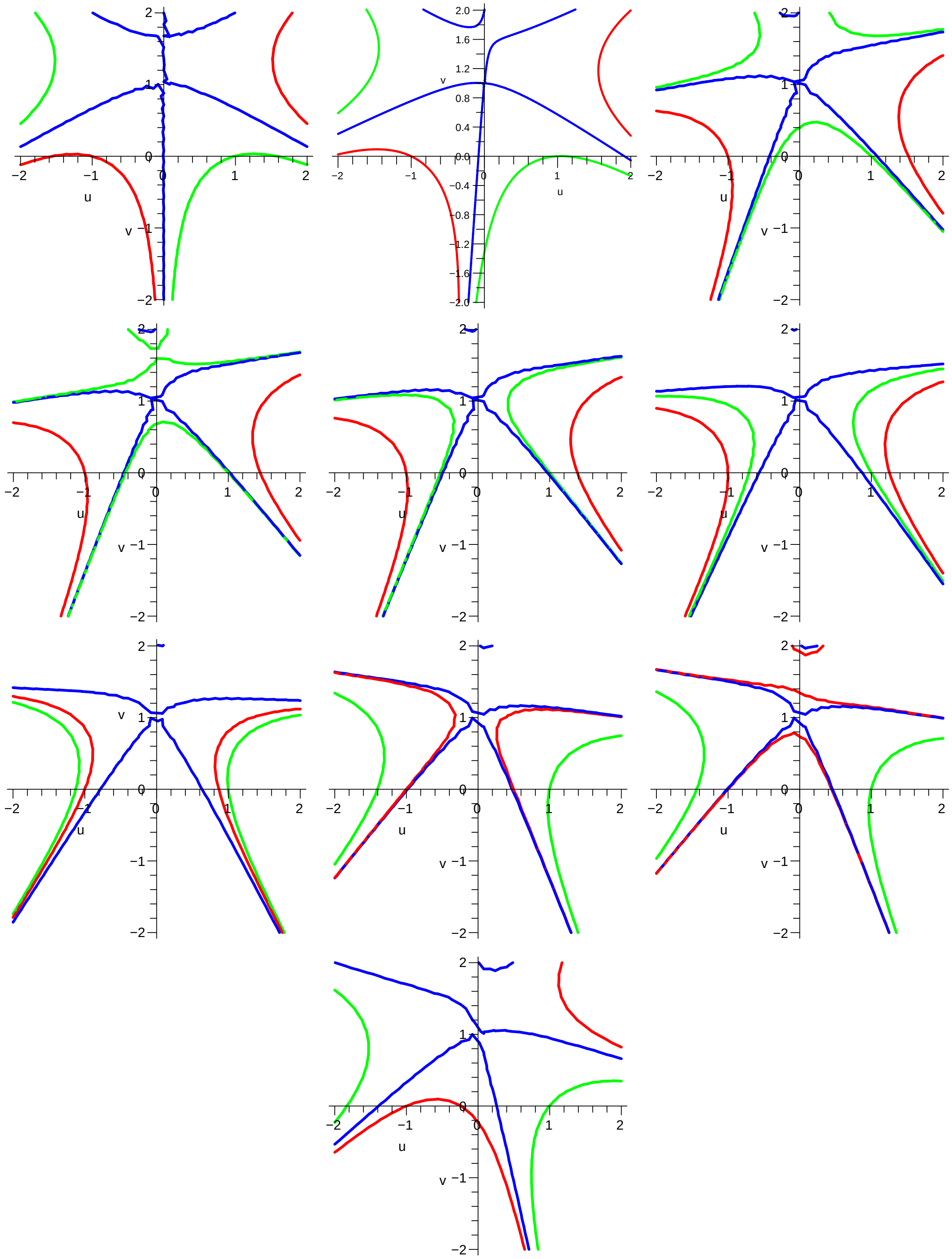

Figure 12: Dependence of contour of steepest descent on $\varphi = \arg x$. $\varphi = 0, 1, 1.08, 1.15, 1.3, 1.7, 2.01, 2.05, 2.5$.

For $|\phi| < \arctan\frac{1}{2} \approx 0.46$ the saddle point dominates the integral. Otherwise the end points.

While for $\varphi < \arctan 4 \approx 1.32$ the descent is towards more negative U it is towards more positive U for $\varphi > \arctan 4$. At $\varphi = \pi - \arctan 2 \approx 2.03$ the two contours reconnect and the contour of steepest descent is asymptotic to that emerging from $t = +1$ and no connecting contour crossing the saddle point arises any more. In this example this reconnection does not affect the leading-order behavior of the integral, though, since the saddle point contributes only a subdominant term for $\varphi > \arctan\frac{1}{2}$. Note that in the computer-generated contour plots the blue lines do not cross at the saddle point, although they do intersect in reality.

2 Forced Oscillators

Resonances in forced oscillations are important in many areas

- dangerous resonances: stability of structures
  - Tacoma Narrows bridge collapse (http://www.youtube.com/watch?v=j-zczJXSxnw or http://www.youtube.com/watch?v=xox9BVSu7Ok)
  - Millenium Bridge swaying (http://www.youtube.com/watch?v=eAXVa__XWZ8)
  - Tae Bo class listening to http://www.youtube.com/watch?v=z33tH-JdPDg makes skyscraper sway
  - 

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Hundreds flee shaking Seoul shopping mall

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South Koreans evacuate from Techno Mart building in Seoul, South Korea, July 5, 2011. The 39-story shopping mall Technomart in the eastern part of Seoul trembled on Tuesday morning, forcing hundreds of people to flee the building, officials said. Between 300 to 500 people fled the Techno Mart building in Seoul's Gwangjin District after the building shook for about 10 minutes on Tuesday morning, fire officials said. No injuries have been reported. (Xinhua/Park Jin Hee)

SEOUL, July 5 (Xinhua) -- A 39-story shopping mall in the eastern part of Seoul trembled on Tuesday morning, forcing hundreds of people to flee the building, officials said.

Between 300 to 500 people fled the Techno Mart building in Seoul's Gwangjin District after the building shook for about 10 minutes on Tuesday morning, fire officials said. No injuries have been reported.

It remains unknown what had caused the shaking of the building. Officials are investigating.

The Gwangjin District Office has ordered all the people working at the building to evacuate. "We will immediately place the order and conduct a thorough safety examination," a district official was quoted as saying by Yonhap news agency.

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THE EXPRESS TRIBUNE WITH THE International Herald Tribune

Gym workout caused tremor at Seoul building: Experts

By AFP
Published: July 20, 2011

SEOUL: South Korean experts said Tuesday a vigorous gym exercise session caused a high-rise building in Seoul to shake for 10 minutes earlier this month, prompting hundreds to flee it in panic.

In front of journalists they re-created the scenario on July 5 – a group of 17 middle-aged people working out to the tune of a pop song, "The Power" by German group Snap – and caused the building to shake in a similar way.

The group had been engaged in Tae Bo – an aerobic exercise routine that combines boxing and martial arts like Taekwondo – in the 12th floor gym of the 39-story TechnoMart mall building.

"Participants in Tuesday's test said they felt the building rolling greatly while others said they had felt lesser oscillations," TechnoMart spokesman Andy Yang told AFP.

Professor Chung Lan of Danguk University in Yongin city, who led the experiment, linked the incident to a physics principle governing the vibration of a structure when it is matched by another source.

He told a radio programme the building, constructed from iron girders and cement, had a characteristic vibration frequency which was "in phase" with the synchronised movements of the Tae Bo practitioners,

After the first occasion the Gwangjin district government ordered the mall closed but reopened it after experts said it was structurally sound.

Cho Byung-Joon, an official with the district government, said earlier Tuesday that depending on the test results, officials might immediately allow the fitness club to reopen.

"But practising Tae Bo in that particular building may have to be banned," Cho said.

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Figure 13: Seoul skyscraper resonance.

http://www.deathandtaxesmag.com/123255/tae-bo-shakes-the-foundation-of-a-korean-skyscraper/
http://www.youtube.com/watch?v=z33tH-JdPDg

- useful resonances: amplification of signals, e.g.
  - electronic circuits
  - double amplification in the ear: two staged oscillators.
    Otoacoustic emissions
    * evoked by sound indicate processing in the inner ear (used as non-invasive test for hearing defects)
    * spontaneous emissions.

Understanding how externally forced oscillators can phase-lock to the forcing and essentially synchronize with it provides also insight into resonantly coupled oscillators, e.g.,

- laser arrays
- heart cells: efficient pumping vs fibrillations
- neurons
  - synchrony can carry additional information
  - too much synchrony may amount to epileptic seizures or Parkinsonian tremor

Types of forcing

- non-parametric forcing: the forcing introduces an additional term in the equation
  Simple example: Pushing on a swing

$$ml\ddot{\theta} + mg\sin\theta = F(t) \qquad \Rightarrow \ddot{u} + \frac{g}{l}\sin\theta = \frac{1}{m}F(t)$$

- parametric forcing: a parameter of the system is modified in time
  Simple example: Pumping on a swing

$$ml(t)\ddot{\theta} + mg\sin\theta = 0 \qquad \Rightarrow \ddot{u} + \frac{g}{l(t)}\sin\theta = 0$$

  with l being the distance of the center of mass to the pivot.
  By shifting his/her center of mass the person changes the effective length l of the pendulum

Useful asymptotic expansions can be obtained for weak forcing near and away from resonances. The expansions and results depend on the type of resonance and the type of forcing, which often reflect the symmetries of the overall system.

This system provides a good example to illustrate

- the important role symmetries can play in the reduction of complex systems
- how important qualitative features of a complex nonlinear system can be extracted by expanding around special (singular) points, i.e.considering distinguished limits.

2.1 Parametrically Forced Oscillators

Consider second-order differential equations with periodically varying coefficients: parametrically forced oscillators.

2.1.1 The Mathieu Equation

Consider the linear differential equation describing a parametrically forced harmonic oscillator (Mathieu equation)

$$\ddot{u} + (\delta + \epsilon \cos 2t)\, u = 0 \qquad \ddot{u} = -\frac{dU}{du} \quad U = \frac{1}{2}(\delta + \epsilon \cos 2t)\, u^2$$

which would model the swing for small angle θ. Even though it is linear it cannot be solved exactly.
Instead of varying the forcing frequency with fixed natural frequency of the unforced oscillator we keep here the forcing frequency fixed, $\omega = 2$, and vary the natural frequency $\sqrt{\delta}$.

The Mathieu equation is at the core of the description of a wide range of forced oscillations. Nonlinear treatments are often based on expansions around the Mathieu equation or variants of it (e.g. Faraday waves on the free surface of vertically vibrated fluid)

Expect: the resonant forcing drives the amplitude of the oscillator to large values.

Goal:

- for *weak forcing* find the curves $\delta(\epsilon)$, $0 \leq \epsilon \ll 1$, for which the Mathieu equation has a periodic solution with period 2π, i.e. twice the period of the forcing.
- the periodic solution is easier to compute than growing or modulated (quasi-periodic) solutions.
- we will find that the periodic solution separates parameter regimes in which the forcing leads to growing rather than quasi-periodic solutions.

Expand:

$$\begin{aligned}\delta &= \delta_0 + \epsilon\delta_1 + \epsilon^2\delta_2 + \ldots \\ u &= u_0 + \epsilon u_1 + \epsilon^2 u_2 + \ldots\end{aligned}$$

Collect

$\mathcal{O}(1)$:

$$\ddot{u}_0 + \delta_0 u_0 = 0$$

$\mathcal{O}(\epsilon)$:

$$\ddot{u}_1 + \delta_0 u_1 = -u_0(\delta_1 + \cos 2t)$$

$\mathcal{O}(\epsilon^2)$:

$$\ddot{u}_2 + \delta_0 u_2 = -\delta_2 u_0 - u_1(\delta_1 + \cos 2t)$$

For the solution to have period 2π we need

$$\delta_0 = n^2 \qquad n = 0, 1, 2, \ldots$$

Note:

- for $n = 0$ the unforced solution does not represent an oscillator; it is constant and its value is arbitrary.
- for $n \geq 1$ the minimal period of the unforced solution is $2\pi/n$.

i) Case $n = 0$, i.e. $\delta_0 = 0$

$$\delta = \epsilon\delta_1 + \epsilon^2\delta_2 + \ldots$$

$\mathcal{O}(1)$**:**

$$u_0 = c_0 \equiv 1$$

we can choose the constant amplitude arbitrarily. Since the equation is linear the overall amplitude depends on the initial conditions and is in that sense arbitrary.

$\mathcal{O}(\epsilon)$**:**

$$\ddot{u}_1 = -\delta_1 - \cos 2t$$

to eliminate secular terms we need to choose $\delta_1 = 0$

$$u_1(t) = c_1 + \frac{1}{4}\cos 2t$$

The constant c_1 modifies the arbitrarily chosen constant amplitude c_0 and can also be chosen arbitrarily. Set $c_1 = 0$.
We want to have the first term in δ that actually depends on ϵ: need to go to next order yet.

$\mathcal{O}(\epsilon^2)$:

$$\begin{aligned} \ddot{u}_2 &= -\delta_2 - \left(c_1 + \frac{1}{4}\cos 2t\right)\cos 2t \\ &= -\delta_2 - \frac{1}{8} - c_1\cos 2t - \frac{1}{8}\cos 4t \end{aligned}$$

to eliminate secular terms we need to choose $\delta_2 = -\frac{1}{8}$.
Thus

$$\delta = -\frac{1}{8}\epsilon^2 + \mathcal{O}(\epsilon^3)$$

Notes:

- $\delta < 0$: to get a periodic solution in the presence of forcing the potential $U(u,t)$ is modulated around a maximum rather than a minimum.
- $\epsilon \gg |\delta|$: during the forcing the potential switches periodically from having a maximum to having a minimum and back.

- the periodic orbit corresponds to balancing a stick on its end (upside-down pendulum) not in vertical position, but at an angle ($u_0 \neq 0$).

ii) Case $n = 1$, i.e. $\delta_0 = 1$

$\mathcal{O}(1)$:

$$u_0 = c_1 \cos t + c_2 \sin t$$

$\mathcal{O}(\epsilon)$:

$$\begin{aligned}\ddot{u}_1 + u_1 &= -\left(c_1 \cos t + c_2 \sin t\right)\left(\delta_1 + \cos 2t\right) \\ &= c_1 \cos t \left(-\delta_1 - \frac{1}{2}\right) + c_2 \sin t \left(-\delta_1 + \frac{1}{2}\right) - c_1 \frac{1}{2} \cos 3t - c_2 \frac{1}{2} \sin 3t\end{aligned}$$

using

$$\cos\alpha\cos\beta = \frac{1}{2}\left\{\cos\left(\alpha+\beta\right) + \cos\left(\alpha-\beta\right)\right\} \qquad \sin\alpha\cos\beta = \frac{1}{2}\left\{\sin\left(\alpha+\beta\right) + \sin\left(\alpha-\beta\right)\right\}$$

To eliminate secular terms one of two cases need to be satisfied

- $\delta_1 = \frac{1}{2}$ and $c_1 = 0$
  $u_0 = c_2 \sin t$, $u_1 = c_3 \cos t + c_4 \sin t + c_2 \frac{1}{16} \sin 3t$
- $\delta_1 = -\frac{1}{2}$ and $c_2 = 0$
  $u_0 = c_1 \cos t$, $u_1 = c_5 \cos t + c_6 \sin t + c_1 \frac{1}{16} \cos 3t$

$\mathcal{O}(\epsilon^2)$:

- case $\delta_1 = +\frac{1}{2}$

$$\begin{aligned}\ddot{u}_2 + u_2 &= -\delta_2 c_2 \sin t - \left(c_3 \cos t + c_4 \sin t + c_2 \frac{1}{16} \sin 3t\right)\left(\delta_1 + \cos 2t\right) \\ &= -\sin t \left(\delta_2 c_2 + c_4 \delta_1 - \frac{1}{2} c_4 + \frac{1}{2} c_2 \frac{1}{16}\right) - \cos t \left(c_3 \delta_1 + \frac{1}{2} c_3\right) \\ &\quad - \cos 3t \left(\frac{1}{2} c_3\right) - \sin 3t \left(\frac{1}{2} c_4 + \frac{1}{16} c_2 \delta_1\right) \\ &\quad - \sin 5t \left(\frac{1}{2}\frac{1}{16} c_2\right)\end{aligned}$$

to avoid secular terms need

$$c_3 = 0 \qquad \delta_2 = -\frac{1}{32}$$

Thus

$$\begin{aligned}\delta_{odd} &= 1 + \frac{1}{2}\epsilon - \frac{1}{32}\epsilon^2 + \mathcal{O}(\epsilon^3) \\ u &= c_2 \sin t + \epsilon\left(c_4 \sin t + c_2 \frac{1}{16} \sin 3t\right) + \ldots\end{aligned}$$

Notes:

  - As in a nonlinear equation, each additional order of the expansion introduces higher-order harmonics.
  - can again set $c_4 = 0$ since it merely adds to the undetermined amplitude c_2

- case $\delta_1 = -\frac{1}{2}$

$$\begin{aligned}
\ddot{u}_2 + u_2 &= -\delta_2 c_1 \cos t - \left(c_5 \cos t + c_6 \sin t + c_1 \frac{1}{16} \cos 3t \right) (\delta_1 + \cos 2t) \\
&= -\cos t \left(\delta_2 c_1 + c_5 \delta_1 + \frac{1}{2} c_5 + \frac{1}{2} c_1 \frac{1}{16} \right) - \sin t \left(c_6 \delta_1 - \frac{1}{2} c_6 \right) \\
&\quad - \cos 3t \left(\frac{1}{16} c_1 \delta_1 + \frac{1}{2} c_5 \right) - \sin 3t \left(\frac{1}{2} c_6 \right) \\
&\quad - \cos 5t \left(\frac{1}{2} \frac{1}{16} c_1 \right)
\end{aligned}$$

now we need

$$c_6 = 0 \qquad \delta_2 = -\frac{1}{32}$$

Thus

$$\begin{aligned}
\delta_{even} &= 1 - \frac{1}{2}\epsilon - \frac{1}{32}\epsilon^2 + \mathcal{O}(\epsilon^3) \\
u &= c_1 \cos t + \epsilon \left(c_5 \cos t + c_1 \frac{1}{16} \cos 3t \right) + \ldots
\end{aligned}$$

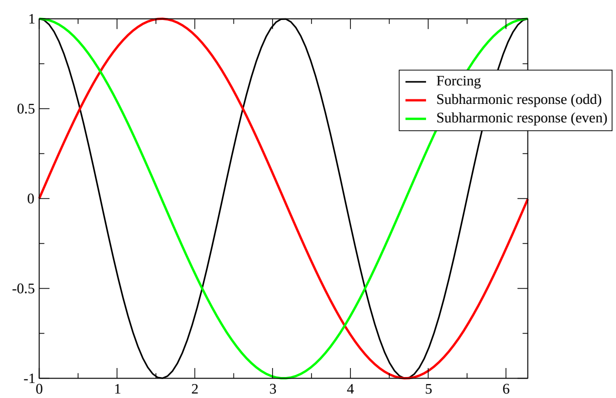


Figure 14: Even and odd solutions for the subharmonic response.

Notes:

- in both cases the solutions have a period that is twice as long as that of the forcing: *subharmonic* response.
- $\delta - \delta_0 = \epsilon\delta_1 + \epsilon^2\delta_2 + \ldots$ characterizes the *detuning* between the forcing frequency and the natural frequency of the unforced oscillator

- for $\delta_1 = +\frac{1}{2}$ the solution contains only $\sin nt$ to the order considered, i.e. it is *odd*, and for $\delta_1 = -\frac{1}{2}$ the solution is *even.*
- for the forcing the transformation $\epsilon \to -\epsilon$ is equivalent to $t \to t + \frac{\pi}{2}$. Shifting the time by $\frac{\pi}{2}$ interchanges the even and the odd solution: $\sin t \to \cos t$ and $\cos t \to -\sin t \Rightarrow$

$$\delta_{even}(\epsilon) = \delta_{odd}(-\epsilon)$$

  The two solutions arise for different values of δ given ϵ.

- Is there something special about the Mathieu equation that the solutions seem to be either even or odd in t?

Time Reversal Symmetry:

- For any n (or δ_0) the *Mathieu equation is even under time reversal*, i.e. the equation does not change form under the replacement

$$t \to -t \equiv \bar{t}$$

  since only even derivatives arise and the coefficients are even in t.
  I.e. if $u(t)$ is a solution, so is $u(-t)$

$$\begin{aligned} \frac{d^2}{dt^2}u(-t) + (\delta + \epsilon \cos(2t))\, u(-t) &= \frac{d^2}{d(-t)^2}u(-t) + (\delta + \epsilon \cos(-2t))\, u(-t) \\ &= \frac{d^2}{d(\bar{t})^2}u(\bar{t}) + (\delta + \epsilon \cos(2\bar{t}))\, u(\bar{t}) = 0 \end{aligned}$$

- If we were to find a solution $u(t)$ that is of mixed parity (neither even nor odd) then
  - $u_e \equiv u(t) + u(-t)$ is even
  - $u_o \equiv u(t) - u(-t)$ is odd

  and since the Mathieu equation is linear, u_e and u_o are also solutions
  Any solution can be written in terms of u_e and u_o

$$u(t) = \frac{1}{2}(u_e + u_o)$$

Thus:

- Because of the time-reversal symmetry we can assume from the start that $u(t)$ is either even or odd and include only $\sin$-functions or $\cos$-functions

Note:

- the Mathieu equation is linear, therefore the amplitude of the solution is undetermined (in the absence of initial conditions)

– we do not need to keep the higher-order homogenous solution since they only change the overall amplitude of the solution.

– fix the amplitude of the leading-order solution by a suitable normalization condition

$$\begin{cases} \int_0^{2\pi} u_0(t)\;\cos t\, dt = \pi & \text{for } \delta = +\frac{1}{2} \\ \int_0^{2\pi} u_0(t)\;\sin t\, dt = \pi & \text{for } \delta = -\frac{1}{2} \end{cases}$$

iii) Case $n = 2$, $\delta_0 = 4$

Note:

- the oscillation period is the same as that of the forcing: *harmonic response*

We can assume the solution is either even or odd.
Use the normalization conditions

$$\begin{cases} \int_0^{2\pi} u_0(t) \sin 2t\, dt = \pi & \int_0^{2\pi} u_{k\geq 1}(t) \sin 2t\, dt = 0 \quad \text{odd} \\ \int_0^{2\pi} u_0(t) \cos 2t\, dt = \pi & \int_0^{2\pi} u_{k\geq 1}(t) \cos 2t\, dt = 0 \quad \text{even} \end{cases}$$

i.e. choose amplitude of u_0 such that the higher-order terms $u_{k\geq 1}$ do not contribute to the amplitude of the fundamental mode.

$\mathcal{O}(1)$:

$$u_0(t) = \begin{cases} \sin 2t & \text{odd solution, out of phase with respect to the forcing} \\ \cos 2t & \text{even solution, in phase with respect to the forcing} \end{cases}$$

with amplitudes fixed by the normalization

Odd solution:

$\mathcal{O}(\epsilon)$:

$$\begin{aligned} \ddot{u}_1 + 4u_1 &= -\delta_1 \sin 2t - \cos 2t\; \sin 2t \\ &= -\delta_1 \sin 2t - \frac{1}{2} \sin 4t \end{aligned}$$

thus

$$\delta_1 = 0 \qquad u_1 = \frac{1}{24} \sin 4t$$

Note:

- because of the normalization we do not keep the homogeneous solution

$\mathcal{O}(\epsilon^2)$:

$$\begin{aligned} \ddot{u}_2 + 4u_2 &= -\delta_2 \sin 2t - \cos 2t\, \frac{1}{24} \sin 4t \\ &= -\sin 2t \left(+\delta_2 + \frac{1}{2}\frac{1}{24} \right) - \frac{1}{2}\frac{1}{24} \sin 6t \end{aligned}$$

avoid secular terms

$$\delta_2 = -\frac{1}{48}$$

Thus

$$\begin{aligned}\delta &= 4 - \frac{1}{48}\epsilon^2 + \ldots \\ u &= \sin 2t + \epsilon\frac{1}{24}\sin 4t + \ldots\end{aligned}$$

Even solution:

$\mathcal{O}(\epsilon)$:

$$\begin{aligned}\ddot{u}_1 + 4u_1 &= -\delta_1 \cos 2t - \cos^2 2t \\ &= -\delta_1 \cos 2t\delta_1 - \frac{1}{2} - \frac{1}{2}\cos 4t\end{aligned}$$

leading to

$$\delta_1 = 0 \qquad u_1 = -\frac{1}{8} + \frac{1}{24}\cos 4t$$

$\mathcal{O}(\epsilon^2)$:

$$\begin{aligned}\ddot{u}_2 + 4u_2 &= -\delta_2 \cos 2t - \cos 2t\left(-\frac{1}{8} + \frac{1}{24}\cos 4t\right) \\ &= \cos 2t\left(-\delta_2 + \frac{1}{8} - \frac{1}{48}\right) - \frac{1}{48}\cos 6t\end{aligned}$$

leading to

$$\begin{aligned}\delta &= 4 + \frac{5}{48}\epsilon^2 + \ldots \\ u &= \cos 2t + \epsilon\left(-\frac{1}{8} + \frac{1}{24}\cos 4t\right) + \ldots\end{aligned}$$

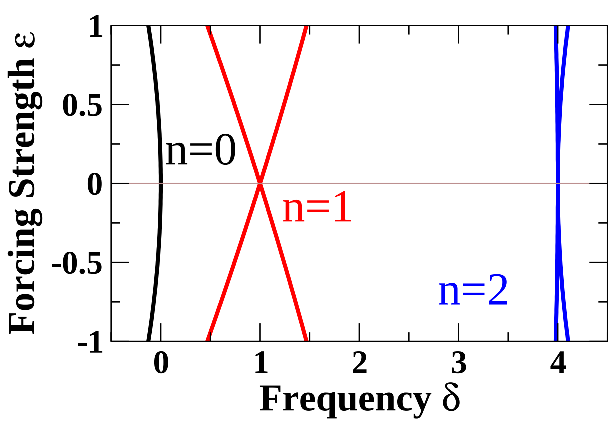


Figure 15: Periodic solutions of the Mathieu equation. The growth and decay of solutions away from the lines is discussed later.

Notes:

- the even harmonic solution is *in phase* with the forcing, the odd one is *out of phase*.
- the even harmonic solution has also a non-zero mean.

2.1.2 Floquet Theory

In the discussion of the Mathieu equation we only obtained period solutions, which in turn required a specific combination of the forcing and the frequency, $\delta = \delta(\epsilon)$.

We would like to have information about the solutions also away from these lines in parameter space. Can we write them in some specific form?

Consider more generally

$$\ddot{u} + Q(t)u = 0 \tag{18}$$

with $Q(t)$ a T-periodic function

$$Q(t+T) = Q(t)$$

Notes:

- the solutions to (18) need not be periodic:
  - for $Q = c < 0$ the solutions would be growing and decaying
    $\Rightarrow$ expect that this character can persist even with Q time-dependent
  - for $Q = c > 0$ the solutions would be periodic with a period that need not be related to the period T of $Q(t)$
    $\Rightarrow$ expect that the underlying period persists to some extent with Q time-dependent
    $\Rightarrow$ expect that quasi-periodic solutions are possible with two incommensurate frequencies

Symmetries:

- for $Q = const.$
  - (18) is invariant for translations in time by *any amount:*
    for *any* Δt the function $u(t+\Delta t)$ is a solution if $u(t)$ is a solution
    the two solutions $u(t)$ and $u(t+\Delta t)$ are not the same.
  - the solution is given by a complex exponential, $u \propto e^{i\alpha t}$, $\alpha \in \mathbb{C}$.
    For arbitrary Δt one has then

$$u(t+\Delta t) = e^{i\alpha\Delta t}u(t) \tag{19}$$

- for T-periodic $Q(t)$
  - eq.(18) is invariant only under translations by an *integer multiple* of T
    $\Rightarrow$ with $u(t)$ also $u(t+\Delta t)$ is a solution *if* $\Delta t = nT$, n integer
    again, $u(t)$ and $u(t+n\Delta t)$ are not the same solutions.
  - If one considers the discrete temporal evolution $u(t=n\Delta t)$ of eq.(18) with $\Delta t = T$ the coefficient is constant, $Q(t+nT)$.

  – does one have a relation like (19) if $Q(t)$ is T-periodic and one considers only shifts $\Delta t = T$?

We show now:
The discrete time translation symmetry together with the linearity of the equation allows solutions $u(t)$ that satisfy a simple relationship between $u(t)$ and $u(t+T)$

$$u(t+T) = \rho\, u(t), \qquad \rho \in \mathbb{C} \tag{20}$$

We want to calculate $u(t)$.

Note:

- not all solutions have the Floquet form (20), but they can be written as superpositions of Floquet solutions

The basic idea is to use the general solution $u(t) = c_1 u_1(t) + c_2 u_2(t)$ to express also the shifted solution $u(t+T)$ in terms of $u_1(t)$ and $u_2(t)$.

Consider the two linearly independent solutions

$$u_1(t) \quad \text{with} \quad \begin{cases} u_1(0) = 1 \\ \dot{u}_1(0) = 0 \end{cases} \qquad\qquad u_2(t) \quad \text{with} \quad \begin{cases} u_2(0) = 0 \\ \dot{u}_2(0) = 1 \end{cases}$$

Since $u_{1,2}$ are linearly independent their Wronskian is non-zero:

$$W(u_1(t), u_2(t)) = \begin{vmatrix} u_1(t) & u_2(t) \\ \dot{u}_1(t) & \dot{u}_2(t) \end{vmatrix}$$

and has the same sign as

$$W(u_1(0), u_2(0)) = \begin{vmatrix} u_1(0) & u_2(0) \\ \dot{u}_1(0) & \dot{u}_2(0) \end{vmatrix} = 1$$

In fact, here

$$\frac{d}{dt} W(u_1(t), u_2(t)) = u_1 \ddot{u}_2 - u_2 \ddot{u}_1 = -Q\,(u_1 u_2 - u_2 u_1) = 0$$

i.e. $W(u_1(t), u_2(t)) = 1$.

Any solution $u(t)$ of eq.(18) can be written in terms of $u_1(t)$ and $u_2(t)$

$$u(t) = c_1 u_1(t) + c_2 u_2(t)$$

where the $c_{1,2}$ are given by the initial conditions

$$c_1 = u(0) \qquad c_2 = \dot{u}(0) \tag{21}$$

We are looking for a specific solution, i.e. we need to determine $c_{1,2}$.

Now consider $u(t+T) = c_1 u_1(t+T) + c_2 u_2(t+T)$. Can we express this as a multiple of $u(t)$

$$u(t+T) = c_1 u_1(t+T) + c_2 u_2(t+T) \overbrace{=}^{?} \rho\left(c_1 u_1(t) + c_2 u_2(t)\right)$$

To investigate this equation we need to write $u_j(t+T)$ in terms of $u_1(t)$ and $u_2(t)$. Because of the discrete time-translation symmetry $t \to t + nT$, the shifted functions $u_{1,2}(t+T)$ are also solutions of (18) and can therefore be expanded in terms of $u_{1,2}(t)$

$$u_j(T+t) = u_j(T)u_1(t) + \dot{u}_j(T)u_2(t) \qquad j = 1,2$$

using (21).

Note: For arbitrarily shifted functions this would not be possible.

This gives

$$\begin{aligned} u(t+T) &= c_1 u_1(t+T) + c_2 u_2(t+T) \\ &= c_1\left[u_1(T)u_1(t) + \dot{u}_1(T)u_2(t)\right] + c_2\left[u_2(T)u_1(t) + \dot{u}_2(T)u_2(t)\right] \end{aligned}$$

Try to write this as $u(t+T) = \rho u(t)$

$$c_1\left[u_1(T)u_1(t) + \dot{u}_1(T)u_2(t)\right] + c_2\left[u_2(T)u_1(t) + \dot{u}_2(T)u_2(t)\right] = \rho\left[c_1 u_1(t) + c_2 u_2(t)\right]$$

Recall: we are seeking coefficients c_i such that $u(t)$ satisfies this condition. Since $u_1(t)$ and $u_2(t)$ are linearly independent we get by collecting coefficients of $u_1(t)$ and of $u_2(t)$

$$\begin{aligned} c_1 u_1(T) + c_2 u_2(T) &= \rho c_1 \\ c_1 \dot{u}_1(T) + c_2 \dot{u}_2(T) &= \rho c_2 \end{aligned}$$

To solve for c_1 and c_2 this requires

$$\begin{vmatrix} u_1(T) - \rho & u_2(T) \\ \dot{u}_1(T) & \dot{u}_2(T) - \rho \end{vmatrix} = 0$$

$$\rho^2 - \rho\underbrace{\left(u_1(T) + \dot{u}_2(T)\right)}_{\equiv 2K} + \underbrace{u_1(T)\dot{u}_2(T) - \dot{u}_1(T)u_2(T)}_{W=1} = 0$$

$$\begin{aligned} \rho^2 - 2K\rho + 1 &= 0 \\ \rho_{1,2} &= K \pm \sqrt{K^2 - 1} \end{aligned} \tag{22}$$

Notes:

- Shifting the solution by a period amounts indeed to the multiplication with a complex number.
- ρ is called the Floquet multipler
- since $K \in \mathbb{R}$ both Floquet multipliers are either real or complex conjugates of each other

- it is also convenient to introduce α via $\rho_1 = e^{i\alpha T}$ which characterizes the Floquet exponent $i\alpha T$. Because $W = 1$ one has $\rho_1\rho_2 = 1$ implying $\rho_2 = e^{-i\alpha T}$.
- For each ρ satisfying (22) we found a solution satisfying $u(t+T) = \rho u(t)$.

The evolution of $u(t)$ during one period of $Q(t)$ is therefore determined by K

1. $K = 1$: $\rho_{1,2} = 1 \quad \Rightarrow \quad$ one has a T-periodic solution.
2. $K = -1$: $\rho_{1,2} = -1 \quad \Rightarrow \quad$ one has a $2T$-periodic (subharmonic) solution
$$u(t+2T) = -u(t+T) = u(T)$$
3. $|K| < 1$: $\rho_{1,2}$ are complex with $|\rho_{1,2}| = 1$, $\alpha \in \mathbb{R}$
4. $|K| > 1$: $|\rho_1| > 1$ and $|\rho_2| < 1$ are real and $\alpha \in i\mathbb{R}$

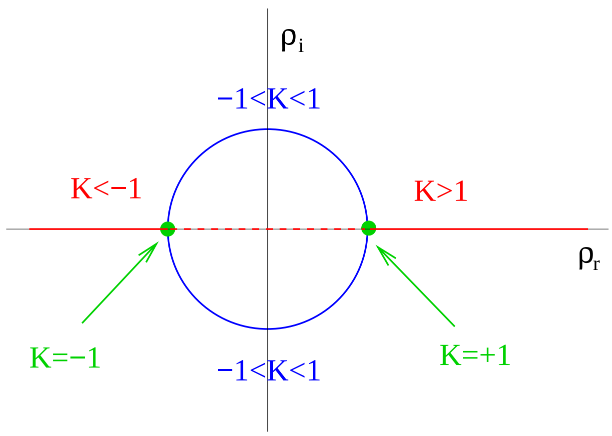


Figure 16: Dependence of real and imaginary parts of $\rho_{1,2}$ on K. As K is increased from $K < -1$ to $K > +1$ the two real and negative $\rho_{1,2}$ split into a complex pair and then merge again to form two real and positive values.

Linear Independence for $|K| \neq 1$.
For $|K| \neq 1$ we found two solutions $u^{(1,2)}(t)$ satisfying the Floquet condition (20) with different values of ρ,

$$\begin{aligned} u^{(1)}(t+T) &= \rho_1 u^{(t)}(t) = e^{i\alpha T} u^{(1)}(t) \\ u^{(2)}(t+T) &= \rho_2 u^{(2)}(t) = e^{-i\alpha T} u^{(2)}(t) \end{aligned}$$

Factoring out the Floquet multiplier $e^{i\alpha T}$ via

$$u^{(1)}(t) = e^{i\alpha t} U^{(1)}(t)$$

we get

$$U^{(1)}(t+T) = u^{(1)}(t+T) e^{-i\alpha(t+T)} = e^{i\alpha T} u^{(1)}(t) e^{-i\alpha(t+T)} = U^{(1)}(t)$$

i.e. $U^{(1)}(t)$ is T-periodic. Analogously

$$u^{(2)}(t) = e^{-i\alpha t}U^{(2)}(t)$$

$U^{(1)}(t)$ and $U^{(2)}(t)$ are T-periodic.

The $u^{(1,2)}(t)$ are independent linear combinations of the two linearly independent solutions $u_1(t)$ and $u_2(t) \Rightarrow u^{(1,2)}(t)$ are also linearly independent.

More formally:
assume there are $k_1 \neq 0 \neq k_2$ with

$$k_1 u^{(1)}(t) + k_2 u^{(2)}(t) = 0 \qquad \text{for all } t$$

$$\begin{aligned} k_1 e^{i\alpha t}U^{(1)}(t) + k_2 e^{-i\alpha t}U^{(2)}(t) &= 0 \\ k_1 e^{i\alpha(t+T)}U^{(1)}(t+T) + k_2 e^{-i\alpha(t+T)}U^{(2)}(t+T) &= 0 \end{aligned}$$

Using $U^{(1,2)}(t+T) = U^{(1.2)}(t)$ non-zero k_i requires the determinant to vanish

$$e^{i\alpha t}U^{(1)}(t)e^{-i\alpha(t+T)}U^{(2)}(t) - e^{-i\alpha t}U^{(2)}(t)e^{i\alpha(t+T)}U^{(1)}(t) = 0$$

i.e.

$$e^{-i\alpha T} - e^{+i\alpha T} = 0 \qquad \rho_1 = \rho_2$$

which is a contradiction. Therefore $k_1 = 0 = k_2$, showing the linear independence.

Thus we have:
For $|K| \neq 1$ the **general solution** $u(t)$ to (18) can be written in the form

$$u(t) = c_1 e^{i\alpha t}U^{(1)}(t) + c_2 e^{-i\alpha t}U^{(2)}(t)$$

with $U^{(1,2)}(t)$ being T-periodic.

1. for $|K| < 1$ the solutions are quasi-periodic since $\alpha \in \mathbb{R}$
   Since $u(t)$ is real

$$\begin{aligned} 2u(t) &= c_1 e^{i\alpha t}U^{(1)}(t) + c_2 e^{-i\alpha t}U^{(2)}(t) + c.c. \\ &= e^{i\alpha t}\left\{c_1 U^{(1)} + c_2^* U^{(2)*}\right\} + e^{-i\alpha t}\left\{c_1^* U^{(1)*} + c_2 U^{(2)}\right\} \\ &= e^{i\alpha t}\left\{c_1 U^{(1)} + c_2^* U^{(2)*}\right\} + c.c. \end{aligned}$$

   Therefore we can write in general

$$u(t) = c\, e^{i\alpha t}U(t) + c.c.$$

   with $U(t+T) = U(t)$.

2. for $|K| > 1$ we have $\alpha \in i\mathbb{R}$ and there are two linearly independent real solutions:
   one solution grows exponentially, while the other decays exponentially.
   The growing solution renders the state $u(t) = 0$ linearly *unstable*.

2.1.3 Stability and Instability in the Mathieu Equation

Use the result from Floquet theory to determine the stability properties of $u(t) = 0$ in the Mathieu equation with respect to a subharmonic instability, $\rho \approx -1$.

Consider

$$\ddot{u} + (\delta + \epsilon \cos 2t)\, u = 0 \tag{23}$$

The general solution can be written as

$$u(t) = c_1 e^{i\alpha t}\psi_1(t) + c_2 e^{-i\alpha t}\psi_2(t) \qquad \psi_i(t+\pi) = \psi_i(t)$$

because the forcing period is $T = \pi$.

Equation is linear: can determine the two components $c_i e^{\pm i\alpha t}\psi_i$ separately. .

Note:

- for any values of δ and ϵ (23) has the base solution $u(t) = 0$. If (23) has also exponentially growing solutions the base solution is linearly *unstable*.

Goal: determine $\alpha(\delta, \epsilon)$.

For a response that is subharmonic with respect to the period of the forcing one has $\rho = e^{i\alpha\pi} = -1$, i.e. $\alpha = 1$. We therefore expand

$$\alpha = 1 + \epsilon\alpha_1 + \epsilon^2\alpha_2 + \ldots$$

$$u = e^{i\alpha t}\psi(t) = e^{i\left(\epsilon\alpha_1 + \epsilon^2\alpha_2 t + \ldots\right)t} \underbrace{e^{it}\psi(t)}_{\phi(t)}$$

with

$$\begin{aligned} \phi(t) &= \phi_0(t) + \epsilon\phi_1(t) + \ldots \\ \delta &= 1 + \epsilon\delta_1 + \ldots \end{aligned}$$

$$\begin{aligned} \ddot{u} = & -\left(\epsilon\alpha_1 + \epsilon^2\alpha_2 + \ldots\right)^2 e^{i\left(\epsilon\alpha_1+\epsilon^2\alpha_2+\ldots\right)t}\phi + 2i\left(\epsilon\alpha_1 + \epsilon^2\alpha_2 + \ldots\right) e^{i\left(\epsilon\alpha_1+\epsilon^2\alpha_2+\ldots\right)t}\dot{\phi} + \\ & + e^{i\left(\epsilon\alpha_1+\epsilon^2\alpha_2+\ldots\right)t}\ddot{\phi} \end{aligned}$$

Insert

$\mathcal{O}(1)$:

$$\ddot{\phi}_0 + \phi_0 = 0 \quad \Rightarrow \quad \phi_0(t) = c_1 \cos t + c_2 \sin t$$

$\mathcal{O}(\epsilon)$:

$$\ddot{\phi}_1 + \phi_1 = -2i\alpha_1\dot{\phi}_0 - \delta_1\phi_0 - \cos 2t\, \phi_0 \equiv F_1(t)$$

$$
\begin{aligned}
F_1(t) &= -2i\alpha_1\left(-c_1\sin t + c_2\cos t\right) - \delta_1\left(c_1\cos t + c_2\sin t\right) - \cos 2t\left(c_1\cos t + c_2\sin t\right) \\
&= \cos t\left(-2i\alpha_1 c_2 - \delta_1 c_1 - \frac{1}{2}c_1\right) + \sin t\left(2i\alpha_1 c_1 - \delta_1 c_2 + \frac{1}{2}c_2\right) \\
&\quad -\frac{1}{2}c_1\cos 3t - \frac{1}{2}c_2\sin 3t
\end{aligned}
$$

To avoid secular terms we need

$$
\begin{aligned}
-2i\alpha_1 c_2 - \delta_1 c_1 - \frac{1}{2}c_1 &= 0 \\
2i\alpha_1 c_1 - \delta_1 c_2 + \frac{1}{2}c_2 &= 0
\end{aligned}
$$

$$
\begin{vmatrix} \delta_1 + \frac{1}{2} & 2i\alpha_1 \\ 2i\alpha_1 & -\delta_1 + \frac{1}{2} \end{vmatrix} = 0
$$

$$
-\delta_1^2 + \frac{1}{4} + 4\alpha_1^2 = 0
$$

$$
\alpha_1^{(\pm)} = \pm\frac{1}{2}\sqrt{\delta_1^2 - \frac{1}{4}} = \pm i\frac{1}{2}\sqrt{\frac{1}{4} - \delta_1^2}
$$

Insert

$$
\left(\delta_1 + \frac{1}{2}\right)c_1 \mp \sqrt{\frac{1}{4} - \delta_1^2}\,c_2 = 0
$$

$$
\sqrt{\frac{1}{2} + \delta_1}\,c_1 \mp \sqrt{\frac{1}{2} - \delta_1}\,c_2 = 0
$$

$$
c_1 = A_\pm\sqrt{\frac{1}{2} - \delta_1} \qquad c_2 = \pm A_\pm\sqrt{\frac{1}{2} + \delta_1}
$$

Thus

$$
\begin{aligned}
u_\pm(t) &= A_\pm e^{i\epsilon\alpha_1^{(\pm)}t}\left(\sqrt{\frac{1}{2} - \delta_1}\,\cos t \pm \sqrt{\frac{1}{2} + \delta_1}\,\sin t\right) \\
&= A_\pm e^{\mp\frac{1}{2}\sqrt{\frac{1}{4} - \delta_1^2}\,\epsilon t}\left(\sqrt{\frac{1}{2} - \delta_1}\,\cos t \pm \sqrt{\frac{1}{2} + \delta_1}\,\sin t\right)
\end{aligned}
$$

The general solution is given by

$$
u(t) = A_+u_+(t) + A_-u_-(t)
$$

with the amplitude $A_\pm$ determined by initial conditions.

Notes:

- for $\delta_1 = \pm\frac{1}{2}$ the solution reduces to the periodic solution obtained in Sec.2.1.1.
- for $\delta_1^2 > \frac{1}{4}$ the solutions remain bounded. The solutions are quasiperiodic, i.e. they exhibit two unrelated frequencies $\omega = 1$ and $\Omega = \sqrt{\delta_1^2 - \frac{1}{4}}$.

- for $\delta_1^2 < \frac{1}{4}$ one solution blows up exponentially: the state $u(t) = 0$ is unstable.
- the Mathieu equation is linear $\Rightarrow$ the exponential growth does not saturate
- recall: δ_1 characterizes the leading-order detuning between the forcing and the subharmonic resonance
  - weak detuning ($\delta_1^2 < \frac{1}{4}$) $\Rightarrow$ resonant driving leads to exponential growth of one mode
  - strong detuning ($\delta_1^2 > \frac{1}{4}$) $\Rightarrow$ driving is out of resonance and is not able to pump in energy to generate growth
  - at the border between these two regimes the solution is periodic

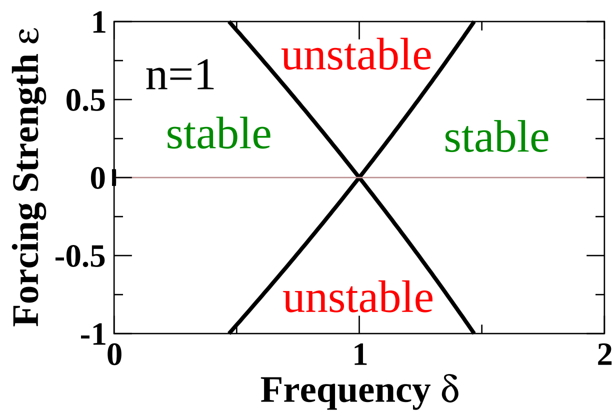


Figure 17: Stability and instability regions of Mathieu equation at the 2:1 resonance (subharmonic response).

2.2 Nonlinear Oscillators: Forced Duffing Oscillator

Nonlinearity can saturate the growth of the oscillations and can affect the resonance frequency

As an example of a forced, weakly nonlinear oscillator consider the Duffing equation

$$\ddot{\hat{y}} + \underbrace{\hat{\beta}\dot{\hat{y}}}_{\text{damping}} + \underbrace{\omega_0^2}_{\text{natural frequency}} \hat{y} + \underbrace{\hat{\alpha}\hat{y}^3}_{\text{nonlinearity}} = \underbrace{\hat{f}\cos\omega t}_{\text{forcing}} \tag{24}$$

Note:

- here the forcing is taken to be *non-parametric*; in the swing picture it would correspond to a person pushing $\Rightarrow$ with forcing the solution $\hat{y} = 0$ does not exist any more

2.2.1 Linear Case

i) linear, undamped case: $\hat{\beta} = 0$, $\hat{\alpha} = 0$

in this case one can easily consider the general case of $\mathcal{O}(1)$-forcing

$$\ddot{y} + \omega_0^2 y = \hat{f}\cos\omega t \qquad y(0) = y_i, \quad \dot{y}(0) = 0$$

For $\omega \neq \omega_0$

$$y(t) = c_1 \cos\omega_0 t + c_2 \sin\omega_0 t + \frac{\hat{f}}{\omega_0^2 - \omega^2}\cos\omega t$$

using initial conditions

$$y(0) = c_1 + \frac{\hat{f}}{\omega_0^2 - \omega^2} \overbrace{=}^{!} y_i \qquad \dot{y}(0) = c_2\omega_0 \overbrace{=}^{!} 0$$

we get

$$\begin{aligned} y(t) &= y_i \cos\omega_0 t + \frac{\hat{f}}{\omega_0^2 - \omega^2}\left(\cos\omega t - \cos\omega_0 t\right) \\ &= y_i \cos\omega_0 t + \underbrace{\frac{\hat{f}}{\omega_0 + \omega}\frac{2}{\omega_0 - \omega}\sin\frac{\omega_0 - \omega}{2}t}_{A(t)}\,\sin\frac{\omega_0 + \omega}{2}t \end{aligned}$$

For $\omega_0 - \omega \ll \omega_0$ the amplitude $A(t)$ varies much more slowly than $\sin\frac{1}{2}(\omega_0 + \omega)\,t$: ***beating***.

For $\omega \to \omega_0$ one gets

$$y(t) \to y_i \cos\omega_0 t + \frac{\hat{f}}{2\omega}\, t \sin\omega t$$

Notes:

- at the resonance $\omega = \omega_0$ the forcing leads to a linear, unbounded growth of the oscillations

ii) linear case with damping: $\hat{\alpha} = 0$

$$\ddot{y} + \hat{\beta}\dot{y} + \omega_0^2 y = \hat{f}\cos\omega t \qquad y(0) = y_i, \quad \dot{u}(0) = 0$$

General solution

$$y(t) = \underbrace{y_h(t)}_{\to 0 \text{ for } t \to \infty} + \hat{f}\,\frac{(\omega_0^2 - \omega^2)\cos\omega t + \hat{\beta}\omega\sin\omega t}{\left(\omega_0^2 - \omega^2\right)^2 + \hat{\beta}^2\omega^2} \qquad (25)$$

Notes:

- the homogeneous solution consists of a damped oscillation

$$y_h(t) = e^{-\sigma t}\left(a\cos\omega_\beta t + b\sin\omega_\beta t\right) \quad \textbf{with} \quad \sigma = \frac{1}{2}\hat{\beta} \quad \omega_\beta = \sqrt{\omega_0^2 - \frac{1}{4}\hat{\beta}^2}$$

  with a and b determined by the initial conditions.

- the damping leads to a lag of the phase of the oscillation relative to the forcing
- for large times the initial condition becomes irrelevant since $y_h \to 0$ for $t \to \infty$
- the approach to the steady state is oscillatory in the amplitude of oscillation, reflecting the beating obtained without damping

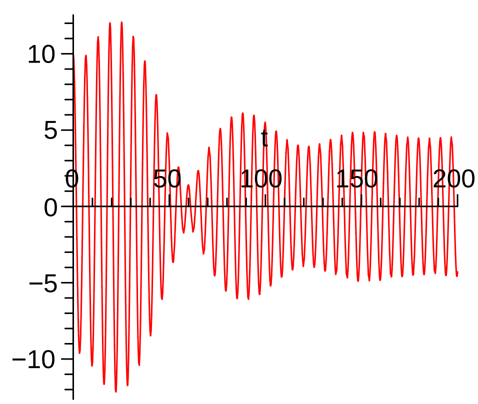


Figure 18: Solution of the linear case with damping for $\beta = 0.05$, $\omega_0 = 1$, $\omega = 1.1$, $\delta = 0.1$, $f = 1$, $\alpha = 0$.

In the steady state reached for $t \to \infty$ the amplitude of the oscillation is given by

$$R_\infty = \frac{\hat{f}}{\left(\omega_0^2 - \omega^2\right)^2 + \hat{\beta}^2\omega^2}\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + \hat{\beta}^2\omega^2} = \frac{\hat{f}}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + \hat{\beta}^2\omega^2}}$$

Frequency of maximal amplitude

$$-4\omega_{max}\left(\omega_0^2 - \omega_{max}^2\right) + 2\hat{\beta}^2\omega_{max} = 0 \qquad \omega_{max}^2 = \omega_0^2 - \frac{1}{2}\hat{\beta}^2$$

$$R_{max} = \frac{\hat{f}}{\hat{\beta}\sqrt{\omega_0^2 - \frac{1}{4}\hat{\beta}^2}}$$

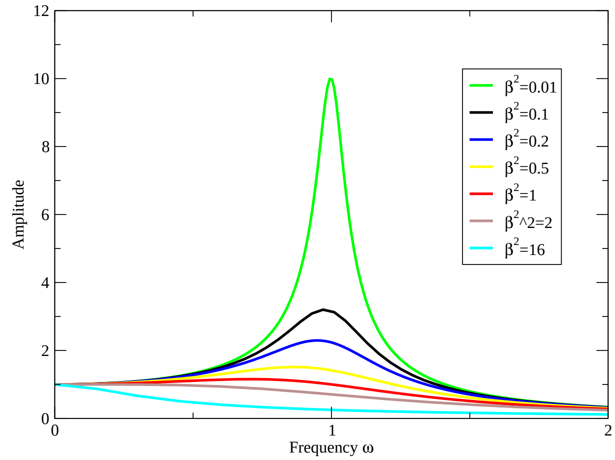


Figure 19: Response curve $R_\infty(\omega)$ for $\omega_0 = 1$.

Notes:

- for non-zero damping
  - the oscillation amplitude does not diverge at the resonance $\omega_0 = \omega$
  - the maximal amplitude is attained at a frequency *below* the natural frequency $\sqrt{\omega_0^2 - \frac{1}{4}\hat{\beta}^2}$
- with decreasing damping the resonance peak of the response curve becomes sharper and taller
- the maximum in the response disappears for $\hat{\beta}^2 > 2\omega_0^2$

2.2.2 Nonlinear Case: 1:1 Forcing

For the Mathieu equation we managed to get an approximate solution for

- weak forcing
- weak detuning

As expected from Floquet theory the solution had the form

$$u_\pm(t) = A_\pm\, e^{i\alpha_1^{(\pm)}T}\,(c_1\,\cos t\,\pm\,c_2\,\sin t)$$

and exhibited

- slow growth or slow oscillation depending on the detuning δ

The slow growth and slow oscillations can be captured by using multiple time scales

$$t \qquad \Rightarrow \qquad \hat{t} = t \quad \text{and} \quad T = \epsilon t$$

We want to extend this approach to include damping and the nonlinearity.

Perturbation approach:

- multiple time scales:
  - eliminate the fast evolution
  - derive equation on a slow time scale

Under what conditions can the damping and the nonlinearity be captured in the slow evolution?

How are they related to time scales?

Time scales in the problem:

- natural frequency of the oscillator
- forcing frequency
- decay time due to damping
- change in frequency due to a change in amplitude through the nonlinearity

Under what conditions can we capture forcing, damping, and nonlinearity with a single equation for the slow evolution?

Consider slow time scales

- weak damping: slow decay
- forcing is close to 1:1 resonance $\omega \approx \omega_0$: small frequency difference
- small amplitude: small change in frequency

We want to capture all three aspects in a single expansion $\Rightarrow$ consider all of them small. How do we have to choose the scaling of the various small quantitites? At this point we just use trial and error. Later we will develop a systematic method to obtain the optimal scaling.

Introducing an explicit $\epsilon \ll 1$ we try

$$\mathcal{O}(\omega - \omega_0) \equiv \epsilon\Omega = \mathcal{O}(\hat{\beta}) \qquad \mathcal{O}(\hat{y}^2) = \epsilon \qquad \mathcal{O}(\hat{f}) = \mathcal{O}(y^3)$$

and rewrite

$$\hat{\beta} = \epsilon\beta \qquad \hat{y} = \epsilon^{\frac{1}{2}} y \qquad \hat{f} = \epsilon^{\frac{3}{2}} f$$

$$\ddot{y} + \underbrace{\epsilon\beta\dot{y}}_{\text{damping}} + \underbrace{\omega_0^2}_{\text{natural frequency}} y + \underbrace{\epsilon\alpha y^3}_{\text{nonlinearity}} = \underbrace{\epsilon f \cos\omega t}_{\text{forcing}} \tag{26}$$

with initial conditions

$$y(0) = y_i \qquad \dot{y}(0) = 0$$

Note:

- in general the balancing of the terms is more subtle. It is not done directly in the original equation that is to be expanded. We will see that symmetries play an important role.

Without loss of generality assume $\omega_0 = 1$

$$\omega = \omega_0 + \epsilon\Omega = 1 + \epsilon\Omega$$

The slow damping, the small frequency difference, and the small change in frequency due to the nonlinearity can be captured with a second, slow time scale using the method of multiple scales

$$y = y(\tilde{t}, T) \qquad \text{with} \qquad T = \epsilon t \qquad \tilde{t} = t$$

Now we simply expand $y(t)$

$$y(t) = y_0(\tilde{t}, T) + \epsilon y_1(\tilde{t}, T) + \ldots$$

and insert into

$$\ddot{y} + \epsilon\beta\dot{y} + y + \epsilon\alpha y^3 = \epsilon f \cos\omega t$$

$\mathcal{O}(1)$:

$$\frac{\partial^2 y_0}{\partial \tilde{t}^2} + y_0 = 0 \qquad y_0(0,0) = y_i, \; \frac{\partial y_0(0,0)}{\partial t} = 0$$

$$y_0(\tilde{t}, T) = A(T)e^{i\tilde{t}} + A^*(T)e^{-i\tilde{t}}$$

Note:

- The complex amplitude can depend slowly on time: this allows slow growth and decay as well as small changes in the frequency of the oscillator

$$A(T) = Re^{i\Omega T} \quad \Rightarrow \quad y_0 = Re^{i(1+\epsilon\Omega)t} + c.c. = R\cos\left((1+\epsilon\Omega)\, t\right)$$

Expand the operators using multiple times:

Using

$$\frac{d}{dt}y(\tilde{t}, T) = \partial_{\tilde{t}} y \frac{d\tilde{t}}{dt} + \partial_T y \frac{dT}{dt} = \partial_{\tilde{t}} y + \epsilon \partial_T y$$

We simply write

$$\frac{\partial}{\partial t} = \partial_{\tilde{t}} + \epsilon\partial_T \qquad \frac{\partial^2}{\partial t^2} = (\partial_{\tilde{t}} + \epsilon\partial_T)^2 = \partial_{\tilde{t}}^2 + 2\epsilon\partial_{\tilde{t}T}^2 + \mathcal{O}(\epsilon^2)$$

$\mathcal{O}(1)$:

$$\partial_{\tilde{t}}^2 y_0 + y_0 \equiv \mathcal{L}y_0 = 0$$

$\mathcal{O}(\epsilon)$:

$$\partial_{\tilde{t}}^2 y_1 + y_1 = -2\partial_{\tilde{t}}\partial_T y_0 - \beta\partial_{\tilde{t}} y_0 - \alpha y_0^3 + f\cos\left((1+\epsilon\Omega)\,\tilde{t}\right) \equiv f_1(\tilde{t}, T)$$

Write as

$$\mathcal{L}y_1 = f_1(\tilde{t}, T)$$

We need

$$
\begin{aligned}
y_0^3 &= \left(A(T)e^{i\tilde{t}} + A^*(T)e^{-i\tilde{t}}\right)^3 = A^3e^{3i\tilde{t}} + 3|A|^2Ae^{\tilde{i}} + 3|A|^2A^*e^{-i\tilde{t}} + A^{*^3}e^{-3i\tilde{t}} \\
\cos\left((1+\epsilon\Omega)\,\tilde{t}\right) &= \frac{1}{2}\left(e^{i\tilde{t}}e^{i\Omega T} + e^{-i\tilde{t}}e^{-i\Omega T}\right)
\end{aligned}
$$

Note:

- the linear operator $\mathcal{L} \equiv \partial_{\tilde{t}}^2 + 1$ is singular with zero-modes $e^{i\tilde{t}}$ and $e^{-i\tilde{t}}$ $\Rightarrow$ secular terms arise from terms $\propto e^{\pm i\tilde{t}}$ in $f_1(\tilde{t},T)$.

Focus only on the terms $\propto e^{i\tilde{t}}$ in f_1

$$
f_1(\tilde{t},T) = \left\{-2i\partial_T A - \beta i A - 3\alpha|A|^2A + \frac{1}{2}fe^{i\Omega T}\right\}e^{i\tilde{t}} + \{\ldots\}\,e^{3i\tilde{t}} + c.c.
$$

The solvability conditions requires

$$
-2i\partial_T A - \beta i A - 3\alpha|A|^2A + \frac{1}{2}fe^{i\Omega T} = 0
$$

i.e.

$$
\partial_T A = -\frac{1}{2}\beta A + \frac{3}{2}i\alpha|A|^2A - \frac{i}{4}fe^{i\Omega T}
$$

Note:

- The solvability condition arising from the secular term $e^{-i\tilde{t}}$ is the complex conjugate of the solvability condition arising from $e^{i\tilde{t}}$. It is therefore equivalent and does not add anything new.

It is convenient to absorb the slow oscillation of the forcing term into a redefined complex oscillation amplitude

$$
\mathcal{A}(T) = e^{-i\Omega T}A(T)
$$

$$
\boxed{\partial_T\mathcal{A} = \left(-\frac{1}{2}\beta - i\Omega\right)\mathcal{A} + \frac{3}{2}i\alpha|\mathcal{A}|^2\mathcal{A} - \frac{i}{4}f} \qquad (27)
$$

It is often also useful to write the complex amplitude equation in terms of a real amplitude and phase

$$
\mathcal{A}(T) = R(T)e^{i\phi(T)}
$$

Thus

$$
\partial_T R + i\partial_T\phi R = \left(-\frac{1}{2}\beta - i\Omega\right)R + \frac{3}{2}i\alpha R^3 - \frac{i}{4}fe^{-i\phi(T)}
$$

Separating into real and imaginary parts

$$
\begin{aligned}
\partial_T R + \frac{1}{2}\beta R &= -\frac{1}{4}f\sin\phi && (28)\\
R\partial_T\phi + \left(\Omega - \frac{3}{2}\alpha R^2\right)R &= -\frac{1}{4}f\cos\phi && (29)
\end{aligned}
$$

Notes:

- Eq.(27) and eqs.(28,29) for the amplitude and phase of the forced oscillator were derived under the assumptions
  - weak damping
  - weak forcing
  - small deviation of the forcing frequency from the natural frequency of the oscillator

  These conditions are satisfied for a system that undergoes a Hopf bifurcation for values of the bifurcation parameter just below the Hopf bifurcation, i.e. in the regime in which the basic state is still linearly stable, but only weakly so ($\Rightarrow$ weak damping). These equations should therefore arise more generally for any such Hopf bifurcation (see later).

Note:

- Drawbacks of the formulation in terms of amplitude and phase.
  - (28,29) are nonlinear even for $\alpha = 0$, i.e. even when the original equations are linear.
  - (28,29) are even *singular* for $R = 0$ (coefficient of ϕ' vanishes). This singularity is only a coordinate singularity, nothing singular happens in the solution.

i) Compare first with previous result for the **linear case** $\alpha = 0$

Steady state (fixed point of (28,29))

$$\frac{1}{4}\beta^2 R_\infty^2 + \Omega^2 R_\infty^2 = \frac{1}{16} f^2 \quad \Rightarrow \quad R_\infty = \frac{1}{2}\frac{f}{\sqrt{\beta^2 + 4\Omega^2}}$$

$$\tan\phi_\infty = \frac{\frac{1}{2}\beta R_\infty}{\Omega R_\infty} \qquad \phi_\infty = \arctan\left(\frac{\beta}{2\Omega}\right)$$

The solution of (25) approaches the steady state in an oscillatory manner in the amplitude R. That was not easily seen in our previous linear solution. Recover that aspect from (28,29) by considering small perturbations around (R_∞, ϕ_∞)

$$R = R_\infty + r(T) \qquad \phi = \phi_\infty + \varphi(T)$$

Insert

$$\begin{aligned} r' + \frac{1}{2}\beta\left(R_\infty + r\right) &= -\frac{1}{4} f\left(\sin\phi_\infty + \cos\phi_\infty\,\varphi\right) \\ R_\infty\varphi' + \Omega\left(R_\infty + r\right) &= -\frac{1}{4} f\left(\cos\phi_\infty - \sin\phi_\infty\,\varphi\right) \end{aligned}$$

and rewrite

$$\begin{pmatrix} r' \\ \varphi' \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}\beta & -\frac{1}{4} f\cos\phi_\infty \\ -\frac{\Omega}{R_\infty} & \frac{1}{4R} f\sin\phi_\infty \end{pmatrix}\begin{pmatrix} r \\ \varphi \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}\beta & +\Omega R_\infty \\ -\frac{\Omega}{R_\infty} & -\frac{1}{2}\beta \end{pmatrix}\begin{pmatrix} r \\ \varphi \end{pmatrix}$$

Using an ansatz

$$\begin{pmatrix} r \\ \varphi \end{pmatrix} = \begin{pmatrix} r_0 \\ \varphi_0 \end{pmatrix} e^{\sigma T}$$

yields

$$\begin{vmatrix} \sigma - \frac{1}{2}\beta & -\Omega R_\infty \\ -\frac{\Omega}{R_\infty} & \sigma - \frac{1}{2}\beta \end{vmatrix} = 0$$

$$\sigma^2 - \beta\sigma + \frac{1}{4}\beta^2 + \Omega^2 = 0 \qquad \sigma = -\beta \pm i\Omega$$

ii) Now the **nonlinear problem**:

Fixed points (critical points):
Start with the easier case without damping: $\beta = 0$

$$\begin{aligned} \beta R_\infty &= -\frac{1}{2} f \sin\phi_\infty \\ 2\Omega R_\infty - 3\alpha R_\infty^3 &= -\frac{1}{2} f \cos\phi_\infty \end{aligned}$$

This yields

$$\sin\phi_\infty = 0 \quad \Rightarrow \quad \phi_\infty = 0 \quad \textbf{or} \quad \phi_\infty = \pi$$

and

$$2\Omega R_\infty - 3\alpha R_\infty^3 = \mp\frac{1}{2} f$$

Since $y = Re^{i\phi}e^{i\tilde{t}} + c.c.$, for $\phi = 0$ the oscillation is in phase with the forcing. For $\phi = \pi$ it is out of phase.

To get an overview of the dependence on the detuning Ω it is easier to solve for Ω than for $R_\infty^{(1,2)}$

In-phase solution $\phi_\infty = 0$:

$$\Omega = \frac{3}{2}\alpha R_\infty^2 - \frac{f}{4R_\infty}$$

Out-of-phase solution $\phi_\infty = \pi$:

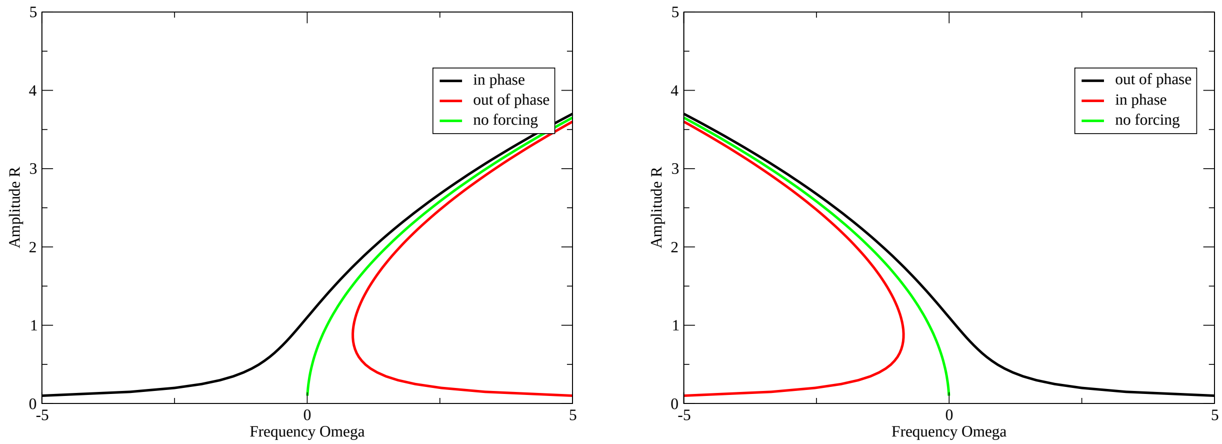
$$\Omega = \frac{3}{2}\alpha R_\infty^2 + \frac{f}{4R_\infty}$$

Notes:

- In the absence of forcing both curves become identical and one recovers the amplitude-dependence of the frequency of the Duffing oscillator

$$\omega = \omega_0 + \frac{3}{2}\alpha R_\infty^2$$

  $\alpha > 0$ corresponds to a hard spring: frequency increases with amplitude
  $\alpha < 0$ corresponds to a soft spring: frequency decreases with amplitude



a) Mon Jan 11 22:34:53 2010 b) Mon Jan 11 22:37:23 2010

Figure 20: Dependence of the amplitude on the forcing frequency for $\alpha > 0$ (a) and $\alpha < 0$ (b).

Characterize $\Omega(R_\infty)$:

$$\frac{d\Omega}{dR_\infty} = 3\alpha R_\infty \pm \frac{f}{4R_\infty^2} \qquad "+" \Leftrightarrow \phi_\infty = 0$$

For $\alpha > 0$ the in-phase solution has a monotonic dependence $\Omega(R_\infty)$ and the out-of-phase solution has a minimum at

$$R_\infty^{(min)} = \left(\frac{f}{12\alpha}\right)^{\frac{1}{3}}$$

and vice versa for $\alpha < 0$ ($f \geq 0$).

Thus for $\alpha > 0$ there are none or two out-of-phase solutions for a given value of Ω and for $\alpha < 0$ there are none or two in-phase solutions.

Linear stability of the various solutions:

$$R = R_\infty + r(T) \qquad \phi = \phi_\infty + \varphi(T)$$

$$\begin{pmatrix} r' \\ \varphi' \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}\beta & -\frac{1}{4}f\cos\phi_\infty \\ -\frac{\Omega}{R_\infty} + \frac{9}{2}\alpha R_\infty & \frac{1}{4R_\infty} f \sin\phi_\infty \end{pmatrix} \begin{pmatrix} r \\ \varphi \end{pmatrix} \overbrace{=}^{\text{no damping}} \begin{pmatrix} 0 & \mp\frac{1}{4}f \\ -\frac{\Omega}{R_\infty} + \frac{9}{2}\alpha R_\infty & 0 \end{pmatrix} \begin{pmatrix} r \\ \varphi \end{pmatrix}$$

with the upper sign (here " $-$ ") corresponding to $\phi_\infty = 0$ and the lower sign (here " $+$ ") to $\phi_\infty = \pi$.

The eigenvalues are

$$\begin{aligned} \sigma^2 &= \pm\frac{1}{4}f\left(\frac{\Omega}{R_\infty} - \frac{9}{2}\alpha R_\infty\right) = \pm\frac{1}{4}f\left(\frac{3}{2}\alpha R_\infty \mp \frac{1}{4}\frac{f}{R_\infty^2} - \frac{9}{2}\alpha R_\infty\right) \\ &= \mp\frac{1}{4}f\left(3\alpha R_\infty \pm \frac{1}{4}\frac{f}{R_\infty^2}\right) = \mp f\frac{d\Omega}{dR_\infty} \end{aligned}$$

Thus both eigenvalues are either real or purely imaginary. One of the real eigenvalues is always positive.

Again, the upper sign (here " $-$ ") corresponds to $\phi_\infty = 0$ and the other one to $\phi_\infty = \pi$.

- $\alpha > 0$:
  - in-phase solution stable: $\sigma \in i\mathbb{R}$ (note $\frac{d\Omega}{dR_\infty} > 0$ on this branch)
  - out-of-phase solution is
    - stable for
$$R_\infty < \left(\frac{f}{12\alpha}\right)^{\frac{1}{3}} = R_\infty^{(min)}$$
    - unstable for $R_\infty > R_\infty^{(min)}$
- $\alpha < 0$:
  - in-phase solution is
    - stable for $R_\infty < R_\infty^{(min)}$
    - unstable for $R_\infty > R_\infty^{(min)}$
  - out-of-phase solution is stable

Notes:

- without damping 'stability' means actually purely oscillatory response to small perturbations, no asymptotic approach to the fixed point.
- expect that with damping the purely imaginary eigenvalues acquire a negative real part and the fixed point becomes asymptotically stable.

Now with damping: $\beta > 0$

Fixed points:

$$\begin{aligned} \beta R_\infty &= -\frac{1}{2} f \sin\phi_\infty \\ \Omega R_\infty - \frac{3}{2}\alpha R_\infty^3 &= -\frac{1}{2} f \cos\phi_\infty \end{aligned}$$

$$\beta^2 R_\infty^2 + \left(\Omega R_\infty - \frac{3}{2}\alpha R_\infty^3\right)^2 = \frac{1}{4} f^2 \qquad (30)$$

Can solve again for Ω

$$\Omega_{1,2} = \frac{3}{2}\alpha R_\infty^2 \pm \frac{1}{2R_\infty}\sqrt{f^2 - 4\beta^2 R_\infty^2} = \frac{3}{2}\alpha R_\infty^2 \pm \frac{1}{2}\sqrt{\frac{f^2}{R_\infty^2} - \beta^2}$$

Now R_∞ is bounded:

- no phase-locked solution for $R_\infty > \frac{f}{\beta}$.

- For $R_\infty = \frac{f}{\beta}$ both branches merge: turning point of $\Omega(R_\infty)$

Turning points of $R_\infty(\Omega)$ correspond to $\frac{d\Omega}{dR_\infty} = 0$

$$\frac{d\Omega_{1,2}}{dR_\infty} = 3\alpha R_\infty \pm \frac{f^2}{4\sqrt{\frac{f^2}{R_\infty^2} - \beta^2}} \frac{-2}{R_\infty^3}$$

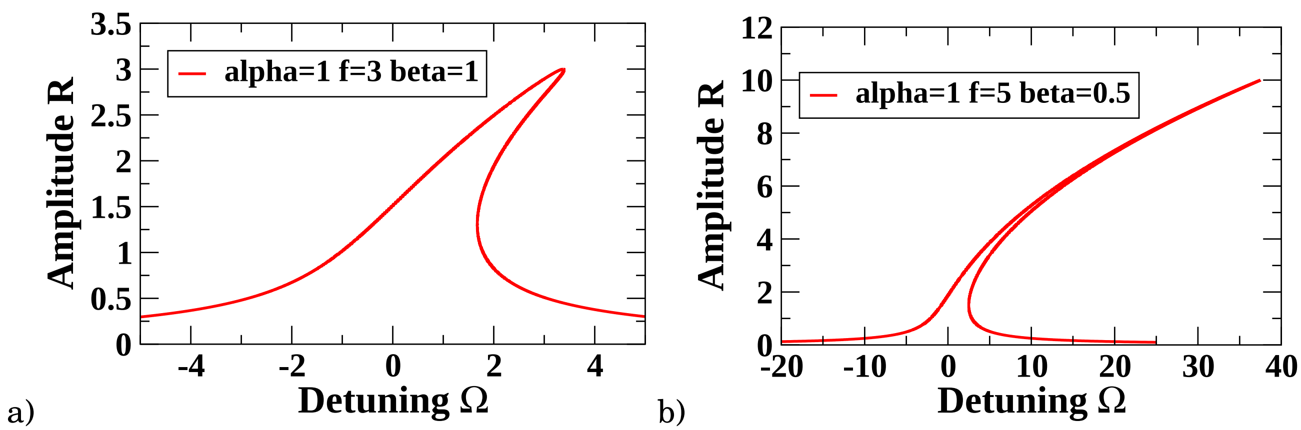
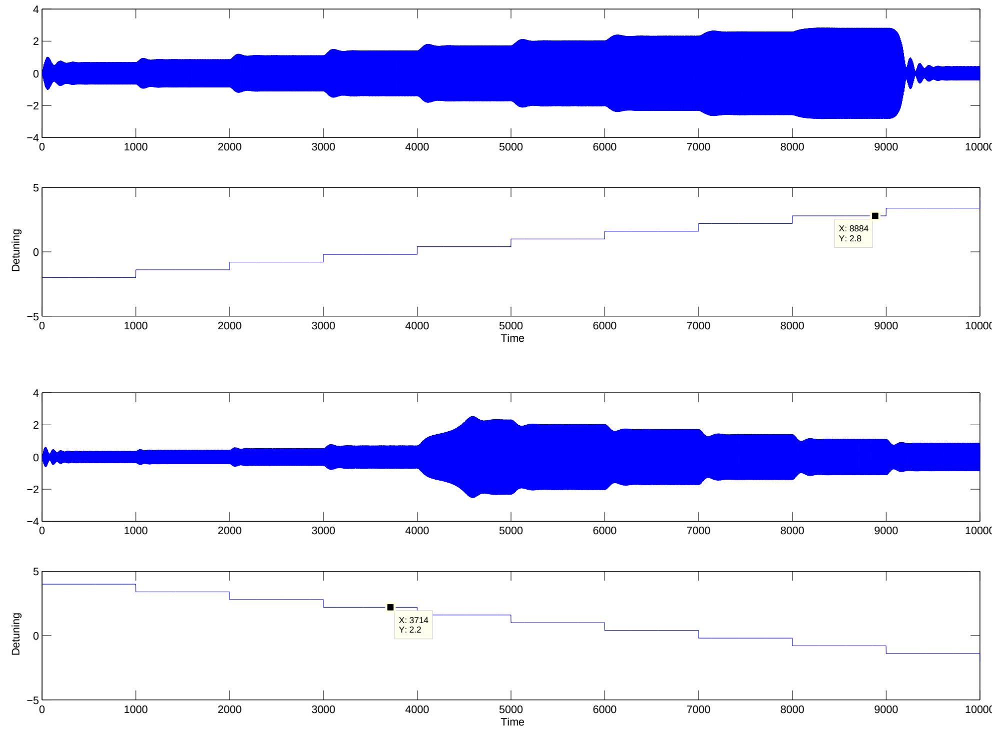


Figure 21: Dependence of the amplitude of the phase-locked solution on the forcing frequency with damping. Strong hysteresis for strong forcing and weak damping (b).

Notes:

- To test the hysteresis need to scan the frequency in steps to be able to assess convergence to a steady state. To make the phase of the forcing continuous across the jumps solve the system

$$\begin{aligned} \dot{\varphi} &= \omega(t) \\ \ddot{y} + \omega_0^2 y + \epsilon \dot{y} + \epsilon \alpha y^3 &= \epsilon f \cos\varphi \end{aligned}$$



a)

b)

Figure 22: Numerical Results: $\alpha = 1$, $f = 3$, $\beta = 1$, $\epsilon = 0.02$. Hysteresis agrees quantitatively with perturbation result (to the extent tested in this short simulation). a) frequency increased in 10 steps, b) frequency decreased in 10 steps.

Notes:

- A review of certain aspects of the dynamics of the Duffing oscillator with $2:1$-forcing is given in [9].

2.2.3 Duffing Oscillator as a System of Equations

For simplicity demonstrate the approach with the Duffing oscillator again near the 1:1-resonance

$$\ddot{y} + \omega_0^2 y + \epsilon \dot{y} + \epsilon \alpha y^3 = \epsilon f \cos \omega t \qquad \omega = \omega_0 + \epsilon \Omega$$

Rewrite in terms of a system of first-order equations with

$$u = y \qquad v = \dot{y}$$

Thus

$$\begin{aligned} \dot{u} - v &= 0 \\ \dot{v} + \omega_0^2 u &= -\epsilon \left\{ \beta v + \alpha u^3 - f \cos \omega t \right\} \end{aligned}$$

Introducing slow and fast times, $\hat{t} = t$, $T = \epsilon t$ we get

$$\begin{aligned} \partial_{\hat{t}} u - v &= -\epsilon\, \partial_T u \equiv \epsilon I_1 \\ \partial_{\hat{t}} v + \omega_0^2 u &= -\epsilon \left\{ \partial_T v + \beta v + \alpha u^3 - f \cos \omega \hat{t} \right\} \equiv \epsilon I_2 \end{aligned}$$

i.e.

$$\mathcal{L}\left(\partial_{\hat{t}}\right)\begin{pmatrix} u \\ v \end{pmatrix} \equiv \begin{pmatrix} \partial_{\hat{t}} & -1 \\ \omega_0^2 & \partial_{\hat{t}} \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \epsilon I_1 \\ \epsilon I_2 \end{pmatrix}$$

Looking for a harmonic response at frequency ω_0, we expand as usually,

$$\begin{pmatrix} u \\ v \end{pmatrix} = A(T)\begin{pmatrix} U_0 \\ V_0 \end{pmatrix} e^{i\omega_0 \hat{t}} + c.c. + \epsilon \begin{pmatrix} u_1(\hat{t},T) \\ v_1(\hat{t},T) \end{pmatrix} + \mathcal{O}(\epsilon^2).$$

We need to determine an evolution equation for the complex amplitude $A(T)$.

Inserting this ansatz will lead to other frequencies through the nonlinear term and the forcing. Since Fourier modes with different frequencies are linearly independent we need to consider only each Fourier mode separately.

For terms proportional to $e^{i\omega\hat{t}}$ the operator $\mathcal{L}\left(\partial_{\hat{t}}\right)$ simply becomes a matrix $\mathcal{L}(\omega)$

$$\mathcal{L}(\omega) \equiv \begin{pmatrix} i\omega & -1 \\ \omega_0^2 & i\omega \end{pmatrix}$$

For the mode $e^{i\omega_0\hat{t}}\begin{pmatrix} U_0 \\ V_0 \end{pmatrix}$ we get

$$\mathcal{L}(\omega_0)\begin{pmatrix} U_0 \\ V_0 \end{pmatrix} \equiv \begin{pmatrix} i\omega_0 & -1 \\ \omega_0^2 & i\omega_0 \end{pmatrix}\begin{pmatrix} U_0 \\ V_0 \end{pmatrix}$$

Consider now each order in ϵ:

$\mathcal{O}(\epsilon^0)$:

$$\mathcal{L}(\omega_0)\begin{pmatrix} U_0 \\ V_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We need a non-trivial solution for this equation. Since $\mathcal{L}(\omega_0)$ will appear at each order of the expansion we look at it in more detail. Consider the eigenvalues of $\mathcal{L}(\omega_0)$

$$\begin{vmatrix} -\sigma + i\omega_0 & -1 \\ \omega_0^2 & -\sigma + i\omega_0 \end{vmatrix} = 0$$

$$\sigma_{1,2} = i\omega_0 \pm i\omega_0 = \begin{cases} 0 \\ 2i\omega_0 \end{cases}$$

with eigenvectors

$$\begin{pmatrix} U_0^{(1)} \\ V_0^{(1)} \end{pmatrix} = \begin{pmatrix} 1 \\ i\omega_0 \end{pmatrix} \quad \text{for} \quad \sigma = 0 \qquad \begin{pmatrix} U_0^{(2)} \\ V_0^{(2)} \end{pmatrix} = \begin{pmatrix} 1 \\ -i\omega_0 \end{pmatrix} \quad \text{for} \quad \sigma = 2i\omega_0$$

To solve the equation at $\mathcal{O}(\epsilon^0)$ we need to take the eigenvector associated with $\sigma_1 = 0$

$$\begin{pmatrix} U_0 \\ V_0 \end{pmatrix} = \begin{pmatrix} 1 \\ i\omega_0 \end{pmatrix}$$

$\mathcal{O}(\epsilon^1)$:

$$\mathcal{L}\left(\partial_{\hat{t}}\right)\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}$$

Again, we can consider each Fourier mode separately.

Since $\mathcal{L}$ is singular we expect that secular terms will arise, which will imply a solvability condition. These terms are associated with the Fourier modes $e^{\pm i\omega_0\hat{t}}$. Thus, we first consider only those terms,

$e^{i\omega_0\hat{t}}$:

$$\mathcal{L}(\omega_0)\begin{pmatrix} U_1 \\ V_1 \end{pmatrix} = \begin{pmatrix} I_1(\omega_0) \\ I_2(\omega_0) \end{pmatrix} \tag{31}$$

where $I_i(\omega_0)$ includes only the terms $\propto e^{i\omega_0\hat{t}}$ in I_i.

Previously, the solvability condition simply amounted to setting the terms proportional to $\cos\omega_0 t$ and to $\sin\omega_0 t$ to 0 (or equivalently $e^{i\omega_0 t}$ and $e^{-i\omega_0 t}$). It *seems* at first glance as if this implied that for each Fourier mode one obtains a solvability condition from each of the equations in (31). This would imply two complex equations for a single complex amplitude: **that cannot be correct.**

To invoke the Fredholm Alternative Theorem we need the left zero-eigenvector $(U^+(t), V^+(t))$ of $\mathcal{L}$,[7]

$$\left(U_0^+, V_0^+\right)\mathcal{L}(\omega_0) = \left(U_0^+, V_0^+\right)\begin{pmatrix} i\omega_0 & -1 \\ \omega_0^2 & i\omega_0 \end{pmatrix} = 0$$

which yields

$$\left(U_0^+, V_0^+\right) = \left(1, -\frac{i}{\omega_0}\right)$$

[7] We could stay within the space of periodic functions and define a suitable scalar product for functions $\mathbf{y}_i(t) \equiv (u_i(t), v_i(t))$

$$\langle \mathbf{y}_1(t), \mathbf{y}_2(t)\rangle \equiv \int_0^{\frac{2\pi}{\omega_0}} (u_1(t)^*, v_1(t)^*)\begin{pmatrix} u_2(t) \\ v_2(t) \end{pmatrix} dt.$$

The left zero-eigenvector $(U^+(t), V^+(t))$ is then defined via the condition

$$\int_0^{\frac{2\pi}{\omega_0}} \left(U^+(t), V^+(t)\right)^* \begin{pmatrix} \partial_t & -1 \\ \omega_0^2 & \partial_t \end{pmatrix}\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = 0$$

for any $\begin{pmatrix} u(t) \\ v(t) \end{pmatrix}$.

Use integration by parts

$$\begin{aligned} 0 &= \int U^{+*}\left(\partial_{\hat{t}}u - v\right) + V^{+*}\left(\omega_0^2 u + \partial_{\hat{t}}v\right) d\hat{t} \\ &= \int -u\partial_{\hat{t}}U^{+*} - U^{+*}v + V^{+*}\omega_0^2 u - v\partial_{\hat{t}}V^{+*} d\hat{t} \\ &= \int \left[\begin{pmatrix} -\partial_{\hat{t}} & \omega_0^2 \\ -1 & -\partial_{\hat{t}} \end{pmatrix}\begin{pmatrix} U^+ \\ V^+ \end{pmatrix}^*\right]^t \begin{pmatrix} u(\hat{t}) \\ v(\hat{t}) \end{pmatrix} d\hat{t} \end{aligned}$$

Thus

$$\begin{pmatrix} U^+ \\ V^+ \end{pmatrix} = \begin{pmatrix} U_0^+ \\ V_0^+ \end{pmatrix} e^{\pm i\omega_0\hat{t}} = \begin{pmatrix} 1 \\ \pm\frac{i}{\omega_0} \end{pmatrix} e^{\pm i\omega_0\hat{t}}.$$

The solvability condition is then given by

$$\int_0^{\frac{2\pi}{\omega_0}} \left(1, -\frac{i}{\omega_0}\right) e^{-i\omega_0\hat{t}}\begin{pmatrix} I_1 \\ I_2 \end{pmatrix} d\hat{t} = 0.$$

Equivalently, we can obtain the left eigenvectors using the adjoint of $\mathcal{L}(\omega_0)$ within the sub-space of functions $\propto e^{i\omega_0 t}$

$$\mathcal{L}^{*t}(\omega_0)\left(\begin{array}{c} U_0^{+^*} \\ V_0^{+^*} \end{array}\right) = \left(\begin{array}{cc} -i\omega_0 & \omega_0^2 \\ -1 & -i\omega_0 \end{array}\right)\left(\begin{array}{c} U_0^{+^*} \\ V_0^{+^*} \end{array}\right) = 0$$

$$\left(\begin{array}{c} U_0^{+^*} \\ V_0^{+^*} \end{array}\right) = \left(\begin{array}{c} 1 \\ \frac{i}{\omega_0} \end{array}\right)$$

Note:

- There is only a single left zero-eigenvector for $e^{i\omega_0 \hat{t}}$, reflecting the fact that there is only a single 0 eigenvalue.
  $\Rightarrow$ only one single solvability condition that combines the equation for u and that for v.
- Important: one does **not** get a separate solvability condition for the u-component and one for the v-component.
- $\mathcal{L}$ is not symmetric, therefore the left and the right 0-eigenvectors differ from each other.
- There is a second left 0-eigenvectors, which is associated with the frequency $-\omega_0$. That eigenvector is the complex conjugate of the eigenvector for $+\omega_0$ because the original equation is real. No additional information is obtained from the Fredholm Alternative Theorem using that eigenvector.

The solvability condition is now given by

$$\left(1, -\frac{i}{\omega_0}\right)\left(\begin{array}{c} I_1(\omega_0) \\ I_2(\omega_0) \end{array}\right) = 0$$

where

$$\begin{aligned} I_1(\omega_0) &= -\partial_T A \\ I_2(\omega_0) &= -i\omega_0\partial_T A - \beta i\omega_0 A - 3|A|^2 A\alpha + \frac{1}{2} f e^{i\Omega T} \end{aligned}$$

The solvability therefore yields

$$-2\partial_T A - \beta A + 3i\frac{\alpha}{\omega_0}|A|^2 A - \frac{i}{2\omega_0} f e^{i\Omega T} = 0$$

i.e.

$$\partial_T A = -\frac{1}{2}\beta A + i\frac{3\alpha}{2\omega_0}|A|^2 A - \frac{i}{4\omega_0} f e^{i\Omega T}$$

Note:

Again we can remove the explicit T-dependence in the equation by writing

$$A = \mathcal{A} e^{i\Omega T}$$

yielding

$$\partial_T \mathcal{A} = \left(-\frac{1}{2}\beta - i\Omega\right)\mathcal{A} + i\frac{3\alpha}{2\omega_0}|\mathcal{A}|^2\mathcal{A} - \frac{i}{4\omega_0}f \tag{32}$$

in agreement with our previous results (27) and (28,29).

Notes:

- using the other left 0-eigenvector would result in an equivalent solvability condition leading to the complex conjugate of (32).

The complex amplitude equation (32) suggests that forcing a general oscillator one would obtain a complex amplitude equation of the form

$$\partial_T \mathcal{A} = \mu\mathcal{A} - \gamma|\mathcal{A}|^2\mathcal{A} + \nu f \tag{33}$$

where $\mu \equiv \mu_r + i\mu_i$, $\gamma \equiv \gamma_r + i\gamma_i$, $\nu \equiv \nu_r + i\nu_i$ are complex coefficients.

Notes:

- Comparing (33) with (32) shows that the Duffing oscillator does not lead to the most general amplitude equation for a forced oscillator:
  - for the Duffing oscillator one has $\gamma_r = 0$:
    no nonlinear dissipation of the oscillation amplitude, only linear damping
    $\Rightarrow$ the saturation of the oscillation amplitude occurs through a change of the natural frequency of the oscillator with increasing amplitude, which renders the forcing less effective with increasing oscillation amplitude
- The fact that $\nu_r = 0$ in (32) is of no significance:
  for arbitrary $\nu \equiv \hat{\nu}e^{i\delta}$ with $\hat{\nu} \in \mathbb{R}$ replacing $\mathcal{A} \to \hat{\mathcal{A}}e^{i\delta}$ leads to

$$\partial_T \hat{\mathcal{A}} = \mu\hat{\mathcal{A}} - \gamma|\hat{\mathcal{A}}|^2\hat{\mathcal{A}} + \hat{\nu} f \tag{34}$$

- Why does the nonlinearity of the amplitude equation (33) have this special form? Why no term like $\mathcal{A}^3$, for instance? Would another nonlinearity in the oscillator equation, e.g. $\dot{y}y^2$ or $\dot{y^2}y$, generate a term like $\mathcal{A}^3$?

2.3 Symmetries

2.3.1 Motivation

How do we know how to scale the various quantities? Review the example in the homework: quadratic oscillator

$$\ddot{u} + u + \beta\dot{u} + u^2 = 0 \tag{35}$$

The naive approach we have used for the Duffing oscillator, where we balanced the nonlinearity directly with the slow time derivative and the damping, suggests balancing

$$\beta \dot{u} \sim u^2 \sim \frac{d}{dT} u$$

suggesting

$$u = \epsilon u_1(\hat{t}, T) + \epsilon^2 u_2(\hat{t}, T) + \ldots \qquad T = \epsilon t, \qquad \beta = \epsilon \beta_1$$

Inserting this expansion yields

$\mathcal{O}(\epsilon^1)$:

$$\frac{\partial^2 u_1}{\partial \hat{t}^2} + u_1 = 0$$

$$u_1 = A(T) e^{i\hat{t}} + A^*(T) e^{-i\hat{t}}$$

$\mathcal{O}(\epsilon^2)$:

$$\frac{\partial^2 u_2}{\partial \hat{t}^2} + u_2 = -2 \frac{\partial^2}{\partial \hat{t} \partial T} u_1 - \beta_1 \frac{\partial u_1}{\partial \hat{t}} + u_1^2$$

$$\frac{\partial^2 u_2}{\partial \hat{t}^2} + u_2 = -2i \frac{dA}{dT} e^{i\hat{t}} + 2i \frac{dA^*}{dT} e^{-i\hat{t}} - \beta_1 i A e^{i\hat{t}} + \beta_1 i A e^{-i\hat{t}} + 2 |A|^2 + A^2 e^{2i\hat{t}} + A^{*2} e^{-2i\hat{t}}.$$

From the terms proportional to $e^{i\hat{t}}$ this leads to the solvability condition

$$\frac{dA}{dT} = -\frac{1}{2} \beta_1 A.$$

This equation does not include any trace of the nonlinearity and is therefore not what we are looking for.

Why is there no quadratic nonlinearity in this amplitude equation? Wouldn't the quadratic nonlinearity of the original equation suggest a quadratic nonlinearity in the amplitude equation? But this quadratic nonlinearity did not generate any terms proportional to $e^{i\hat{t}}$.

We can now try a different scaling in the hope that this will work better. Will the quadratic nonlinearity ever generate secular terms, which then need to be included in the solvability condition?

So, go to higher order in ϵ. To do so we need to solve first for u_2. Inserting the ansatz

$$u_2 = B_0 + B_2 e^{2i\hat{t}} + B_2^* e^{-2i\hat{t}}$$

into the equation at $\mathcal{O}(\epsilon^2)$ we get

$$B_0 = 2 |A|^2 \qquad B_2 = -\frac{1}{3} A^2.$$

Since we are expecting to get a suitable solvability condition at an order beyond $\mathcal{O}(\epsilon^2)$, it would be wise to change the scaling of the damping and the slow time as well, in order to push those to higher order. Try

$$\beta = \epsilon_2 \beta_2 \qquad T = \epsilon^2 t \qquad \frac{d^2}{dt^2} = \frac{\partial^2}{\partial \hat{t}^2} + 2\epsilon^2 \frac{\partial^2}{\partial \hat{t} \partial T} + \mathcal{O}(\epsilon^4).$$

Then on to cubic order

$\mathcal{O}(\epsilon^3)$:

$$\begin{aligned}\frac{\partial^2 u_3}{\partial \hat{t}^2} + u_3 &= -2\frac{\partial^2}{\partial \hat{t}\partial T}u_1 - \beta_2 \frac{\partial u_1}{\partial \hat{t}} + 2u_1 u_2 \\ &= -2i\frac{dA}{dT}e^{i\hat{t}} - \beta_2 i A e^{i\hat{t}} + c.c. + 2\left(Ae^{i\hat{t}} + A^* e^{-i\hat{t}}\right)\left(B_0 + B_2 e^{2i\hat{t}} + B_2^* e^{-2i\hat{t}}\right) \\ &= e^{i\hat{t}}\left\{-2i\frac{dA}{dT} - i\beta_2 A + AB_0 + B_2 A^*\right\} + \text{non-secular terms}\end{aligned}$$

Thus, we get now as solvability condition

$$\frac{dA}{dT} = -\frac{1}{2}\beta_2 A + \frac{1}{2i}\left(2 - \frac{1}{3}\right)|A|^2 A. \qquad (36)$$

So, it seems that the first nonlinear term that arises for this oscillator has the form $|A|^2 A$.

If we had known the form of the amplitude equation from the start, we could have chosen the scaling accordingly and would not have had to guess.

In all generality, our expansion involves essentially a Taylor expansion. It will therefore lead to an equation of the form

$$\frac{dA}{dT} = F(A, A^*)$$

with $F(A, A^*)$ being a general polynomial in its arguments. But apparently the solvability condition does not lead to the most general polynomial: certain terms do not appear. Why not?

Very often, terms in an equation do not appear because of special *symmetries*. For instance, the Taylor expansion of $\sin x$ does not include any even terms because $\sin x$ is even in x. The situation here is analogous: all oscillators have a specific symmetry that eliminates certain terms in the amplitude expansion.

2.3.2 Symmetries, Selection Rule, and Scaling

In the absence of forcing (35) is invariant under arbitrary time translations

$$t \to t + \Delta t$$

i.e. if $y(t)$ is a solution to (35) so is $y(t + \Delta t)$.
This invariance must be reflected in the resulting amplitude equation (36). However, A does not depend on the fast time t. How is it then affected by translations in the fast time t?

Consider a solution $y(t)$ and the time-shifted solution $y(t + \Delta t)$ and their expansions in terms of the complex amplitude A,

$$\begin{aligned} y(t) &= A(T)\, e^{i\omega t} + c.c. + h.o.t. \\ y(t+\Delta t) &= A(T) e^{i\omega(t+\Delta t)} + c.c. + h.o.t. \\ &= A(T) e^{i\omega \Delta t}\, e^{i\omega t} + c.c. + h.o.t.\end{aligned}$$

The expansion implies:

- If $y(t)$ is a solution of the original equation then $A(T)$ is a solution of the amplitude equation and vice versa.
- The time-shifted function $y(t+\Delta t)$ is represented in terms of the complex amplitude by $A(T)e^{i\omega\Delta t}$, i.e. shifting the fast time by Δt is equivalent to rotating the phase of A by $e^{i\omega\Delta t}$.
- If $y(t+\Delta t)$ is a solution of the original equation then $A(T)e^{i\omega\Delta t}$ is a solution of the amplitude equation and vice versa[8].

Using the fact that with $y(t)$ also $y(t+\Delta t)$ is a solution, one obtains the following commutative diagram

$y(t)$ solves the original equation	$\Leftrightarrow$	$A(T)$ solves the amplitude equation
$\Updownarrow$		$\Updownarrow$
$y(t+\Delta t)$ solves the original equation	$\Leftrightarrow$	$A(T)e^{i\omega\Delta t}$ solves the amplitude equation

Note:

- The implication that $y(t)$ solves the original equation if $A(T)$ solves the amplitude equation holds only in the limit of small amplitudes.
- This symmetry argument makes use of the multi-timing assumption that the slow time T and the fast time t are independent variables. That is an approximation: the two time variables are not independent and for solutions that are not strictly periodic, e.g. with a non-periodic time dependence of $A(T)$, the symmetry argument does not hold. In fact, in a center manifold reduction, which does not use multi-timing, additional terms arise in the amplitude equations. They can, however, be removed by *near-identity transformations* of the amplitude, which bring the amplitude equation into what is called its 'normal form'. The equations obtained with our symmetry arguments generate that normal form (in a non-rigorous way).

One says:

- Translations Δt in time induce an *action* on the amplitude:

  $$t \to t+\Delta t \quad \Rightarrow \quad A(T) \to A(T)e^{i\omega\Delta t}$$

  In this case the action corresponds to a phase shift by an arbitrary amount $\Delta\phi = \omega\Delta t$.
- The amplitude equation must be *equivariant* under that action:
  all terms of the amplitude equation must transform the same way under that operation

[8]Using the ansatz for $y(t+\Delta t)$ in the expansion generates exactly the same expressions everywhere as obtained for $y(t)$ with $A(T)$ everywhere replaced by $A(T)e^{i\omega\Delta t}$.

Selection rule

Since the amplitude equation arises in an expansion in terms of the complex amplitude it has the general form

$$\frac{d}{dT}A = \mathcal{F}(A, A^*) = \sum_{m,n} a_{mn} A^m A^{*n} \tag{37}$$

If $A(T)$ is a solution to (37) so must be $A(T)e^{i\Delta\phi}$ for arbitrary $\Delta\phi$.

Thus

$$\frac{d}{dT}Ae^{i\Delta\phi} = \mathcal{F}(Ae^{i\Delta\phi}, A^* e^{-i\Delta\phi}) = \sum_{m,n} a_{mn} A^m A^{*n} e^{i(m-n)\Delta\phi}$$

Inserting dA/dT from (37) we get

$$\sum_{m,n} a_{mn} A^m A^{*n}\, e^{i\Delta\phi} = \sum_{m,n} a_{mn} A^m A^{*n} e^{i(m-n)\Delta\phi}$$

Equating like powers of A and A^*

$$a_{mn} = a_{mn} e^{i\Delta\phi(m-n-1)} \quad \text{for all} \quad \Delta\phi$$

Thus, we get the selection rule

$$\boxed{\text{either} \quad m = n+1 \quad \text{or} \quad a_{mn} = 0}$$

Alternatively, one can express this result also as:

The action induced by the time-translation symmetry transforms the terms in the expansion as

$$A^n A^{*m} \to A^n A^{*m} e^{i\varphi(n-m)}$$

- Equivariance of the amplitude equation under this action requires that for all terms in the amplitude equation the difference $n - m$ must be the same:
  thus
  $$n - m = k \quad \text{for some} \quad k \in \mathbb{N}.$$
- Since the amplitude equation has a term $\frac{d}{dT}A$ in it one has $k = 1$.
  Thus, the only terms allowed are of the form
  $$|A|^{2l}A \qquad 0 \leq l \in \mathbb{N}$$

Scaling

In the weakly nonlinear regime y is small. Since we do not know the proper scaling yet we do not introduce an explicit ϵ but rather assume that the amplitude $A(T)$ is small. To leading order in the amplitudes one therefore gets

$$\partial_T A = \mu A - \gamma |A|^2 A$$

implying the scaling

$$\frac{d}{dT} \sim \mu \sim |A|^2$$

Notes:

- the symmetry condition allows us to write down the form of the resulting amplitude equation without performing the nonlinear expansion in detail.
- of course, to obtain the values of the coefficients one still has to do the algebra.
- but the algebra is simplified since the scaling of the various parameters can be determined before hand (no trial and error needed).

2.3.3 1:1 Forcing

We considered forcing near the 1:1-resonance: $\omega = \omega_0 + \epsilon\Omega$.

The small detuning Ω can be captured through the dependence of the complex amplitude A on the slow time T. Therefore consider a system exactly at the 1:1-resonance.

With 1:1-resonance the system is not invariant for arbitrary time shifts any more, but still for shifts

$$t \to t + \frac{2\pi}{\omega}$$

With the expansion

$$y(t,T) = A(T)e^{i\omega t} + A(T)^* e^{-i\omega t} + \mathcal{O}(\epsilon)$$

time translations induce the action

$$t \to t + \frac{2\pi}{\omega} \quad \Rightarrow \quad A \to Ae^{i\omega\frac{2\pi}{\omega}} = A$$

i.e. the amplitude is *unchanged* by such translations. One says, in this case the action of the symmetry is *trivial*.

Thus

- with 1:1-forcing any polynomial in A and A^* is allowed by symmetries

$$\partial_T A = a_{00} + a_{10}A + a_{01}A^* + a_{20}A^2 + \ldots$$

Why did we not obtain terms like A^* or A^2 in our direct derivation of the amplitude equation?

Scaling

We also assumed *weak* forcing. The amplitude of the forcing has not entered our symmetry consideration at all so far. To include this information it is useful to consider an *extended dynamical system* in which the forcing is considered a dynamical variable rather than an external force

$$\ddot{\hat{y}} + \hat{\beta}\dot{\hat{y}} + \omega_0^2\hat{y} + \alpha\hat{y}^3 - \hat{f} = 0 \quad (38)$$

$$\ddot{\hat{f}} + \omega^2\hat{f} = 0 \quad (39)$$

where we assume $\hat{\beta}$, $\hat{y}$, and $\hat{f}$ are small. At this point it is not clear how these quantities scale with each other.

Expand now

$$\begin{aligned}\hat{y} &= A(T)e^{i\omega_0 t} + A(T)^* e^{-i\omega_0 t} + \ldots \\ \hat{f} &= F(T)e^{i\omega t} + F(T)^* e^{-i\omega t}\end{aligned}$$

Note:

- the expansion for $\hat{f}$ does not have any higher order terms since its evolution (39) is linear and is not coupled to y.

Time translations act on the amplitude $A(T)$ and $F(T)$ as

$$t \to t + \Delta t \quad \Rightarrow \quad A \to Ae^{i\omega_0 \Delta t} \quad F \to Fe^{i\omega \Delta t}$$

Note:

- since the forcing is now part of the dynamical system this extended dynamical system is invariant under *any time translations*.

The expansion will lead to solvability conditions of the type

$$\partial_T A = \sum_{klmn} a_{klmn} A^k A^{*l} F^m F^{*n} \qquad \partial_T F = \sum_{klmn} f_{klmn} A^k A^{*l} F^m F^{*n}$$

For the 1:1-resonance, $\omega = \omega_0$, A and F transform the same way under time translations. The selection rule is

$$k - l + m - n = 1$$

What are the lowest-order terms?

- a nonlinear saturating term needs to be retained. We want to retain a term that is saturating also without forcing. Generically, the leading-order saturating[9] term without forcing is $|A|^2 A$
- the leading-order forcing term is F
- to balance these two essential terms we have

  $$F \sim A^3$$

  $\Rightarrow$ to leading order the only term containing the forcing is F

To leading order we then get

$$\partial_T A = a_{1000} A + a_{0010} F + a_{2100} |A|^2 A$$

in agreement with (34) with $a_{1000} = \mu$, $a_{0010} = \nu$, $a_{2100} = \gamma$.

Consistent scaling requires $\mu = \mathcal{O}(A^2)$, whereas $\nu, \gamma = \mathcal{O}(1)$.

Summary:

[9] Whether this term is actually saturating will depend on the sign of its coefficient.

- oscillation amplitude A and forcing amplitude F are each associated with their respective frequencies ω_0 and ω
- in terms of Fourier modes all terms in the resulting amplitude equation have to correspond to the same frequency, which for the A-equation is ω_0.
- no attention has to be paid to the equation for the forcing amplitude since the equation (39) is not coupled to the oscillation amplitude

2.3.4 3:1 Forcing

Consider now $\omega = 3\omega_0$ and use the expansion

$$\begin{aligned} y &= A(T)e^{i\omega_0 t} + A(T)^* e^{-i\omega_0 t} + \ldots \\ \hat{f} &= F(T)e^{3i\omega_0 t} + F(T)^* e^{-3i\omega t} \end{aligned}$$

It induces the action

$$t \to t + \Delta t \quad \Rightarrow \quad A \to Ae^{i\varphi} \quad F \to Fe^{3i\varphi} \quad \text{with } \varphi = \omega_0 \Delta t$$

$$A^k A^{*l} F^m F^{*n} \to A^k A^{*l} F^m F^{*n} e^{i\varphi(k-l+3(m-n))}$$

Selection Rule

for the equation for A

$$k - l + 3\,(m-n) = 1 \quad \Rightarrow \quad k - l = 1 - 3\,(m-n)$$

Identify the lowest-order terms in the forcing

F:

$$m - n = 1 \quad \Rightarrow \quad k - l = -2 \quad k = 0 \quad l = 2 \quad \Rightarrow \quad FA^{*2}$$

F^*:

$$m - n = -1 \quad \Rightarrow \quad k - l = 4 \quad k = 4 \quad l = 0 \quad \Rightarrow \quad F^* A^4$$

For any small A one has $FA^{*2} \gg F^* A^4$. Therefore use FA^{*2} to balance saturation and forcing

$$A^3 \sim FA^2 \quad \Rightarrow \quad F \sim A$$

Keeping only terms up to cubic order we get therefore the restriction

$$k + l + m + n \leq 3$$

We have already considered the case $m + n = 1$.

Consider now $m + n = 2$:

$$m = 2 \quad n = 0 \quad \Rightarrow \quad k - l = -5 \quad \Rightarrow \quad F^2 A^{*5}$$

$$m = 1 \quad n = 1 \quad \Rightarrow \quad k - l = 1 \quad \Rightarrow \quad |F|^2 A$$

$$m = 0 \quad n = 2 \qquad \Rightarrow \qquad k - l = 7 \quad \Rightarrow \quad F^{*2}A^7$$

For $m + n = 3$ we get the condition

$$k = 0 = l \qquad \Rightarrow \qquad 3(m - n) = 1 \qquad \text{cannot be satisfied}$$

To leading order symmetry and scaling show that the equation has to have the form

$$\partial_T A = \left(\mu + \beta |F|^2\right) A - \gamma |A|^2 A + \delta F A^{*2} \tag{40}$$

Notes:

- Through the term $\beta|F|^2$ the forcing modifies the linear coefficient of the equation
  - depending on the sign of β_r the forcing can enhance or reduce the damping
  - through $\beta_i|F|^2$ the frequency of small-amplitude oscillations are modified by the forcing
- The forcing will lead to *qualitatively new* phenomena only through the term involving A^{*2} because only it breaks the symmetry $A \to Ae^{i\varphi}$ for $\varphi \neq \frac{2\pi}{3}$.
- For consistent scaling we need again $\mu = \mathcal{O}(A^2)$
- (40) captures the weakly nonlinear behavior of all generic, weakly forced oscillators near the 1:3 resonance. Different oscillators only differ in the values of the coefficients.

2.3.5 Non-resonant Forcing

Consider $\omega = \alpha\omega_0$ with α irrational.

Selection Rule

$$k - l + \alpha\,(m - n) = 1 \quad \Rightarrow \quad m = n \quad k = l + 1$$

lowest-order term

$$|F|^2 A$$

$\Rightarrow$ the forcing appears only through $|F|^2$, i.e. the phase of the forcing does not play a role and there is no resonance between the oscillator and the forcing

Balance saturation and forcing

$$A^3 \sim F^2 A \quad \Rightarrow \quad F \sim A$$

resulting in the amplitude equation

$$A' = \left(\mu + |F|^2\right) A - \gamma |A|^2 A$$

Notes:

- Non-resonant forcing does not introduce new terms in the equation of the unforced oscillator (at any order), it only modifies its coefficients. None of the terms are phase-sensitive.
  - in principle, all coefficients depend on $|F|^2$
  - the strongest effect of the forcing is on the bifurcation parameter μ because it is small
  - only for the linear term is the shift of the coefficient of the same order as the coefficient itself and therefore relevant at leading order
- In the resonant cases - as those discussed before - all the coefficients depend also on $|F|^2$, but again most effects are of higher order. More relevant are, however, the new terms that are phase-sensitive.
- Resonant forcing with higher resonances ($m:1$ with $m \geq 4$) does not lead to additional terms in the lowest order amplitude equation (see homework)
  - but it introduces new *higher-order terms* that are *phase-sensitive*
  - to capture aspects of its impact on the system in a leading-order amplitude equation one may have to consider singular limits like $|\gamma| \ll 1$ to make the higher-order phase-sensitive terms of the same order as the formally lower-order nonlinear terms, i.e. consider higher singular points.

2.4 A Quadratic Oscillator with 3:1 Forcing

Consider the forced oscillator

$$\partial_{\tilde{t}}^2 + \hat{\beta}\partial_{\tilde{t}} y + \omega_0^2 y + \alpha y^2 = \hat{f}(\tilde{t})$$

with $\hat{f} \sim \cos\omega\tilde{t}$ where ω is close to $3\omega_0$.

We want to reduce this equation to an amplitude equation using multiple time scales.

For weak forcing and weak damping we expect for the oscillation amplitude A on symmetry grounds the equation

$$\partial_T A = \left(\mu + \beta|F|^2\right) A - \gamma|A|^2 A + \delta F A^{*2}$$

All 5 terms will arise at the same order if the following scaling is satisfied

$$\partial_T = \mathcal{O}(\mu) = \mathcal{O}(A^2) \qquad F = \mathcal{O}(A)$$

Notes:

- the oscillator equation has only a quadratic nonlinearity. How will the cubic nonlinearities be generated that the symmetry arguments predict?
- the forcing is non-parametric. Why is there no inhomogeneous term in the amplitude equation?

Introduce a small parameter ϵ explicitly via

$$y = \epsilon y_1 + \epsilon^2 y_2 + \qquad \ldots$$

and the rescaled variables

$$T = \epsilon^2 t, \qquad \hat{\beta} = \epsilon^2 \beta, \qquad \hat{f} = \epsilon \mathcal{F}, \qquad \omega = 3\left(\omega_0 + \epsilon^2 \Omega\right)$$

The amplitude equation is then expected to arise at $\mathcal{O}(\epsilon^3)$.

We get then

$$\partial_{\tilde{t}} = \partial_t + \epsilon^2 \partial_T \qquad \partial_{\tilde{t}}^2 = \partial_t^2 + \epsilon^2 2 \partial_t \partial_T + \mathcal{O}(\epsilon^4)$$

$\mathcal{O}(\epsilon)$:

$$\partial_t^2 y_1 + \omega_0^2 y_1 = \mathcal{F} \cos\left(3\left(\omega_0 + \epsilon^2 \Omega\right) t\right) = \frac{1}{2} \mathcal{F} \left\{ e^{3i\omega_0 t + 3i\Omega T} + e^{-3i\omega_0 t - 3i\Omega T} \right\}$$

thus the structure of the leading-order equation is

$$\mathcal{L}(\partial_t)\, y_1 = I_1,$$

i.e. the equation is inhomogeneous.

In terms of Fourier modes: $\mathcal{L}(\omega_0)$ is singular, but $\mathcal{L}(3\omega_0)$ is not singular.

The general solution at this order is therefore given by

$$y_1 = A e^{i\omega_0 t} + B e^{3i\omega_0 t} + A^* e^{-i\omega_0 t} + B^* e^{-3i\omega_0 t}$$

with A undetermined at this order and

$$-8\omega_0^2 B = \frac{1}{2} \mathcal{F} e^{3i\Omega T} \qquad B = -\frac{1}{16\omega_0^2} \mathcal{F} e^{3i\Omega T}$$

$\mathcal{O}(\epsilon^2)$:

$$\partial_t^2 y_2 + \omega_0^2 y_2 = -\alpha y_1^2 = -\alpha \left\{ A e^{i\omega_0 t} + B e^{3i\omega_0 t} + A^* e^{-i\omega_0 t} + B^* e^{-3i\omega_0 t} \right\}^2$$

The r.h.s. has no terms proportional $e^{i\omega_0 t}$ or $e^{-i\omega_0 t}$ $\Rightarrow$ no secular terms arise and we can solve for y_2 without any solvability arising.

$$y_2 = C + D e^{2i\omega_0 t} + E e^{4i\omega_0 t} + F e^{6i\omega_0 t} + D^* e^{-2i\omega_0 t} + E^* e^{-4i\omega_0 t} + F^* e^{-6i\omega_0 t}$$

with

$$F = \frac{1}{35} \frac{\alpha B^2}{\omega_0^2} \qquad E = \frac{2}{15} \frac{\alpha A B}{\omega_0^2}$$

$$D = \frac{1}{3} \frac{\alpha}{\omega_0^2} \left\{ A^2 + 2A^* B \right\} \qquad C = -\frac{2\alpha}{\omega_0^2} \left\{ |A|^2 + |B|^2 \right\}$$

$\mathcal{O}(\epsilon^3)$:

$$\partial_t^2 y_3 + \omega_0^2 y_3 = -2\alpha y_1 y_2 - \beta \partial_t y_1 - 2\partial_t \partial_T y_1$$

From the term $y_1 y_2$ secular terms arise

$$y_1 y_2 \sim \ldots AC + \ldots BD^* + \ldots A^* D + \ldots B^* E$$

Collecting these terms with Maple one gets

$$\partial_t^2 y_3 + \omega_0^2 y_3 = e^{i\omega_0 t}\left\{-\partial_T A - \frac{1}{2}\beta A - i\frac{6}{5}\frac{\alpha^2}{\omega_0^3}|B|^2 A - i\frac{5}{3}\frac{\alpha^2}{\omega_0^3}|A|^2 A + i\frac{\alpha^2}{\omega_0^3} B A^{*2}\right\} + \text{non-secular terms}$$

Inserting B we get an equation of the form

$$\partial_T A = \left(\mu + \mu_2|\mathcal{F}|^2\right) A - \gamma |A|^2 A + \delta \mathcal{F} A^{*2} e^{3i\Omega T}$$

Eliminate again the time-dependence of the coefficent of the forcing via

$$A = \mathcal{A} e^{i\Omega T}$$

we get

$$\partial_T \mathcal{A} = \left(\mu + \mu_2|\mathcal{F}|^2\right)\mathcal{A} - \gamma|\mathcal{A}|^2\mathcal{A} + \delta\mathcal{F}\mathcal{A}^{*2} \qquad (41)$$

with the coefficients

$$\mu = -\frac{1}{2}\beta - i\Omega \qquad \mu_2 = -\frac{3}{640} i \frac{\alpha^2}{\omega_0^7} \qquad \gamma = -\frac{5}{3} i \frac{\alpha^2}{\omega_0^3} \qquad \delta = -\frac{1}{16} i \frac{\alpha^2}{\omega_0^5}$$

Notes:

- The nonlinearities of the amplitude equation are *not* directly determined by the nonlinearities of the underlying differential equation from which it is derived. Higher-order nonlinearities can always be generated by cycling through the lower-order nonlinearities.
- The determining factor for the form of the amplitude equation is the *action on the amplitude that is induced by the symmetries* of the underlying equation
- The form of the nonlinearities of the underlying equation may determine aspects of its symmetries. For instance, if the underlying equation is odd in y it induces the additional action $A \to -A$ under which the amplitude equation must be equivariant as well. For (41) this would imply that $\delta = 0$.
- Symmetries can make coefficients zero but not non-zero.

To get steady-state solutions it is better to write (41) in terms of magnitude and phase

$$\mathcal{A} = R(T) e^{i\phi(T)} e^{i\theta}$$

$$\partial_T \mathcal{R} + iR\partial_T\phi = \left(\mu + \mu_2|F|^2\right) R - \gamma|R|^2 R + \delta\, F R^2 e^{-3i\phi} e^{-3i\theta}$$

One can always choose θ to cancel the argument of δ, i.e. effectively one can always choose the phase θ such that the coefficient of the forcing is positive.

Introduce $m = \mu + \mu_2\,|F|^2 \equiv m_r + i m_i$ etc.

$$\partial_T R = m_r R - \gamma_r R^3 + |\delta| F R^2 \cos 3\phi \qquad (42)$$

$$R\partial_T \phi = m_i R - \gamma_i R^3 + |\delta| F R^2 \sin 3\phi \qquad (43)$$

Analyze the fixed points (critical points) of (42,43).

There is always a fixed point

$$R_\infty^{(1)} = 0$$

i.e.

$$y = -\frac{f}{8\omega_0^2}\cos\left(3\left(\omega_0 + \epsilon^2\Omega\right)t\right) + h.o.t.$$

Its linear stability is determined by

$$\partial_T \mathcal{A} = \left(\mu + \mu_2 |F|^2\right)\mathcal{A}$$

it is linearly stable for

$$\mu_r + \mu_{2r}\,|F|^2 < 0$$

with $\mu_{2r} = 0$ and $\mu_r = -\frac{1}{2}\beta$ this is always the case for this oscillator.

For $R_\infty \neq 0$ one gets

$$|\delta|^2 F^2 R^2 = \left(m_r - \gamma_r R^2\right)^2 + \left(m_i - \gamma_i R^2\right)^2$$

i.e.

$$|\gamma|^2 R^4 - \left\{2\left(m_r\gamma_r + m_i\gamma_i\right) + |\delta|^2 F^2\right\} R^2 + |m|^2 = 0 \tag{44}$$

$$R^2 = \frac{\{\} \pm \sqrt{\Delta}}{2\,|\gamma|^2}$$

We need positive solutions for R^2.

Consider the discriminant Δ,

$$\Delta = \left\{2\left(m_r\gamma_r + m_i\gamma_i\right) + |\delta|^2 F^2\right\}^2 - 4|\gamma|^2|m|^2$$

Because $|\gamma|^2|m|^2 > 0$ both solutions have either the same sign or they are complex.

Note:

- In particular, $R = 0$ cannot be a solution of this equation. This is consistent with the fact that the solution $R_\infty^{(1)} = 0$ is linearly stable for all F and therefore does not undergo a (local) bifurcation.

We need $R^2 > 0$. Consider $F^2 \to \infty$:

$$\{\} \to |\delta|^2 F^2 > 0$$

Since the sign of the solution of the biquadratic equation (44) does not change, $R^2 > 0$ for all F for which R^2 is real (i.e. R^2 becomes complex before $\{\}$ becomes negative).

To get any steady-state solutions we need the discriminant to be non-negative

$$\begin{aligned}\Delta &= \left\{2\left(m_r\gamma_r + m_i\gamma_i\right) + |\delta|^2 F^2\right\}^2 - 4|\gamma|^2|m|^2 \\ &= |\delta|^4 F^4 + 4|\delta|^2 F^2\left(m_r\gamma_r + m_i\gamma_i\right) - 4\left(m_r\gamma_i - m_i\gamma_r\right)^2 \overbrace{\geq}^{!} 0\end{aligned}$$

In our case: $\gamma_r = 0$ and $\mu_{2r} = 0$

$$\begin{aligned}\Delta &= |\delta|^4 F^4 + 4|\delta|^2 F^2 m_i \gamma_i - 4 m_r^2 \gamma_i^2 \\ &= \left(|\delta|^4 + 4|\delta|^2 \mu_{2i}\gamma_i\right) F^4 + 4|\delta|^2 F^2 \mu_i \gamma_i - 4\mu_r^2\gamma_i^2\end{aligned}$$

In addition, $\gamma_i < 0$, $\mu_{2i} < 0$. Therefore

$$\begin{aligned}\Delta &< 0 \qquad \text{for } F^2 \to 0 \\ \Delta &> 0 \qquad \text{for } F^2 \to \infty\end{aligned}$$

Thus,

$$\begin{aligned}F &< F_c \qquad \text{no solution} \\ F &> F_c \qquad \text{2 solutions}\end{aligned}$$

with F_c defined via

$$\Delta(F_c) = 0.$$

At $F = F_c > 0$ one has a saddle-node bifurcation. Solving $\Delta = 0$ for $\mu_i = -\Omega$ the line of bifurcations is given by

$$\Omega_{SN} = -\mu_i = \frac{|\delta|^2 + 4\mu_{2i}\gamma_i}{4\gamma_i} F^2 - \frac{\mu_r^2\gamma_i}{|\delta|^2}\frac{1}{F^2}$$

with two solutions appearing for $\Omega > \Omega_{SN}$.

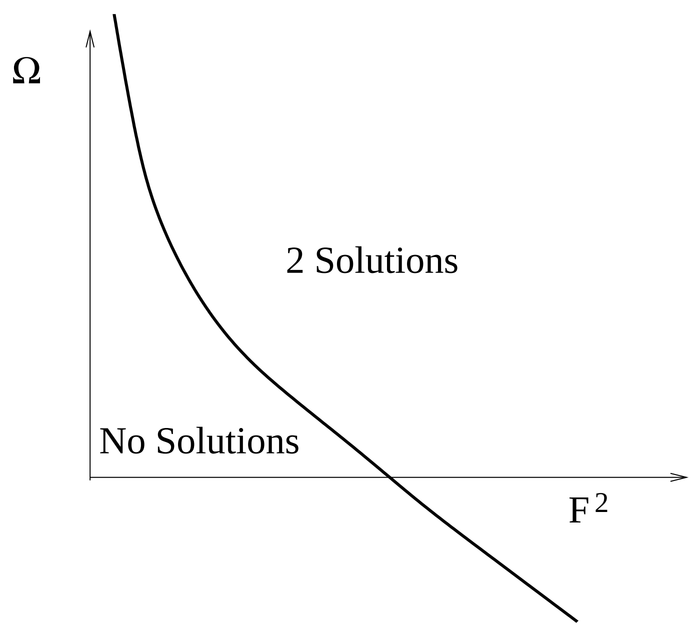


Figure 23: Phase diagram for quadratic oscillator with $\gamma_r = 0 = \mu_{2r}$ and $\gamma_i < 0$ and $\mu_{2i} < 0$. The line denotes a line of saddle-node bifurcations. Note: for all parameter values there is the additional solution $R_\infty^{(1)} = 0$.

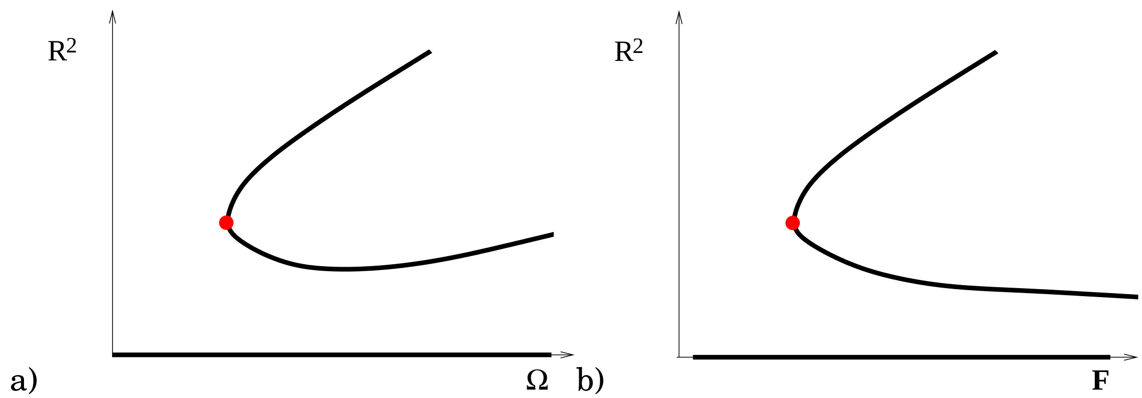


Figure 24: Bifurcation diagram for quadratic oscillator with $\gamma_r = 0 = \mu_{2r}$ obtained by cutting the phase diagram along a line of constant F. The circle denotes the saddle-node bifurcations. Note: for all parameter values there is the additional solution $R_\infty^{(1)} = 0$.

Notes:

- for the 3:1-resonance the phase-locked forced solution does not arise in a bifurcation off the basic solution $y = -\frac{f}{8\omega_0^2}\cos\left(3\left(\omega_0 + \epsilon^2\Omega\right)t\right) + h.o.t.$; instead it arises at finite amplitude through a saddle-node bifurcation.

3 Nonlinear Schrödinger Equation

Consider oscillations in a nonlinear conservative system, i.e. a system without dissipation.

Classic example: pendulum of length L without damping:

$$\partial_t^2\psi = -\omega_0^2 \sin\psi \qquad \text{with} \qquad \omega_0^2 = \frac{g}{L}$$

More generally, the right-hand side could be any function $f(\psi)$ (with $f(0) = 0$)

Consider nonlinear oscillations in a continuum (many coupled pendula)

$$\partial_t^2\psi - c^2\partial_x^2\psi + \omega_0^2 \sin\psi = 0 \tag{45}$$

Note:

- this nonlinear equation is called the sine-Gordon equation in analogy to the linear Klein-Gordon equation
$$\partial_t^2\psi - c^2\partial_x^2\psi + \omega_0^2\psi = 0$$
which is the linearization of the sine-Gordon equation

The Klein-Gordon equation allows simple traveling waves

$$\psi = Ae^{iqx-i\omega t} + A^* e^{-iqx+i\omega t} \qquad \text{with} \qquad \omega^2 = \omega_0^2 + c^2q^2$$

Weakly nonlinear regime: seek traveling waves with slightly different frequency and slightly different wave form.

Aim: weakly nonlinear theory for such waves that allows also spatially slow modulations of the fast (carrier) waves, e.g. spatially varying wavenumbers or *wave packets*

Note:

- expect similarities between these traveling waves and the oscillations and waves arising from a Hopf bifurcation (e.g. in the Belousov-Zhabotinsky reaction in an unstirred reactor)

To derive a weakly nonlinear description we use symmetry arguments:

Ansatz for right-traveling wave

$$\psi = \epsilon A(X,T,\ldots)e^{iqx-i\omega t} + \epsilon A^*(X,,T,\ldots)e^{-iqx+i\omega t} + \epsilon^2\psi_2 + \epsilon^3\psi_3 + \ldots \tag{46}$$

with $T = \epsilon^2 t$, $X = \epsilon x$.

For a wave packet one would have $A \to 0$ for $X \to \pm\infty$.

The sine-Gordon equation (45) is equivariant under

$$\begin{aligned} \text{reflections in space:} &\quad x \to -x \\ \text{reflections in time:} &\quad t \to -t \\ \text{translations in space:} &\quad x \to x + \Delta x \\ \text{translations in time:} &\quad t \to t + \Delta t \end{aligned}$$

As in the forced oscillator case, the time translations imply that the evolution equation for A is equivariant under $A \to Ae^{i\phi}$ for arbitrary ϕ . The spatial translations imply the same equivariance $\Rightarrow$ expect an evolution equation of the form

$$\partial_T A = aA + c|A|^2 A + v\partial_X A + d\partial_X^2 A + \ldots \tag{47}$$

Action of the reflections:

- under spatial reflections (and under reflections in time) a right-traveling wave is transformed into a left-traveling wave
- (46) includes only a right-traveling wave: pure spatial reflections cannot be represented within the class of functions (46)
- *combined* reflections in time and space, however, map a right-traveling wave again into a right-traveling wave $\Rightarrow$ they have a simple action on the amplitude A in the ansatz (46)

$$x \to -x \quad \text{combined with} \quad t \to -t \quad \text{induces} \quad T_2 \to -T_2, \quad X \to -X, \quad A \to A^*$$

Applied to the general evolution equation (47) the transformation yields

$$-\partial_T A^* = aA^* + c|A|^2A^* - v\partial_X A^* + d\partial_X^2 A^* + \dots$$

taking the complex conjugate implies

$$-\partial_T A = a^*A + c^*|A|^2A - v^*\partial_X A + d^*\partial_X^2 A + \dots$$

and

$$a = -a^* \quad d = -d^* \quad c = -c^* \quad v = v^*$$

Thus:

$$\partial_T A = v\partial_X A + i\,\frac{1}{2}\frac{d^2\omega}{dq^2}\,\partial_X^2 A + i\,\frac{1}{4}\frac{\omega_0^2}{\omega}|A|^2A$$

Going into a moving frame $X \to X + vT$ one gets

$$\partial_T A = i\,\frac{1}{2}\frac{d^2\omega}{dq^2}\,\partial_X^2 A + i\,\frac{1}{4}\frac{\omega_0^2}{\omega}|A|^2A$$

Notes:

- this equation is the *nonlinear Schrödinger equation* (NLS)
- the NLS is the generic description for small-amplitude waves in non-dissipative media
- in a multiple-scale analysis one has to introduce actually two slow times $T_1 = \epsilon t$ and $T_2 = \epsilon^2 t$ and gets two non-trivial solvability conditions:
  at $\mathcal{O}(\epsilon^2)$ one gets
  $$\partial_{T_1} A = v\partial_X A$$
  and $\mathcal{O}(\epsilon^3)$ one gets the NLS.
- systems undergoing a Hopf bifurcation to waves are dissipative: expect complex and not purely imaginary coefficients. One obtains then the complex Ginzburg-Landau equation (CGL).

3.1 Some Properties of the NLS

Consider the NLS in the form

$$\partial_t \psi = \frac{i}{2}\partial_x^2\psi + is|\psi|^2\psi \qquad \text{with} \quad s = \pm 1$$

Note:

- the magnitude of the coefficients can be absorbed into the amplitude and the spatial scale
- the overall sign of the r.h.s. can be absorbed by running time backward $t \to -t$

- the relative sign s between $\partial_x^2\psi$ and $|\psi|^2\psi$ cannot be changed by scaling or coordinate transformations
  - $s = +1$: focusing case (spatially homogeneous oscillations linearly unstable, cf. Benjamin-Feir instability of CGL).
  - $s = -1$: defocusing case.

The NLS does not have a Lyapunov functional[10], but is a Hamiltonian system with Hamiltonian (energy) functional

$$\mathcal{H}\{\psi\} = \frac{1}{4}\int_{-\infty}^{+\infty} |\partial_x\psi|^2 - s|\psi|^4 dx$$

i.e.

$$\partial_t\psi_r = \frac{\delta\mathcal{H}\{\psi\}}{\delta\psi_i} \tag{48}$$

$$\partial_t\psi_i = -\frac{\delta\mathcal{H}\{\psi\}}{\delta\psi_r} \tag{49}$$

This is seen by employing the basic property of functional derivatives,

$$\frac{\delta\psi(x)}{\delta\psi(x')} = \delta(x - x'),$$

which is analogous to the derivative of a vector function v with respect to one of its components

$$\frac{\partial v_i}{\partial v_j} = \delta_{ij}$$

with $\delta(x - x')$ the Dirac $\delta-$function and δ_{ij} the Kronecker δ,

[10] A differential equation $du/dt = f(u)$ arises from a Lyapunov function if it can be written as

$$\frac{du}{dt} = -\frac{\partial V(u)}{\partial u} \quad \text{with} \quad V \geq V_0$$

Then it cannot have any persistent dynamics

$$\frac{dV}{dt} = \frac{\partial U}{\partial u}\frac{du}{dt} = -\left(\frac{du}{dt}\right)^2 \leq 0$$

because the Lyapunov function, which is bounded from below, is monotonically decreasing as long as u evolves in time.

and using integration by parts

$$\begin{aligned}
\frac{\delta\mathcal{H}\left\{\psi\right\}}{\delta\psi_r(x)} &= \frac{1}{4}\int\frac{\delta}{\delta\psi_r(x)}\left[\left(\partial_x\psi_r(x')\right)^2+\left(\partial_x\psi_i(x')\right)^2-s\left(\psi_r^2(x')+\psi_i^2(x')\right)^2\right]dx' \\
&= \frac{1}{4}\int 2\partial_x\psi_r(x')\partial_x\left(\delta(x-x')\right)-2s\left(\psi_r^2(x')+\psi_i^2(x')\right)2\psi_r(x')\delta(x-x')dx' \\
&\underbrace{=}_{i.b.p.} -\frac{1}{2}\int\partial_x^2\psi_r(x')\delta(x-x')dx'-s\left|\psi(x)\right|^2\psi_r(x) \\
&= -\frac{1}{2}\partial_x^2\psi_r(x)-s\left|\psi(x)\right|^2\psi_r(x) \\
&= -\partial_t\psi_i(x)
\end{aligned}$$

and analogously for $\partial_t\psi_r$.

Note:

- the structure (48,49) is a continuum version of the Hamiltonian structure of Newtonian mechanics
$$\frac{d}{dt}x=\frac{\partial H(x,p)}{\partial p}\qquad\frac{d}{dt}p=-\frac{\partial H(x,p)}{\partial x}$$
where $H(x,p)$ is the total energy of the system
$$H(x,p)=E_{kin}(x,p)+E_{pot}(x,p)$$
- a characteristic, essential feature of (48,49) is its *symplectic* structure, i.e. the opposite sign in (48) and (49)

Conserved Quantities:

- in any Hamiltonian system the total energy $\mathcal{H}$ is conserved:
$$\begin{aligned}
\frac{d}{dt}\mathcal{H} &= \int\frac{\delta\mathcal{H}\{\psi_r,\psi_i\}}{\delta\psi_r(x')}\partial_t\psi_r(x')+\frac{\delta\mathcal{H}\{\psi_r,\psi_i\}}{\delta\psi_i(x')}\partial_t\psi_i(x')\,dx' \\
&= \int\frac{\delta\mathcal{H}\{\psi_r,\psi_i\}}{\delta\psi_r(x')}\frac{\delta\mathcal{H}\{\psi_r,\psi_i\}}{\delta\psi_i(x')}+\frac{\delta\mathcal{H}\{\psi_r,\psi_i\}}{\delta\psi_i(x')}\left(-\frac{\delta\mathcal{H}\{\psi_r,\psi_i\}}{\delta\psi_i(x')}\right)dx' \\
&= 0
\end{aligned}$$
i.e. think of $\psi(x,t)$ as a vector with components labeled by x, each of which depends on t, i.e. $\psi(x,t)\sim\psi_x(t)$, and then use chain rule on $\mathcal{H}\{\psi,\psi^*\}$.
- for the NLS there are additional conserved quantities:

e.g. L_2-norm of ψ: $\mathcal{N}=\int|\psi|^2dx$

$$\frac{d}{dt}\mathcal{N} = 2\int\psi_r(x')\partial_t\psi_r(x')+\psi_i(x')\partial_t\psi_i(x')\,dx' = \tag{50}$$

$$= \int\psi_r(x')\left[-\frac{1}{2}\partial^2_{x'}\psi_i(x')-s\left|\psi(x')\right|^2\psi_i(x')\right]+ +\psi_i(x')\left[\frac{1}{2}\partial^2_{x'}\psi_r(x')+s\left|\psi(x')\right|^2\psi_r(x')\right]\,dx' \tag{51}$$

$$\underbrace{=}_{\text{i.b.p.}} \frac{1}{2}\int\partial_{x'}\psi_r(x')\partial_{x'}\psi_i(x')-\partial_{x'}\psi_i(x')\partial_{x'}\psi_r(x')\,dx'=0$$

11

Notes:

[11]Alternative, more compact formulation of the Hamiltonian for the NLS using the Wirtinger calculus for complex derivatives (not exactly the same as the usual complex derivative), in which ψ and ψ^* are independent of each other :
The NLS does not have a Lyapunov functional, but is a Hamiltonian system with Hamiltonian (energy) functional

$$\mathcal{H}\{\psi,\psi^*\}=\frac{1}{2}\int|\partial_x\psi|^2-|\psi|^4dx$$

i.e.

$$\partial_t\psi=-i\frac{\delta\mathcal{H}\{\psi,\psi^*)}{\delta\psi^*} \tag{52}$$

since using integration by parts and employing the basic property of functional derivatives,

$$\frac{\delta\psi(x)}{\delta\psi(x')}=\delta(x-x'),$$

one gets

$$\frac{\delta\mathcal{H}\{\psi,\psi^*)}{\delta\psi^*}=\frac{\delta}{\delta\psi^*}\frac{1}{2}\int-\partial_x^2\psi\,\psi^*-\psi^2\psi^{*2}\,dx=-\frac{1}{2}\partial_x^2\psi-\psi^2\psi^*$$

Conserved Quantities:

- L_2-norm of ψ: $\mathcal{N}=\int|\psi|^2dx$

$$\frac{d}{dt}\mathcal{N} = \int\partial_t\psi\,\psi^*+\psi\,\partial_t\psi^*dx = \tag{53}$$

$$= \int\left(\frac{i}{2}\partial_x^2\psi+i|\psi|^2\psi\right)\psi^*-\psi\left(\frac{i}{2}\partial_x^2\psi^*+i|\psi|^2\psi^*\right)dx$$

$$\underbrace{=}_{\text{integration by parts}} \int\frac{i}{2}\partial_x^2\psi\,\psi^*-\left(\frac{i}{2}\partial_x^2\psi\right)\psi^*dx=0$$

- total energy $\mathcal{H}$
  to compute $\frac{d}{dt}\mathcal{H}$ note that (53) can be written as

$$\int\partial_t\psi\,\psi^*+\psi\partial_t\psi^*dx=\int\partial_t\psi\,\frac{\delta\mathcal{N}\{\psi,\psi^*)}{\delta\psi(x)}+\frac{\delta\mathcal{N}\{\psi,\psi^*)}{\delta\psi^*(x)}\partial_t\psi^*\,dx$$

  i.e. think of $\psi(x,t)$ as a vector with components labeled by x, each of which depends on t, i.e. $\psi(x,t)\sim\psi_x(t)$, and then use chain rule on $N\{\psi,\psi^*\}$.

- because of the factor i in (48) the energy of the system does not decrease with time as it does in systems with a Lyapunov functional. Instead it is conserved.
- for a system with a Lyapunov functional (variational system) one would have

$$\partial_t \psi = -\frac{\delta \mathcal{F}\{\psi\}}{\delta \psi} \qquad \text{with} \qquad \mathcal{F} \in \mathbb{R}$$

resulting in a non-increasing dependence of $\mathcal{F}$ on t:

$$\frac{d}{dt}\mathcal{F}\{\psi\} = \frac{\delta \mathcal{F}}{\delta \psi}\partial_t \psi = -\left(\frac{\delta \mathcal{F}}{\delta \psi}\right)^2 \leq 0$$

Significance of conserved quantities:

Example:

- Newton's equation of motion conserves total energy

$$m\frac{d^2}{dt^2}x = F(x) = -\frac{d}{dx}V(x)$$

multiply by the integrating factor $\frac{d}{dt}x$

$$\begin{aligned} m\frac{d}{dt}x\frac{d^2}{dt^2}x &= -\frac{d}{dt}x\,\frac{d}{dx}V(x) \\ \frac{1}{2}m\frac{d}{dt}\left(\left(\frac{d}{dt}x\right)^2\right) &= -\frac{d}{dt}V(x) \end{aligned}$$

i.e.

$$\frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2 + V(x)\right) = 0$$

$$\frac{1}{2}m\dot{x}^2 + V(x) = E = const.$$

$$\dot{x} = \sqrt{\frac{2}{m}\left(E - V(x)\right)} \qquad \Rightarrow \qquad t = \int \frac{dx}{\sqrt{\frac{2}{m}\left(E - V(x)\right)}}$$

Thus:

  - because of energy conservation the order of the differential equation can be reduced:
    expresses $\dot{x}$ as a function of x

Analogously,

$$\begin{aligned} \frac{d}{dt}\mathcal{H} &= \int \frac{\delta \mathcal{H}\{\psi,\psi^*\}}{\delta \psi}\,\partial_t\psi + \frac{\delta \mathcal{H}\{\psi,\psi^*\}}{\delta \psi^*}\,\partial_t\psi^*\,dx = \\ &= \int -i\partial_t\psi^*\,\partial_t\psi + i\partial_t\psi\,\partial_t\psi^*\,dx = 0 \end{aligned}$$

– solution can be obtained by simple integration (quadrature): the system is called *integrable*

– for two interacting particles $x_1(t)$ and $x_2(t)$ energy conservation alone leads to a single relation between the two velocities

$$\frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 + V(x_1, x_2) = E = const.$$

to express each velocity $\dot{x}_i$ in terms of the positions x_i we would need a second equation: a second conserved quantity

– in general: for Newton's equations of motion with N degrees of freedom to be integrable one needs N independent, conserved quantities.

Notes:

- Hamiltonian systems with N degress of freedom are integrable if they have N independent conserved quantities.
- In an integrable system all dynamics are on N-dimensional tori.
- the NLS has infinitely many degrees of freedom and *infinitely many* conserved quantities.
  It can be shown to be integrable. Exact solutions can be obtained by the inverse scattering transform (well beyond this class).
- the existence of completely integrable nonlinear systems like the NLS was found after a lot of effort in the wake of the numerical simulations by Fermi, Pasta, and Ulam of a nonlinear one-dimensional lattice model in which they were trying to identify the approach of such a system to thermal equilibrium. They found, however, that the nonlinear system they investigated did not approach equilibrium, but instead the system repeatedly returned to a state very close to the initial condition. This was quite surprising since it often had been assumed that even a small nonlinearity would generically make systems non-integrable and have them approach states corresponding to thermal equilibrium. For some more details and a historical overview have a look at the overviews given in [12, 4].

3.2 Soliton Solutions of the NLS

For $s = +1$ the NLS has exact localized solution of the form

$$\psi(x,t) = \lambda \frac{1}{\cosh \rho x} e^{i\omega t}$$

Inserting into NLS yields

$$\begin{aligned} \partial_t \psi - \frac{i}{2}\partial_x^2 \psi - si|\psi|^2\psi &= \lambda \left(i\omega \frac{1}{\cosh \rho x} - \frac{i}{2}\rho^2 \frac{\cosh^2 \rho x - 2}{\cosh^3 \rho x} - si\lambda^2 \frac{1}{\cosh^3 \rho x} \right) e^{i\omega t} = \\ &= \lambda \frac{i}{\cosh^3 \rho x} \left(\cosh^2 \rho x \left(\omega - \frac{1}{2}\rho^2 \right) + \rho^2 - s\lambda^2 \right) e^{i\omega t} \end{aligned}$$

Thus we need $s=+1$ and

$$\rho=\lambda \qquad \omega=\frac{1}{2}\lambda^2 \quad \Rightarrow \quad \psi(x,t)=\lambda\frac{1}{\cosh\lambda\,(x-x_0)}e^{i\frac{1}{2}\lambda^2(t-t_0)} \tag{54}$$

Note:

- the parameter λ is *arbitrary*: there is a one-parameter continuous family of solutions with different amplitudes and associated different frequencies and width
- the soliton solution is *not* an attractor: small perturbations *do not* relax and the solution does not come back to the unperturbed solution
- in fact, due to translation symmetry in space and time, i.e. x_0 and t_0 are arbitrary, (54) corresponds already to a three-parameter continuous family of solutions

NLS also allows transformations into moving frames of reference (*boosts*): $u=x-ct$

Consider

$$\tilde{\psi}(x,t)=\psi(t,x-ct)e^{iqx+i\omega t}$$

where $\psi(t,x)$ is a solution. For $\tilde{\psi}$ to be a solution, as well, q and ω have to satisfy certain conditions. Insert $\tilde{\psi}$ into NLS

$$\begin{aligned}\partial_t\tilde{\psi}-\frac{i}{2}\partial_x^2\tilde{\psi}-si|\tilde{\psi}|\tilde{\psi} &= \left(\partial_t\psi-c\partial_u\psi+i\omega\psi-\frac{i}{2}\left(\partial_u^2\psi+2iq\partial_u\psi-q^2\psi\right)-si|\psi|^2\psi\right)e^{iqx+i\omega t}= \\ &= \left(\partial_u\psi\,(-c+q)+i\psi\left(\omega+\frac{1}{2}q^2\right)\right)e^{iqx+i\omega t}\end{aligned}$$

using that $\psi(t,u)$ satisfies NLS. Require

$$c=q \qquad \omega=-\frac{1}{2}q^2$$

Note:

- the boost velocity c or the background wavenumber q is a free parameter generating a continuous family of solutions

Thus:

The focusing Nonlinear Schrödinger equation has a *four-parameter family of solutions* of solitons

$$\psi(x,t)=\lambda\frac{1}{\cosh\left(\lambda(x-qt-x_0)\right)}e^{iqx+i\frac{1}{2}(\lambda^2-q^2)t+i\phi_0}$$

After a perturbation (change) in any of the four parameter q, λ, x_0, ϕ the solution does not relax back to the unperturbed solution but gets shifted along the corresponding family of solutions.

Note:

- Surprising feature of solitons:
  during collisions solutions become quite complicated, but after the collisions the solitons emerge unperturbed except for a shift in position x_0 and the phase ϕ_o. In particular, the other two parameters, λ and q, are unchanged, although there is also no 'restoring force' to push them back to the values before the collision.

- general solution can be described in terms of a nonlinear superposition of many interacting solitons and periodic waves (captured by inverse scattering theory).

3.3 Perturbed Solitons

Real systems usually have some dissipation. A beautiful example of solitary waves in a dissipative system that can be described as weakly perturbed NLS-solitons are waves observed in convection of mixtures (Fig.25).

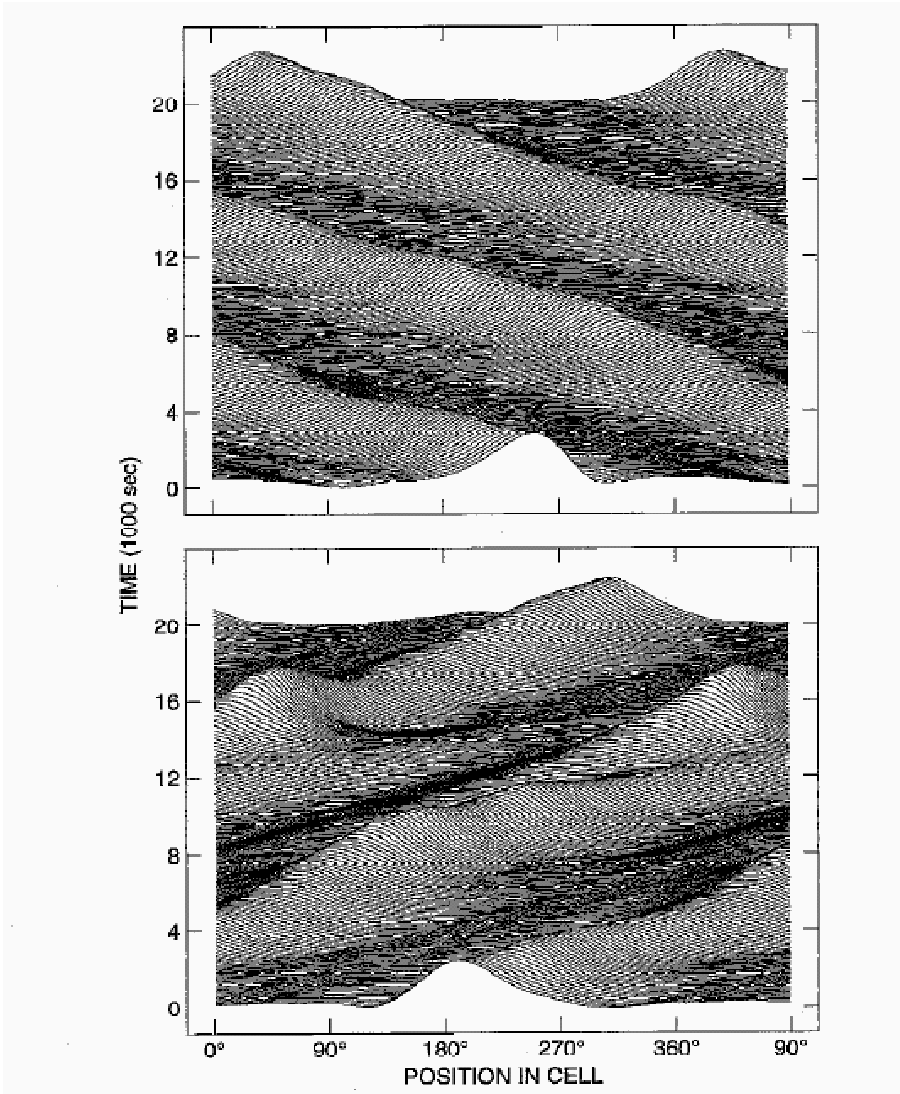


Figure 25: Localized wave trains in convection of water-alcohol mixtures. Space-time plot of the envelope of the left- (top panel) and the right-traveling wave component (bottom panel). In this parameter regime the localized convection waves are not stable but evolve chaotically ('dispersive chaos') [6, 7].

Consider soliton-like solutions of the (focusing) NLS with small dissipative perturbations

$$\partial_t \Psi - \frac{i}{2}\partial_x^2 \Psi - i|\Psi|^2\Psi = \epsilon P(\Psi, \partial_x \Psi, \ldots)$$

For small perturbations expect slow evolution along the family of solutions:

$$T = \epsilon t \qquad \lambda = \lambda(T) \qquad q = q(T) \qquad x_0 = x_0(T) \qquad \phi_0 = \phi_0(T)$$

For simplicity: focus on perturbations for which soliton remains stationary: $c = 0 = q$

Slow changes in the amplitude λ or the wavenumber q imply also slow changes in the frequency $\omega = \frac{1}{2}(\lambda^2 - q^2)$: introduce a phase ϕ

$$\phi = \int^t \omega(\epsilon t')dt' \qquad \frac{d}{dt}\phi(t) = \omega(T).$$

Ansatz for the expansion

$$\Psi(t,T)=\psi(\Theta,T)\,e^{i\phi}=\left[\psi_0(\Theta,T)+\epsilon\psi_1(\Theta,T)+\ldots\right]e^{i\phi(t,T)}$$

where

$$\psi_0(\Theta)=\lambda(T)\frac{1}{\cosh\left(\Theta-\Theta_0\right)}e^{i\phi_0},\qquad \Theta=\lambda(T)x\qquad \Theta_0,\phi_0=const.$$

and

$$\lambda(T)=\lambda_0(T)+\epsilon\lambda_1(T)+\ldots,\qquad \omega(T)=\omega_0(T)+\epsilon\omega_1(T)+\ldots$$

with the relation (from the unperturbed case)

$$\omega_0(T)=\frac{1}{2}\lambda_0(T)^2$$

Note:

- this is not a weakly nonlinear analysis: the amplitude λ is not assumed small, $\lambda=\mathcal{O}(1)$
- the phase ϕ evolves on the $\mathcal{O}(1)$ time scale, $\frac{d}{dt}\phi=\omega$, but the frequency changes on the slow time scale as the solution evolves along the family of solutions
- in the general case ($q\neq 0$) one would have to introduce a spatial phase $\theta(x,t)$ as well

$$\Theta=\lambda(T)\theta(x,t)$$

Rewrite NLS in terms of ψ rather than Ψ

$$i\omega\psi-\frac{i}{2}\lambda^2\partial_\Theta^2\psi-i|\psi|^2\psi=\epsilon\left(-\partial_T\psi+e^{-i\phi}P(\Psi,\partial_x\Psi,\ldots)\right)$$

Insert expansion of ψ

$\mathcal{O}(\epsilon^0)$:

$$i\underbrace{\omega_0}_{\frac{1}{2}\lambda_0^2}\psi_0-\frac{i}{2}\lambda_0^2\partial_\Theta^2\psi_0-i|\psi_0|^2\psi_0=0$$

confirms

$$\omega_0=\frac{1}{2}\lambda_0^2$$

$\mathcal{O}(\epsilon)$:

$$\begin{aligned}\mathcal{L}\psi_1 &\equiv i\omega_0\psi_1-\frac{i}{2}\lambda_0^2\partial_\Theta^2\psi_1-i\left(2|\psi_0|^2\psi_1+\psi_0^2\psi_1^*\right)\\ &= -\partial_T\psi_0-i\omega_1\psi_0+i\lambda_0\lambda_1\partial_\Theta^2\psi_0+e^{-i\phi}P\end{aligned}$$

Essential question for the perturbation expansion is whether the linearized operator $\mathcal{L}$ is invertible or singular.

The unperturbed soliton ψ_0 is part of a four-parameter family of solutions (here we are keeping the parameter $q = 0$ fixed) ,

$$\psi_0 = \psi_0(x, t; \Theta_0, \phi_0, \lambda, q).$$

The linear operator $\mathcal{L}$ is therefore **singular**:

Take derivatives of the $\mathcal{O}(1)$-equation

$$i\frac{1}{2}\lambda_0^2\psi_0 - \frac{i}{2}\lambda_0^2\partial_\Theta^2\psi_0 - i|\psi_0|^2\psi_0 = 0$$

with respect to the parameters Θ_0, ϕ_0:

$$\begin{aligned}\partial_{\Theta_0}\left\{i\frac{1}{2}\lambda_0^2\psi_0 - \frac{i}{2}\lambda_0^2\partial_\Theta^2\psi_0 - i|\psi_0|^2\psi_0\right\} &= i\frac{1}{2}\lambda_0^2\partial_{\Theta_0}\psi_0 - \frac{i}{2}\lambda_0^2\partial_\Theta^2\partial_{\Theta_0}\psi_0 - 2i|\psi_0|^2\partial_{\Theta_0}\psi_0 - i\psi_0^2\partial_{\Theta_0}\psi_0^* \\ &= \mathcal{L}\left(\partial_{\Theta_0}\psi_0\right) = 0\end{aligned}$$

and

$$\partial_{\phi_0}\left\{i\frac{1}{2}\lambda_0^2\psi_0 - \frac{i}{2}\lambda_0^2\partial_\Theta^2\psi_0 - i|\psi_0|^2\psi_0\right\} = \mathcal{L}\left(\partial_{\phi_0}\psi_0\right) = \mathcal{L}i\psi_0 = 0$$

Thus:

- $\mathcal{L}$ is singular and has two eigenvectors with vanishing eigenvalues.

Broken Continuous Symmetries and 0 Eigenvalues

- The unperturbed solution breaks a continuous symmetry $\Leftrightarrow$ the unperturbed solution depends continuously on a parameter that does not appear in the equation.
- Here: the solution ψ_0 *breaks* two *continuous* symmetries of the original system:

$$\Theta \to \Theta + \Delta\theta \qquad \text{translation symmetry in space}$$

$$\phi \to \phi + \Delta\phi \qquad \text{translation symmetry in time}$$

- **General**:
  - If a solution breaks a continuous symmetry of a system, then the linear operator obtained from linearizing around this solution has 0 eigenvalues.
  - The associated eigenvectors (modes) are called *translation modes* or *Goldstone modes*.
- The existence of a continuous symmetry $\phi \to \phi + \Delta\phi$ alone does not imply a vanishing eigenvalue:
  if the solution ψ_0 does not break the continuous symmetry, then $\partial_\phi\psi_0 = 0$ and $\partial_\phi\psi_0$ does not represent an eigenvector.

- Breaking a discrete symmetry leads to a discrete family of solutions. This does not imply that the linearization around that solution has a vanishing eigenvalue since one cannot go continuously from one of the solutions in the family to another and therefore no derivative with respect to a continuous parameter can be taken.
  Intuitively: in a discrete family of solutions each solution is generically either linearly stable or linearly unstable $\Rightarrow$ restoring or repelling 'force'

Note:

- The full soliton solution of the NLS is part of a four-parameter familty because of the continuous dependence on the amplitude and on the velocity of the solution $\rightarrow$ additional vanishing eigenvalues.
  it turns out that two of the 0-eigenvalues of $i\mathcal{L}$ are associated with *proper eigenvectors*, whereas the other two have *generalized eigenvectors*.

$$\Psi_{\phi_0}: \quad i\mathcal{L}i\psi_0 = 0 \qquad \Psi_{x_0}: \quad i\mathcal{L}\partial_\Theta\psi_0 = 0$$
$$\Psi_\lambda: \quad i\mathcal{L}\left(\Theta\partial_\Theta\psi_0 + \psi_0\right) = i\lambda_0^2\psi_0 \qquad \Psi_q: \quad i\mathcal{L}i\Theta\psi_0 = -\lambda_0^2\partial_\Theta\psi_0$$

Since the linearized operator $\mathcal{L}$ is singular, the equation at $\mathcal{O}(\epsilon)$ can again only be solved if solvability conditions are satisfied (Fredholm alternative). To get the solvability conditions we need to project the $\mathcal{O}(\epsilon)$-equation onto the relevant left eigenvectors.

Projections need a scalar product. For functions the scalar product typically involves some integral over the domain. Here we can make $i\mathcal{L}$ (not $\mathcal{L}$, though) self-adjoint[12] by a suitable choice of the scalar product. Choose

$$\langle\psi_1,\psi_2\rangle = \Re\left(\int_{-\infty}^{\infty}\psi_1^*\psi_2 d\Theta\right)$$

Then

$$\begin{aligned}
\langle\psi_1, i\mathcal{L}\psi_2\rangle &= \Re\left(\int_{-\infty}^{\infty}\psi_1^* i\mathcal{L}\psi_2 d\Theta\right) = \\
&= \Re\left(\int \psi_1^*\left(-\omega_0\psi_2 + \frac{1}{2}\lambda_0^2\partial_\Theta^2\psi_2 + \left(2|\psi_0|^2\psi_2 + \psi_0^2\psi_2^*\right)\right)d\Theta\right) = \\
&\underbrace{=}_{i.b.p.} \Re\left(\int -\omega_0\psi_1^*\psi_2 + \frac{1}{2}\lambda_0^2\partial_\Theta^2\psi_1^*\,\psi_2 + 2|\psi_0^2|\psi_1^*\,\psi_2 + \underbrace{\psi_0^2\psi_1^*\,\psi_2^*}_{\left(\psi_0^{*2}\psi_1\,\psi_2\right)^*}\;d\Theta\right) \\
&\underbrace{=}_{\text{exploiting }\Re} \Re\left(\int\left(i\mathcal{L}\psi_1\right)^*\;\psi_2\,d\Theta\right) = \langle i\mathcal{L}\psi_1,\psi_2\rangle
\end{aligned}$$

In the last step was used $\Re\left(\psi_0^{*2}\psi_1\psi_2\right) = \Re\left(\left(\psi_0^{*2}\psi_1\psi_2\right)^*\right)$.
Thus, with this scalar product the left eigenvectors are identical to the right eigenvectors.

Note:

[12]Note because of the terms $i\omega_0$ and $\frac{1}{2}i\lambda_0^2\partial_\Theta^2$ $\mathcal{L}$ is not self-adjoint, but suggest that $i\mathcal{L}$ may be self-adjoint.

- of course $\mathcal{L}i\psi_0 = 0$ as well. While $i\psi_0$ is a not left-eigenvector of $\mathcal{L}$; it is one of $i\mathcal{L}$
- $(i\mathcal{L})^2\, i\Theta\psi_0 = 0$ and $(i\mathcal{L})^2\,(\Theta\partial_\Theta\psi_0 + \psi_0) = 0$ as expected of generalized eigenvectors

Focus here on a simple, purely *dissipative* perturbation ($\mu, \alpha, \gamma \in \mathbb{R}$),

$$P(\Psi) = \mu\Psi + \alpha|\Psi|^2\Psi + \gamma|\Psi|^4\Psi \tag{55}$$

Then we need only the eigenvector Ψ_ϕ associated with the phase invariance $\phi \to \phi + \Delta\phi$

$$\Psi_{\phi_0} = i\psi_0$$

Thus, using $i\mathcal{L}$, i.e. after multiplying $\mathcal{O}(\epsilon)$-equation by i,

$$\begin{aligned} 0 &= \langle i\psi_0, i\mathcal{L}\psi_1\rangle = \Re \int -i\psi_0^*\, i\left(-\partial_T\psi_0 - i\omega_1\psi_0 - i\lambda_0\lambda_1\partial_\Theta^2\psi_0 + e^{-i\phi}P\right) d\Theta = \\ &= \int \psi_0^*\, \left(-\partial_T\psi_0 + \mu\psi_0 + \alpha\,|\psi_0|^2\,\psi_0 + \gamma\,|\psi_0|^4\,\psi_0\right) d\Theta \end{aligned}$$

Use $\psi_0 = \lambda\frac{1}{\cosh\Theta}e^{i\phi_0}$ and

$$\int \frac{1}{\cosh^2\Theta}d\Theta = 2 \qquad \int \frac{1}{\cosh^4\Theta}d\Theta = \frac{4}{3} \qquad \int \frac{1}{\cosh^6\Theta}d\Theta = \frac{16}{15}$$

to get

$$\frac{d}{dT}\lambda = \mu\lambda + \frac{2}{3}\alpha\lambda^3 + \frac{8}{15}\gamma\lambda^5 \tag{56}$$

Notes:

- the dissipative perturbations P lead to a slow evolution of the amplitude of the perturbed soliton close to the soliton family of solutions: slow manifold
- with increasing amplitude the perturbed soliton becomes narrower
- non-trivial fixed points
  - $\alpha < 0$:
    supercritical pitch-fork bifurcation

    $$\lambda^2 = \frac{3}{2}\frac{\mu}{\alpha} + h.o.t. \qquad \text{if } \mu > 0.$$

    Within the amplitude equation (56) the fixed point is stable.
    However: do not expect this localized soliton-like solution to be stable within the full NLS since $\Psi = 0$ is unstable for $\mu > 0$: perturbations will grow far away from the soliton

- $\alpha > 0$:
  subcritical pitch-fork bifurcation

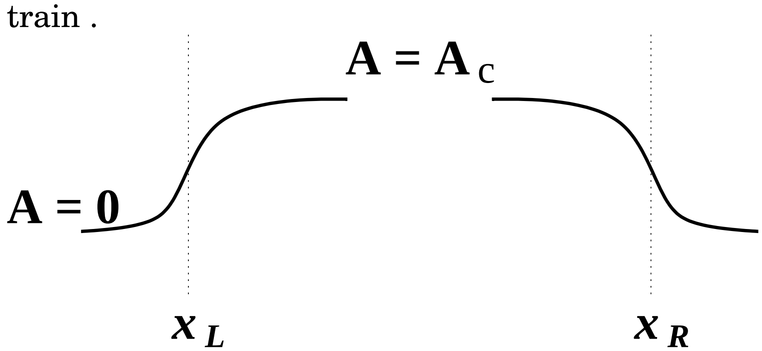
$$\lambda_{1,2}^2 = -\frac{5}{8}\frac{\alpha}{\gamma} \pm \frac{15}{16\gamma}\sqrt{\frac{4}{9}\alpha^2 - \frac{32}{15}\mu\gamma}$$

  two soliton-like solutions created in saddle-node bifurcation at $\alpha^2 = \frac{24}{5}\mu\gamma$
  within (56) the one with larger amplitude is stable, the other unstable.
  Background state $\Psi = 0$ is linearly stable for $\mu < 0$.

- full solution consists of four coupled evolution equations for λ, q, x_0, ϕ_0:
  - would have to check that for the perturbation (55) the equations for q, x_0, and ϕ_0 have *stable* fixed points with $q = 0$, $x_0 = const.$ and $\phi_0 = const.$
  - a general perturbation can make the soliton travel, $q \neq 0$, $\frac{d}{dT}x_0 \neq 0$.

Notes:

- experiments in convection of water-alcohol mixtures: onset of convection via a subcritical Hopf bifurcation
- quintic complex Ginzburg-Landau equation
  - for strong dispersion, i.e. large α and β, the complex Ginzburg-Landau equation can be considered as a perturbed NLS: expect localized solutions in the form of perturbed solitons [10].
  - for weak dispersion perturbation approach via *interacting fronts [8]*:
    subcritical bifurcation (cf. Sec.4).
    * bistability between conductive state ($\Psi = 0$) and convective state ($\Psi = \Psi_0 \neq 0$)
    * front solutions $\Psi_\pm(x,t) \to 0$ for $x \to \mp\infty$ and $\Psi_\pm(x,t) \to \Psi_0$ for $x \to \pm\infty$
    * fronts Ψ_+ and Ψ_- can interact and form a stable pair: wide localized wave train .



4 Fronts and Their Interaction

Consider nonlinear PDEs with spatial translation symmetry that have multiple stable spatially homogeneous solutions

- $\Rightarrow$ there must be also solutions that connect the stable states: fronts or kinks
- these fronts are *heteroclinic* in space: they connect two different fixed points for $x \to \pm\infty$. They are topologically stable: they cannot disappear except at infinity or by collision with 'anti-fronts'.
- This is to be compared to *homoclinic* solutions which connect to the same fixed point for $x \to \pm\infty$, i.e. localized 'humps' (like the solitons). They are not topologically stable since they can disappear, e.g., due to a sufficiently large perturbation.

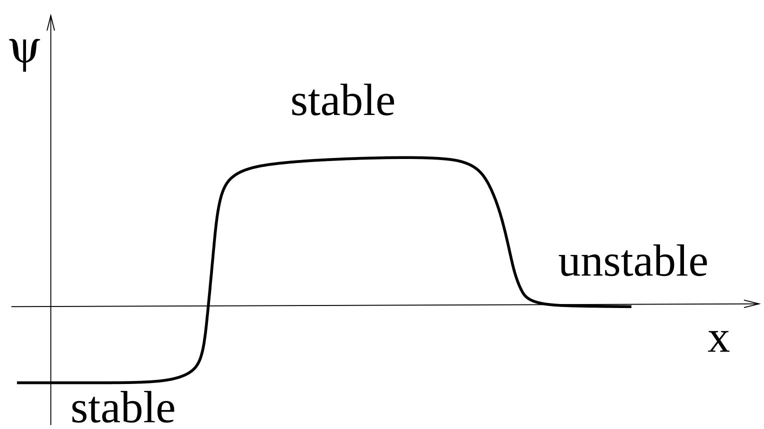
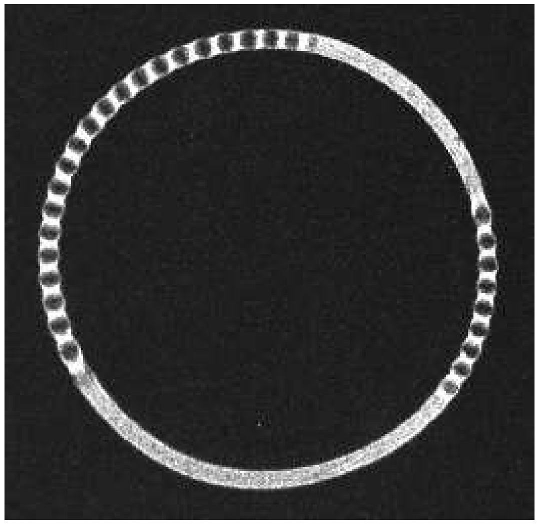


Figure 26: Fronts connecting two stable and one unstable spatially homogeneous state.

Questions:

- Do such fronts travel? What determines their speed?
- How do the fronts interact? Can they from stable bound states: localized domains?

a) 

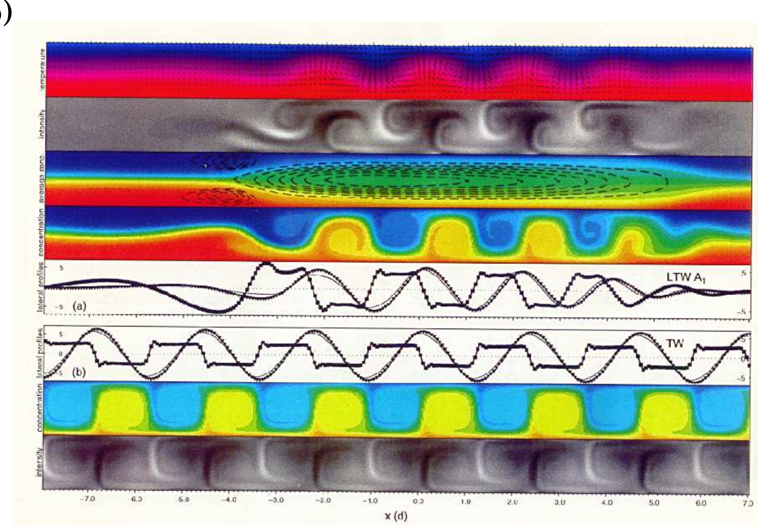
b) 

Figure 27: Localized wave trains in convection of water-alcohol mixtures. a) Top view of annular convection cell. In this regime the localized waves are spatially more extended, resembling bound states of fronts. Two slowly drifting, stable localized wave trains are seen [5]. b) Numerical simulations of localized states and of extended traveling waves [1, 2].

4.1 Single Fronts Connecting Stable States

Consider a simple nonlinear diffusion equation

$$\partial_t \psi = \partial_x^2 \psi + f(\psi) \equiv \partial_x^2 \psi - \partial_\psi V(\psi;\lambda)$$

where λ is a control parameter of the system.

This equation can be written in variational form

$$\partial_t \psi = -\frac{\delta \mathcal{V}\{\psi\}}{\delta \psi} \qquad \text{with} \qquad \mathcal{V}\{\psi\} = \int \frac{1}{2}\left(\partial_x \psi\right)^2 + V(\psi;\lambda) dx$$

Assume $V(\lambda;\psi)$ has two minima at $\psi = \psi_{1,2}$, corresponding to stable, spatially homogeneous solutions.

Look for 'wave solution', i.e. a steadily propagating front solution

$$\psi = \psi(\zeta) \qquad \text{with} \qquad \zeta = x - vt$$

which satisfies

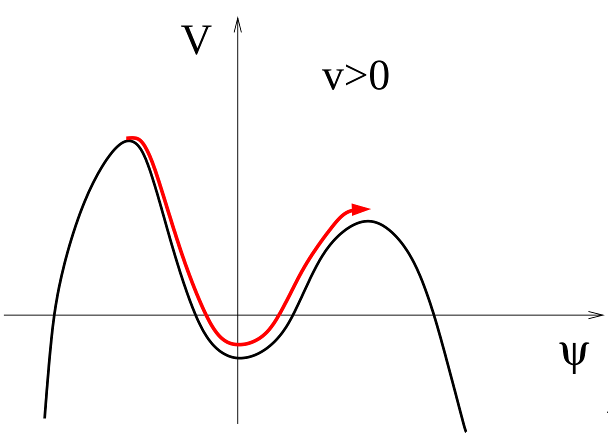
$$\partial_\zeta^2 \psi + v\partial_\zeta \psi = +\partial_\psi V(\psi) \equiv -\partial_\psi \hat{V}(\psi) \qquad \text{with} \qquad \hat{V}(\psi;\lambda) = -V(\psi;\lambda).$$

Notes:

- this equation can be read as describing the position ψ of a particle in the potential $\hat{V}(\psi)$ and experiencing friction with coefficient v.
- we are interested in solutions that start at ψ_1 and end at ψ_2

$$\psi(\zeta) \to \psi_1 \quad \text{for} \quad \zeta \to -\infty \qquad \psi(\zeta) \to \psi_2 \quad \text{for} \quad \zeta \to +\infty$$

- since in terms of $\hat{V}(\psi)$ the 'positions' $\psi_{1,2}$ are actually maxima, the 'friction' v must be tuned exactly such that the particle, starting at one maximum, stops at the other maximum:
  $\Rightarrow$ the velocity is uniquely determined.
- depending on the relative heights of the maxima the 'friction' may be negative.
- for fronts connecting a stable state with an unstable state the velocity ('friction') is not uniquely determined: the unstable state corresponds to the minimum of the potential for the 'particle' and the 'particle' will end up in that minimum for a wide range of friction values. The velocity selection in this situation is an interesting problem (e.g. [11]).

a) 

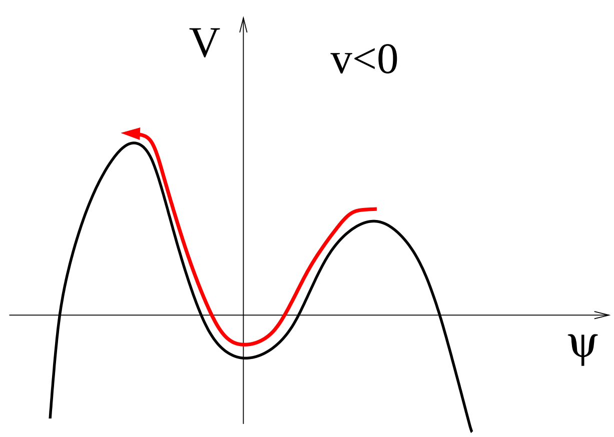
b) 

Figure 28: Fronts correspond to a particle moving in a potential with friction. a) friction positive. b) friction negative.

4.1.1 Perturbation Calculation of the Front Velocity

Assume there is a parameter value, $\lambda = \lambda_0 \equiv 0$, for which the front solution $\psi(x; \lambda = 0)$ is stationary. If one has access to that solution one can obtain the front velocity for close-by parameter values perturbatively.

Expand

$$\lambda = \epsilon\lambda_1 \qquad v = \epsilon v_1 + \epsilon^2 v_2 + h.o.t. \qquad \psi = \psi_0(x) + \epsilon\psi_1 + h.o.t.$$

$\mathcal{O}(\epsilon^0)$:

$$\partial_x^2\psi_0 + \partial_\psi\hat{V}(0;\psi_0) = 0$$

yields the equation for the stationary front

$\mathcal{O}(\epsilon)$:

$$\underbrace{\partial_x^2\psi_1 + \partial_\psi^2\hat{V}(0;\psi_0)\Big|}_{\mathcal{L}}\psi_1 = -v_1\partial_x\psi_0 - \lambda_1\,\partial_\lambda\partial_\psi\hat{V}(\lambda;\psi)_{\lambda=0,\psi=\psi_0} \tag{57}$$

Can we invert the operator $\mathcal{L}$ and solve directly for ψ_1?
The system is invariant under spatial translations: take the x-derivative of the equation at $\mathcal{O}(\epsilon^0)$:

$$\partial_x\left[\partial_x^2\psi_0 + \partial_\psi\hat{V}(0;\psi_0)\right] = \partial_x^2\partial_x\psi + \partial_\psi^2\hat{V}(0;\psi)\Big|_{\psi=\psi_0}\partial_x\psi_0 = \mathcal{L}\,\partial_x\psi_0$$

Since $\psi_0(x)$ breaks the continuous translation symmetry, $\partial_x\psi_0 \neq 0$ and is a proper eigenvector of $\mathcal{L}$ with eigenvalue 0.
Thus, $\mathcal{L}$ is singular and the eigenvector associated with the 0 eigenvalue is the translation mode $\partial_x\psi_0$.

Note:

- if ψ_0 did not break the translation symmetry, $\partial_x\psi_0$ would vanish and not represent an eigenvector and there would be no 0 eigenvalue associated with the translation symmetry and $\mathcal{L}$ could be invertible.

$\mathcal{L}$ is self-adjoint $\Rightarrow$ $\partial_x\psi_0$ is also its left 0-eigenvector.

Project (57) on $\partial_x\psi_0$

$$0 = \int_{-\infty}^{+\infty}\partial_x\psi_0\left[-v_1\partial_x\psi_0 - \lambda_1\partial_\lambda\partial_\psi\;\hat{V}(\lambda;\psi)\Big|_{\lambda=0,\psi=\psi_0}\right]dx$$

$$v_1\int_{-\infty}^{\infty}(\partial_x\psi_0)^2\,dx = -\lambda_1\partial_\lambda\int_{-\infty}^{+\infty}\underbrace{\partial_\psi\hat{V}(0,\psi_0)\,\partial_x\psi_0}_{\partial_x\hat{V}(0;\psi_0(x))}\,dx$$

Thus

$$v_1\int_{-\infty}^{\infty}(\partial_x\psi_0)^2\,dx = -\lambda_1\;\partial_\lambda\left[V(\lambda;\psi_0(x))\right]\big|_{\lambda=0}\big|_{x=-\infty}^{+\infty} \approx -\,V(\lambda_1;\psi_0(x))\big|_{x=-\infty}^{+\infty}$$

Notes:

- The l.h.s of the equation can be read as the amount of work performed by the friction

$$\int_{-\infty}^{+\infty}\left(\beta\frac{dx}{dt}\right)\underbrace{\frac{dx}{dt}dt}_{dx}$$

- The r.h.s of the equation can be read as the difference in potential energy between initial and final state
- Important: the perturbation method does not rely on the existence of a potential $\Rightarrow$ it works also when there are multiple coupled components $\psi_j(x,t)$ satisfying nonlinear PDEs that cannot be derived from a potential.

4.2 Interaction between Fronts

Consider fronts of the nonlinear diffusion equation

$$\partial_t\psi = \partial_x^2\psi - \psi + c\psi^3 - \psi^5$$

Notes:

- the coefficients of $\partial_x^2\psi$, ψ, and of ψ^5 can be chosen to have magnitude 1 by rescaling of space, time and ψ.
- the coefficient of ψ is chosen negative: $\psi = 0$ is linearly stable
- the coefficient of ψ^5 is chosen negative: saturation at large values of ψ

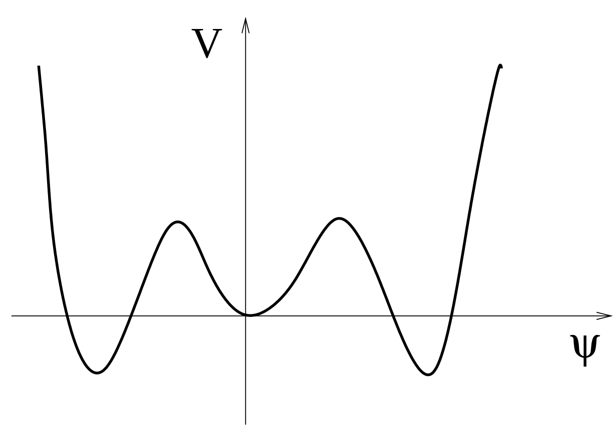


Figure 29: Potential with minima at ψ_0 and $\pm\psi_0$.

Homogeneous stationary states:

linearly stable

$$\psi = 0 \qquad \text{or} \qquad \psi_0^2 = \frac{c+\sqrt{c^2-4}}{2}$$

linearly unstable

$$\psi_u^2 = \frac{c - \sqrt{c^2 - 4}}{2}$$

Consider two fronts that connect $\psi = 0$ with $\psi = \psi_0$

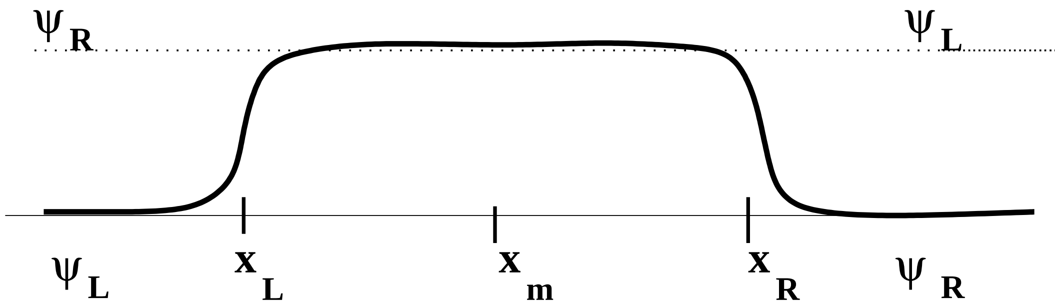


Figure 30: Front positions.

Goal:
Obtain evolution equations for the positions x_L and x_R. These equations would describe the interaction between the two fronts and reduce the PDE to coupled ODEs.

For such a reduction we need a separation of time scales

- The relaxation of ψ to the equilibrium values ψ_0 and $\psi = 0$ should be much faster the motion of the individual fronts.
- For the motion to be slow in the presence of the interaction between the fronts the interaction must be weak:
  consider widely separated fronts, $x_R - x_L$ large, then the fronts deform each other only weakly.

Note:

- Since the fronts approach their asymptotic value exponentially fast, it turns out that the interaction is exponentially weak in the distance between the fronts.

Ansatz:

$$\psi = \psi_L + \psi_R \qquad \underbrace{-\psi_0}_{\text{subtract common part}} \qquad + \epsilon\psi_1 + \ldots$$

with

$$\psi_L = \psi_F\left(x - x_L(T)\right) \qquad \psi_R = \psi_F\left(x_R(T) - x\right)$$

$$T = \epsilon t \qquad c = c_0 + \epsilon c_1 \qquad c_0 = \frac{4}{\sqrt{3}}$$

$$\psi_F(\zeta) = \psi_0\sqrt{\frac{1}{2}\left(1 + \tanh\zeta\right)} \qquad \psi_0 = 3^{\frac{1}{4}}$$

Notes:

- c_0 can be determined as the point where the potential $V(\psi) = +\frac{1}{2}\psi^2 - \frac{1}{4}c\psi^4 + \frac{1}{6}\psi^6$ has the same value at the two minima $\psi = 0$ and $\psi = \psi_0$ (cf. Sec.4.1.1).

Denote

$$\psi'_L \equiv \left.\frac{d\psi_F(\xi_L)}{d\xi_L}\right|_{\xi_L = x - x_L} \qquad \psi'_R \equiv \left.\frac{d\psi_F(\xi_R)}{d\xi_R}\right|_{\xi_R = x_R - x}$$

Then

$$\partial_t \psi_L = \psi'_L \frac{d\xi_L}{dt} = -\epsilon \psi'_L \partial_T x_L \qquad \partial_t \psi_R = \psi'_R \frac{d\xi_R}{dt} = +\epsilon \psi'_R \partial_T x_R$$

Note: $\frac{d\psi_L}{dx} = \psi'_L$, but $\frac{d\psi_R}{dx} = -\psi'_R$.

Insert expansion

$$\begin{aligned} 0 &= \epsilon \left\{\psi'_L\, \partial_T x_L - \psi'_R\, \partial_T x_R\right\} + \psi''_L + \psi''_R + \epsilon \psi''_1 - \psi_L - \psi_R + \psi_0 - \epsilon\psi_1 \\ &\quad + (c_0 + \epsilon c_1) \left\{\psi_L + \psi_R - \psi_0 + \epsilon\psi_1\right\}^3 - \left\{\psi_L + \psi_R - \psi_0 + \epsilon\psi_1\right\}^5 \end{aligned}$$

For $x < x_m \equiv \frac{1}{2}(x_L + x_R)$ we have $x_R - x \gg 1$:

$$\begin{aligned} \psi_R - \psi_0 &= \psi_0 \left\{ \sqrt{\frac{1}{2}\left(1 + \tanh\left(x_R - x\right)\right)} - 1 \right\} \\ &= \psi_0 \left\{ \sqrt{\frac{1}{2} \frac{e^{x_R - x} + e^{x - x_R} + e^{x_R - x} - e^{x - x_R}}{e^{x_R - x} + e^{x - x_R}}} - 1 \right\} \\ &= \psi_0 \left\{ \frac{1}{\sqrt{1 + e^{-2(x_R - x)}}} - 1 \right\} \to -\frac{1}{2}\psi_0 e^{-2(x_R - x)} \qquad \text{for} \quad x_R - x \to \infty \end{aligned}$$

Analogously for $x > x_m$ we have $x - x_L \gg 1$:

$$\psi_L - \psi_0 \to -\frac{1}{2}\psi_0 e^{-2(x - x_L)}$$

Consider now the expansion separately for $x < x_m$ and $x > x_m$.

For $x < x_m$:

$$\left\{\psi_L + (\psi_R - \psi_0) + \epsilon\psi_1\right\}^3 = \psi_L^3 + 3\psi_L^2 \left(\psi_R - \psi_0\right) + +3\epsilon\psi_1\psi_L^2 + \mathcal{O}\left(\left(\psi_R - \psi_0\right)^2, \epsilon\left(\psi_R - \psi_0\right), \epsilon^2\right)$$

and

$$\left\{\psi_L + (\psi_R - \psi_0) + \epsilon\psi_1\right\}^5 = \psi_L^5 + 5\psi_L^4 \left(\psi_R - \psi_0\right) + +5\epsilon\psi_1\psi_L^4 + \mathcal{O}\left(\left(\psi_R - \psi_0\right)^2, \epsilon\left(\psi_R - \psi_0\right), \epsilon^2\right)$$

Using that $\psi_{L,R}$ satisfy the $\mathcal{O}(\epsilon^0)$ equations

$$\psi''_L - \psi_L + c_0\psi_L^3 - \psi_L^5 = 0 \qquad \psi''_R - \psi_R + c_0\psi_R^3 - \psi_R^5 = 0$$

we get for $x < x_m$

$$\begin{aligned} -\epsilon \underbrace{\left\{\psi''_1 - \psi_1 + 3c_0\psi_L^2\psi_1 - 5\psi_L^4\psi_1\right\}}_{\mathcal{L}_L\psi_1} &= \psi''_R + (\psi_R - \psi_0)\left\{-1 + 3c_0\psi_L^2 - 5\psi_L^4\right\} + \\ &\quad +\epsilon\left\{c_1\psi_L^3 + \partial_T x_L \psi'_L - \partial_T x_R \psi'_R\right\} \end{aligned}$$

with

$$\mathcal{L}_L = \partial_x^2 - 1 + 3c_0\psi_L^2 - 5\psi_L^4$$

For $x > x_m$

$$\begin{aligned} -\epsilon\mathcal{L}_R\psi_1 &= \psi_L'' + (\psi_L - \psi_0)\left\{-1 + 3c_0\psi_R^2 - 5\psi_R^4\right\} + \\ &+\epsilon\left\{c_1\psi_R^3 + \partial_T x_L\psi_L' - \partial_T x_R\psi_R'\right\} \end{aligned}$$

with

$$\mathcal{L}_R = \partial_x^2 - 1 + 3c_0\psi_R^2 - 5\psi_R^4$$

To obtain evolution equations for x_L and x_R we need *two solvability conditions*

Translation symmetry:

- single front: $\partial_x\psi_{L,R}$ is a 0-eigenvector
- two interacting fronts: there is only one exactly vanishing eigenvalue with the eigenvector arising from the double-front solution $\partial_x\psi_{LR}$, for which the $x_R - x_L$ is not growing or shrinking

Note:

- the double-front solution is stationary for c slightly different than c_0 due to the interaction between the two fronts.

How do we get a second solvability condition?

We want the perturbation expansion to remain well-ordered in the limit $x_R - x_L \to \infty$, i.e. even if the fronts are infinitely far apart, ψ_1 has to remain small compared to $\psi_L+\psi_R-\psi_0$

- for any finite L: only 1 translation mode, which is (approximately) $\partial_x\left(\psi_L + \psi_R - \psi_0\right)$
- for $L = \infty$: 2 independent fronts $\Rightarrow$ expect 2 translation modes

$$\mathcal{L}_L\partial_x\psi_L = 0 \qquad \mathcal{L}_R\partial_x\psi_R = 0$$

Project in the two domains $x < x_m$ and $x > x_m$ separately onto the two translation modes $\partial_x\psi_{L,R}$, respectively.

$x < x_m$:

$$\begin{aligned} -\epsilon\int_{-\infty}^{x_m}\psi_L'\mathcal{L}_L\psi_1 dx &= \epsilon\partial_T x_L\int_{-\infty}^{x_m}\psi_L'^2 dx - \epsilon\partial_T x_R\int_{-\infty}^{x_m}\psi_L'\psi_R' dx + \int_{-\infty}^{x_m}\psi_L'\psi_R'' dx + \\ &+\int_{-\infty}^{x_m}\psi_L'\left\{-1 + 3c_0\psi_L^2 - 5\psi_L^4\right\}(\psi_R - \psi_0)\,dx + \epsilon c_1\int_{-\infty}^{x_m}\psi_L'\psi_L^3 dx \end{aligned}$$

For $x_m \to \infty$ the operator $\mathcal{L}_L$ is self-adjoint and we could roll it over to ψ_L' and the l.h.s. would vanish. For finite x_m boundary terms arise.

Integrate the l.h.s by parts

$$\int_{-\infty}^{x_m}\psi_L'\mathcal{L}_L\psi_1 dx = \psi_L'\psi_1'\big|_{-\infty}^{x_m} - \underbrace{\int_{-\infty}^{x_m}\psi_L''\psi_1' dx}_{\psi_L''\psi_1\big|_{\infty}^{x_m} - \int_{-\infty}^{x_m}\partial_x^2(\partial_x\psi_L)\psi_1 dx} + \int_{-\infty}^{x_m}\psi_L'\left\{-\psi_1 + 3c_0\psi_L^2\psi_1 - 5\psi_L^4\psi_1\right\}dx$$

ψ_L' and ψ_L'' are exponentially small at x_m and for $x \to -\infty \Rightarrow$ boundary terms are exponentially small and can be ignored at this order since they are already multiplied by ϵ. The remainder of the lhs is $\psi_1 \mathcal{L}_L \partial_x \psi_L$, which vanishes since $\mathcal{L}_L \partial_x \psi_L = 0 \Rightarrow$ we obtain a solvability condition.

To estimate and evaluate the integrals rewrite in terms of

$$s = e^{x-x_L} \qquad \text{and} \qquad L = x_R - x_L$$

$$dx = \frac{1}{s} ds \qquad \int_{-\infty}^{x_m} \ldots dx = \int_0^{e^{\frac{L}{2}}} \ldots \frac{1}{s} ds$$

$$\psi_L = \psi_0 \sqrt{\frac{1}{2}\left(1 + \frac{s - \frac{1}{s}}{s + \frac{1}{s}}\right)} = \psi_0 \sqrt{\frac{1}{2} \frac{1 + s^2 + s^2 - 1}{1 + s^2}} = \psi_0 \frac{s}{\sqrt{1+s^2}}$$

$$\psi_R - \psi_0 \to -\frac{1}{2}\psi_0 e^{-2(x_R - x)} = -\frac{1}{2}\psi_0 e^{-2x_R} e^{2(x-x_L)} e^{2x_L} = -\frac{1}{2}\psi_0 e^{-2L}\, s^2$$

$$\psi_R' \to -\psi_0 s \frac{ds}{dx} e^{-2L} = -\psi_0 s^2 e^{-2L} \qquad\qquad \psi_R'' \to -2\psi_0 s^2 e^{-2L}$$

$$\partial_s \psi_L = \psi_0 \left\{ \frac{1}{\sqrt{1+s^2}} - \frac{s^2}{\sqrt{1+s^2}^3} \right\} = \psi_0 \frac{1}{\sqrt{1+s^2}^3} \qquad \Rightarrow \qquad \psi_L' = \psi_0 \frac{s}{\sqrt{1+s^2}^3}$$

We get then

$$\begin{aligned} 0 \;=\; & \epsilon \partial_T x_L \psi_0^2 \int_0^{e^{\frac{L}{2}}} \frac{s^2}{(1+s^2)^3} \frac{1}{s} ds + \epsilon \partial_T x_R \psi_0^2 \int_0^{e^{\frac{L}{2}}} \frac{s}{\sqrt{1+s^2}^3} \left(-s^2\right) e^{-2L} \frac{1}{s} ds + \\ & + \psi_0^2 \int_0^{e^{\frac{L}{2}}} \frac{s}{\sqrt{1+s^2}^3} \left(-2s^2\right) e^{-2L} \frac{1}{s} ds + \\ & + \psi_0^2 \int_0^{e^{\frac{L}{2}}} \frac{s}{\sqrt{1+s^2}^3} \left\{ -1 + 3c_0 \psi_0^2 \frac{s^2}{1+s^2} - 5\psi_0^4 \frac{s^4}{(1+s^2)^2} \right\} \frac{-s^2}{2} e^{-2L} \frac{1}{s} ds \\ & + \epsilon c_1 \frac{1}{4} \underbrace{\psi_L^4 \Big|_{x=-\infty}^{x=x_m}}_{\psi_0^4 + h.o.t.} \end{aligned} \tag{58}$$

Analogously for $x > x_m$:

$$\begin{aligned} 0 \;=\; & \epsilon \partial_T x_L \int_{x_m}^{\infty} \psi_L' \psi_R' dx - \epsilon \partial_T x_R \int_{x_m}^{\infty} \psi_R'^2 dx + \int_{x_m}^{\infty} \psi_R' \psi_L'' dx + \\ & + \int_{x_m}^{\infty} \psi_R' \left\{-1 + 3c_0 \psi_R^2 - 5\psi_R^4\right\} (\psi_L - \psi_0)\, dx + \epsilon c_1 \int_{x_m}^{\infty} \psi_R' \psi_R^3 dx \end{aligned} \tag{59}$$

Rewrite these integrals in terms of

$$u = e^{x_R - x}$$

$$dx = -\frac{1}{u} du \quad \frac{du}{dx} = -u \qquad \psi_R = \psi_0 \frac{u}{\sqrt{1+u^2}} \qquad \partial_u \psi_R = \psi_0 \frac{1}{\sqrt{1+u^2}^3}$$

$$\psi_R' = -\frac{d\psi_R}{dx} = u\frac{d\psi_R}{du} = \psi_0 \frac{u}{\sqrt{1+u^2}^3}$$

$$\psi_L - \psi_0 \to -\frac{1}{2}\psi_0 u^2 e^{-2L} \qquad \psi_L' \to \psi_0 u^2 e^{-2L} \qquad \psi_L'' \to -2\psi_0 u^2 e^{-2L}$$

Since

$$\int_{x_m}^{\infty} \ldots dx \to \int_{e^{x_R - x_m}}^{0} \ldots \left(-\frac{1}{u}\right) du = \int_0^{e^{\frac{L}{2}}} \ldots \frac{1}{u} du$$

each integral in the expression for $x > x_m$ has a corresponding integral for $x < x_m$ and their magnitudes are the same.

Add (59) and (58)

$$\begin{aligned} 0 &= \epsilon\left(\partial_T x_L - \partial_T x_R\right)\psi_0^2 \left\{ \int_0^{e^{\frac{L}{2}}} \frac{s^2}{(1+s^2)^3}\frac{1}{s} ds - \int_0^{e^{\frac{L}{2}}} \frac{s}{\sqrt{1+s^2}^3}\left(-s^2\right) e^{-2L}\frac{1}{s} ds \right\} + \\ &\quad + 2e^{-2L}\psi_0^2 \int_0^{e^{\frac{L}{2}}} \frac{s}{\sqrt{1+s^2}^3}\left(-2s^2\right)\frac{1}{s} ds + \\ &\quad + 2e^{-2L}\psi_0^2 \int_0^{e^{\frac{L}{2}}} \frac{s}{\sqrt{1+s^2}^3}\left\{-1 + 3c_0\psi_0^2\frac{s^2}{1+s^2} - 5\psi_0^4\frac{s^4}{(1+s^2)^2}\right\}\frac{-s^2}{2}\frac{1}{s} ds + \\ &\quad + \epsilon c_1 \frac{1}{2}\psi_0^4 \end{aligned}$$

For large s all integrands decay at least as $\frac{1}{s} \Rightarrow$ the integrals are at most $\mathcal{O}(\ln s) = \mathcal{O}(L)$:

- We can therefore ignore the second integral with respect to the first integral in the first term
- ϵ must be exponentially small in L to balance the terms

Relevant integrals:

$$\begin{aligned} \int_0^{e^{\frac{L}{2}}} \frac{s}{(1+s^2)^3} ds &= \frac{1}{4}\left(1 - \frac{1}{(1+e^L)^2}\right) \\ \int_0^{e^{\frac{L}{2}}} \frac{s^2}{\sqrt{1+s^2}^3} ds &= \frac{1}{2}L + \ln 2 - 1 + \mathcal{O}(e^{-L}) \\ \int_0^{e^{\frac{L}{2}}} \frac{s^4}{\sqrt{1+s^2}^3} ds &= \frac{1}{2}L + \ln 2 - \frac{4}{3} + \mathcal{O}(e^{-L}) \\ \int_0^{e^{\frac{L}{2}}} \frac{s^6}{\sqrt{1+s^2}^3} ds &= \frac{1}{2}L + \ln 2 - \frac{23}{15} + \mathcal{O}(e^{-L}) \end{aligned}$$

Thus

$$\partial_T L = -16\frac{e^{-2L}}{\epsilon} + 2\sqrt{3}c_1 \tag{60}$$

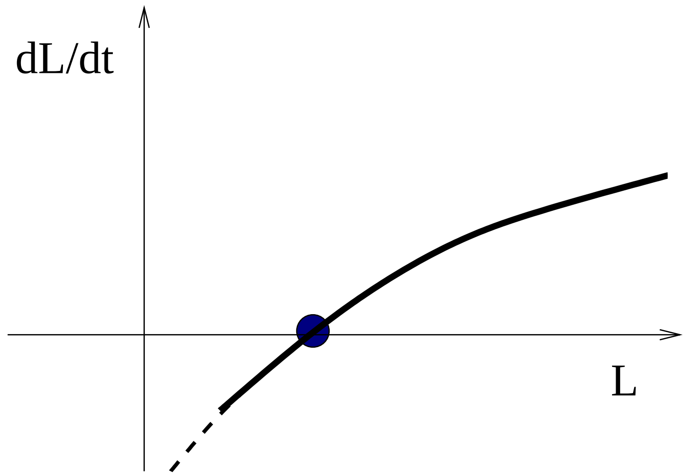


Figure 31: Dependence of growth rate of domain on domain size L.

Notes:

- Interaction
  - attactive $\Rightarrow$ fixed point $L = L_0$ for $c > c_0$, i.e. without interaction the fronts would be drifting apart
  - decays with distance
  - $\Rightarrow$ localized state is *unstable*:
    for $L > L_0$ the attraction is insufficient and the fronts drift apart
    for $L < L_0$ the attraction is too strong and the fronts annihilate each other.
- This localized state corresponds to a *critical droplet* in a first-order phase transition
  - $\psi = 0$ corresponds to the gas phase, say, and $\psi = \psi_0$ to the liquid phase
  - $L = 0$ corresponds to a pure gas phase, $L \to \infty$ to a pure liquid phase.
  - the localized state separates these two stable phases $\Rightarrow$ if there is only one such localized state it has to be unstable.
- the interaction between the fronts is exponential and *monotonic*
- in a more general system the interaction could be non-monotonic, e.g.,

$$\frac{dL}{dt} = a \cos \kappa L \, e^{-\alpha L} + bc_1$$

  then there are multiple localized states, alternating stable and unstable

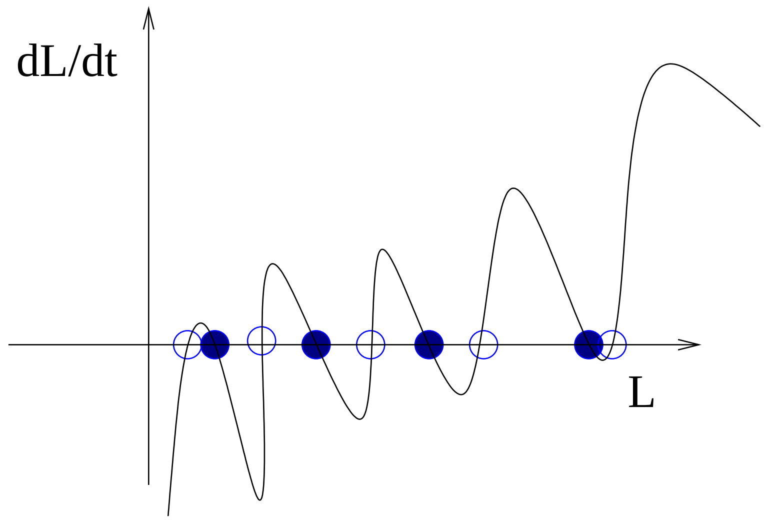


Figure 32: Oscillatory interaction between fronts would allow multiple localized states, stable and unstable.

- for oscillatory interaction fronts can ‘lock’ into each other at multiple positions, arrays of fronts can be spatially chaotic [3].

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[13]**Integration by parts of Laplace integrals** $I(x) = \int_a^b f(t)e^{x\phi(t)}dt$
If $\phi'(t) \neq 0$ for all $t \in [a,b]$ one can again employ integration by parts using $e^{x\phi} = \frac{1}{x}\frac{1}{\phi'}\frac{d}{dt}e^{x\phi}$

$$I(x) = f(t)\frac{1}{x}\frac{1}{\phi'(t)}e^{x\phi(t)}\Big|_a^b - \frac{1}{x}\int_a^b \frac{d}{dt}\left(\frac{f(t)}{\phi'(t)}\right)e^{x\phi(t)}dt$$

This is useful if the resulting integral is small compared to the boundary terms, because then

$$I(x) \sim \frac{1}{x}\frac{f(b)}{\phi'(b)}e^{x\phi(b)} - \frac{1}{x}\frac{f(a)}{\phi'(a)}e^{x\phi(a)}$$

For instance:
For $\phi'(t) > 0$ for $t \in [a,b]$, $f(b) \neq 0$ (or $\phi'(t) < 0$ for $t \in [a,b]$, $f(a) \neq 0$), and $f(t)$ and $\phi(t)$ sufficiently regular one can estimate the integral as follows

$$\begin{aligned}\left|\int_a^b \frac{d}{dt}\left(\frac{f(t)}{\phi'(t)}\right)e^{x\phi(t)}dt\right| &\leq \left|\int_a^{b-\Delta}\frac{d}{dt}\left(\frac{f(t)}{\phi'(t)}\right)e^{x\phi(t)}dt\right| + \left|\int_{b-\Delta}^b \frac{d}{dt}\left(\frac{f(t)}{\phi'(t)}\right)e^{x\phi(t)}dt\right| \\ &\leq e^{x(\phi(b)-\Delta)}\left|\int_a^{b-\Delta}\frac{d}{dt}\left(\frac{f(t)}{\phi'(t)}\right)dt\right| + \Delta \max_{t\in[b-\Delta,b]}\left|\frac{d}{dt}\left(\frac{f(t)}{\phi'(t)}\right)\right| e^{x\phi(b)}\end{aligned}$$

Now choose Δ sufficiently small to make the second integral small but not too small to make the first integral small for large x, $\Delta = x^{-\alpha}$, with $0 < \alpha < 1$. Then

$$\left|\int_a^b \frac{d}{dt}\left(\frac{f(t)}{\phi'(t)}\right)e^{x\phi(t)}dt\right| \leq e^{-x^{1-\alpha}}e^{x\phi(b)}\left|\int_a^{b-\Delta}\frac{d}{dt}\left(\frac{f(t)}{\phi'(t)}\right)dt\right| + x^{-\alpha}\max_{t\in[b-\Delta,b]}\left|\frac{d}{dt}\left(\frac{f(t)}{\phi'(t)}\right)\right| e^{x\phi(b)}$$

Both terms are small compared to the dominant boundary term.
More generally one can show the the boundary term at $t = b$ dominates the integral for $\Re(\phi(t)) < \Re(\phi(b))$ for $a \leq t < b$ and $\Re(\phi'(b)) \neq 0$ and $f(b) \neq 0$.

A Forced Duffing Oscillator II

Consider now explicitly the Duffing oscillator with forcing near $3\omega_0$

$$\ddot{y} + \epsilon\beta\dot{y} + \omega_0^2 y + \epsilon\alpha y^3 = f\cos\omega t$$

with initial conditions $y(0) = \delta$, $\dot{y}(0) = 0$.

To get an overview of possible resonances consider first general ω and $\mathcal{O}(1)$-forcing.

$\mathcal{O}(1)$:

$$\ddot{y}_0 + \omega_0^2 y_0 = f\cos\omega t$$

$$y_0(t) = \delta\cos\omega_0 t + \frac{f}{\omega_0^2 - \omega^2}\left(\cos\omega t - \cos\omega_0 t\right)$$

$\mathcal{O}(\epsilon)$:

$$\ddot{y}_1 + \omega_0^2 y_1 = -\beta\dot{y}_0 - y_0^3 \equiv f_1$$

Insert y_0 into f_1 using

$$K \equiv \frac{f}{\omega_0^2 - \omega^2}$$

$$\begin{aligned}
f_1 &= \beta\left\{\omega_0\sin\omega_0 t + K\left[\omega\sin\omega t - \omega_0\sin\omega_0 t\right]\right\} - \left\{K\cos\omega t + (1-K)\cos\omega_0 t\right\}^3 \\
&= \beta\omega_0\left(1-K\right)\sin\omega_0 t + \beta\omega\sin\omega t - \\
&\quad -\left\{K^3\cos^3\omega t + 3K^2\left(1-K\right)\cos^2\omega t\cos\omega_0 t + 3K\left(1-K\right)^2\cos\omega t\cos^2\omega_0 t + \left(1-K\right)^3\cos^3\omega_0 t\right\}
\end{aligned}$$

use

$$\begin{aligned}
\cos^3\omega t &= \frac{3}{4}\cos\omega t + \frac{1}{4}\cos 3\omega t \\
\cos^2\omega t\cos\omega_0 t &= \frac{1}{4}\cos\left(\left(2\omega - \omega_0\right)t\right) + \frac{1}{4}\cos\left(\left(2\omega + \omega_0\right)t\right) + \frac{1}{2}\cos\omega_0 t \\
\cos\omega t\cos^2\omega_0 t &= \frac{1}{4}\cos\left(\left(\omega - 2\omega_0\right)t\right) + \frac{1}{4}\cos\left(\left(\omega + 2\omega_0\right)t\right) + \frac{1}{2}\cos\omega t \\
\cos^3\omega_0 t &= \frac{3}{4}\cos\omega_0 t + \frac{1}{4}\cos 3\omega_0 t
\end{aligned}$$

Note:

- if one uses complex exponentials one does not have to deal with trig identities
- the cubic nonlinearity generates all frequencies that can be obtained by adding the frequencies of three of the four terms in

$$\left\{e^{i\omega_0 t} + e^{-i\omega_0 t} - \left(e^{i\omega t} + e^{-i\omega t}\right)\right\}^3$$

Each term in f_1 with frequency $\tilde{\omega}$ drives a specific part of the particular solution for y_1

$$\cos\tilde{\omega}t \quad \rightarrow \quad \frac{1}{-\tilde{\omega}^2 + \omega_0^2}$$

i.e.

$$\begin{array}{rcl}
\cos\left(\left(2\omega-\omega_0\right)t\right) & \rightarrow & \dfrac{1}{4\omega^2-4\omega\omega_0}=\dfrac{1}{4}\dfrac{1}{\omega\left(\omega-\omega_0\right)} \\
\cos\left(\left(2\omega+\omega_0\right)t\right) & \rightarrow & \dfrac{1}{4\omega^2+4\omega\omega_0}=\dfrac{1}{4}\dfrac{1}{\omega\left(\omega+\omega\right)} \\
\cos\left(\left(2\omega_0-\omega\right)t\right) & \rightarrow & \dfrac{1}{3\omega_0^2-4\omega_0\omega+\omega^2}=\dfrac{1}{\left(\omega-\omega_0\right)\left(\omega-3\omega_0\right)} \\
\cos\left(\left(2\omega_0+\omega\right)t\right) & \rightarrow & \dfrac{1}{3\omega_0^2+4\omega_0\omega+\omega^2}=\dfrac{1}{\left(\omega+\omega_0\right)\left(\omega+3\omega_0\right)} \\
\cos 3\omega t & \rightarrow & \dfrac{1}{\left(\omega_0-3\omega\right)\left(\omega_0+3\omega\right)}
\end{array}$$

With this forcing one can drive resonances at

$$\begin{aligned}
\omega &= 0 \\
\omega &= \omega_0 \\
\omega &= 3\omega_0 \\
\omega &= \frac{1}{3}\omega_0
\end{aligned}$$

As our symmetry analysis showed, depending on the ratio ω/ω_0 the perturbation analysis leads to different amplitude equations.

A.1 3:1 Forcing of Duffing Oscillator

Consider

$$\omega_0=1 \qquad \omega=3\left(1+\Omega\right)$$

with $\mathcal{O}(1)$ forcing (cf. in Sec.2.2.2 where the forcing was $\mathcal{O}(\epsilon)$).

As in the case $\omega\approx\omega_0$ write[14]

$$y(t,T)=y_0(\psi,T)+\epsilon y_1(\psi,T)+\ldots \qquad \psi=t+\Omega T-\phi(T)$$

with initial conditions

$$y(0,0)=\delta \qquad \frac{dy}{dt}=0$$

Again the derivative ∂_t^2 is expanded as

$$\begin{aligned}
\frac{\partial}{\partial t} &= \left(1+\epsilon\left(\Omega-\partial_T\phi\right)\right)\partial_\psi+\epsilon\partial_T \\
\frac{\partial^2}{\partial t^2} &= \partial_\psi^2+\epsilon\left\{2\left(\Omega-\partial_T\phi\right)\partial_\psi^2+2\partial_{\psi T}^2\right\}+\mathcal{O}(\epsilon^2)
\end{aligned}$$

and one obtains

[14] It might be easier to use complex exponentials as in the symmetry analysis.

$\mathcal{O}(1)$:

$$\partial_\psi^2 y_0 + y_0 = f\cos\left(3\left(\psi+\phi\right)\right)$$

with

$$y_0(0,0)=\delta \qquad \partial_t y_0(0,0)=0$$

yields

$$y_0(\psi,T) = R(T)\cos\psi + K\cos\left(3\left(\psi+\phi\right)\right) \quad \text{with} \quad K=-\frac{f}{8}$$

$$R(0)\cos\psi(0)=\delta \quad R(0)\sin\psi(0)=0 \quad \Rightarrow \quad R(0)=\delta \quad \psi(0)=0$$

$\mathcal{O}(\epsilon)$:

$$\partial_\psi^2 y_1 + y_1 = -\beta\partial_\psi y_0 - \left\{2\left(\Omega-\phi'\right)\partial_\psi^2 + 2\partial_{\psi T}^2\right\} y_0 - y_0^3 \equiv f_1$$

$$\begin{aligned}
f_1 &= \beta\left[R\sin\psi + 3K\sin\left(3\left(\psi+\phi\right)\right)\right] + 2\left(\Omega-\phi'\right)\left[R\cos\psi + 9K\cos\left(3\left(\psi+\phi\right)\right)\right] \\
&\quad +2\left[R'\sin\psi + 9\phi' K\cos\left(3\left(\psi+\phi\right)\right)\right] \\
&\quad -R^3\cos^3\psi - 3R^2K\cos^2\psi\cos\left(3\left(\psi+\phi\right)\right) - 3RK^2\cos\psi\cos^2\left(3\left(\psi+\phi\right)\right) - K^3\cos^3\left(3\left(\psi+\phi\right)\right) \\
&= \beta\left[R\sin\psi + 3K\sin\left(3\left(\psi+\phi\right)\right)\right] + 2\left(\Omega-\phi'\right)R\cos\psi + 18\Omega K\cos\left(3\left(\psi+\phi\right)\right) + 2R'\sin\psi \\
&\quad -\frac{3}{4}R^3\cos\psi - \frac{3}{4}R^2K\underbrace{\cos\left(2\psi-3\left(\psi+\phi\right)\right)}_{\cos\psi\cos3\phi-\sin\psi\sin3\phi} - \frac{3}{2}RK^2\cos\psi + \text{higher harmonics}
\end{aligned}$$

The solvability condition results then in

$$2R' + \beta R + \frac{3}{4}R^2K\sin3\phi = 0 \tag{61}$$

$$2\left(\Omega-\phi'\right)R - \frac{3}{4}R^3 - \frac{3}{4}R^2K\cos3\phi - \frac{3}{2}RK^2 = 0 \tag{62}$$

Notes:

- the result agrees in its form with that obtained using symmetry and scaling arguments (cf. (40))
- compared to the general equation not only the term R^3 but also the term RK^2 are missing in the equation for the amplitude R:
  the forcing is not modifying the linear damping of the amplitude

One fixed point is at $R_\infty^{(1)}=0$. It corresponds to the solution

$$y^{(1)} = K\cos\left(3\left(\psi+\phi\right)\right) + \mathcal{O}(\epsilon)$$

which does not excite the resonance near ω_0.

For $R_\infty \neq 0$ one has

$$\begin{aligned}
\beta &= -\frac{3}{4}R_\infty K\sin3\phi_\infty \\
2\Omega - \frac{3}{4}R_\infty^2 - \frac{3}{2}K^2 &= \frac{3}{4}R_\infty K\cos3\phi_\infty
\end{aligned}$$

Thus

$$\beta^2 + \left(2\Omega - \frac{3}{4}R_\infty^2 - \frac{3}{2}K^2\right)^2 = \frac{9}{16}R_\infty^2 K^2$$

$$\frac{9}{16}R_\infty^4 + R_\infty^2\left(2\frac{3}{4}\left(\frac{3}{2}K^2 - 2\Omega\right) - \frac{9}{16}K^2\right) + \left(2\Omega - \frac{3}{2}K^2\right)^2 + \beta^2 = 0$$

$$R_\infty^4 + R_\infty^2\left(3K^2 - \frac{16}{3}\Omega\right) + \left(\frac{8}{3}\Omega - 2K^2\right)^2 + \frac{16}{9}\beta^2 = 0$$

We need a real and positive solution of this bi-quadratic equation.
Discriminant

$$\begin{aligned}\Delta &= \left(3K^2 - \frac{16}{3}\Omega\right)^2 - 4\left(\frac{8}{3}\Omega - 2K^2\right)^2 - \frac{64}{9}\beta^2 \\ &= -7K^4 + \frac{32}{3}\Omega K^2 - \frac{64}{9}\beta^2\end{aligned}$$

Solving for Ω we find that to have real solutions ($\Delta \geq 0$) we need

$$\Omega \geq \frac{21}{32}K^2 + \frac{2}{3}\frac{\beta^2}{K^2} \equiv \Omega_0(K)$$

For $\Omega \geq \Omega_0(K)$ we have

$$\begin{aligned}-\left(3K^2 - \frac{16}{3}\Omega\right) &\geq \frac{16}{3}\frac{21}{32}K^2 - 3K^2 + \frac{32}{9}\frac{\beta^2}{K^2} = \\ &\quad \left(\frac{21}{6} - 3\right)K^2 + \frac{32}{9}\frac{\beta^2}{K^2} > 0\end{aligned}$$

Moreover, $\Delta \leq \left(3K^2 - \frac{16}{3}\Omega\right)^2$.
Thus, if $\Delta > 0$ both solutions $R_{1,2}^2$ are positive and we conclude

Ω	$>$	$\Omega_0(K)$	2 positive solutions
Ω	$<$	$\Omega_0(K)$	no positive solution

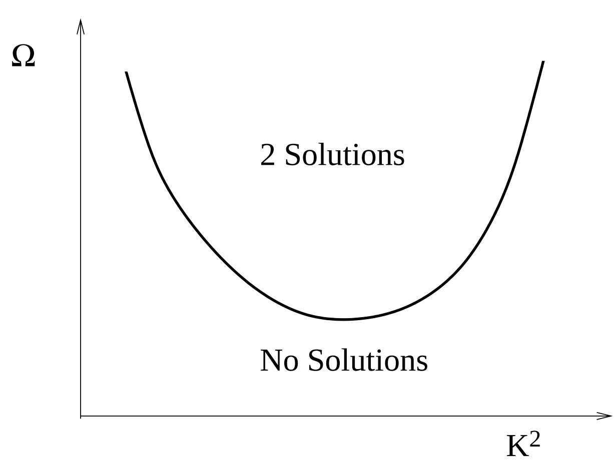

Figure 33: Phase diagram: existence of solutions as a function of Ω and K. The line marks $\Delta = 0$.

For fixed Ω non-trivial solutions $R > 0$ exist only in a range $K_{min} \leq K \leq K_{max}$

$$K^2_{min,max} = \frac{16}{21}\left\{\Omega \pm \sqrt{\Omega^2 - \frac{7}{4}\beta^2}\right\}$$

At K_{min} and K_{max} two solutions come into existence or disappear in a saddle-node bifurcation. The amplitudes at these points are

$$R^2_{min,max} = \frac{1}{2}\left\{\frac{16}{3}\Omega - 3K^2_{min,max}\right\} > 0$$

Note:

- These solutions come into existence at finite amplitude and *do not* bifurcate off the trivial solution $R^{(1)} = 0$.

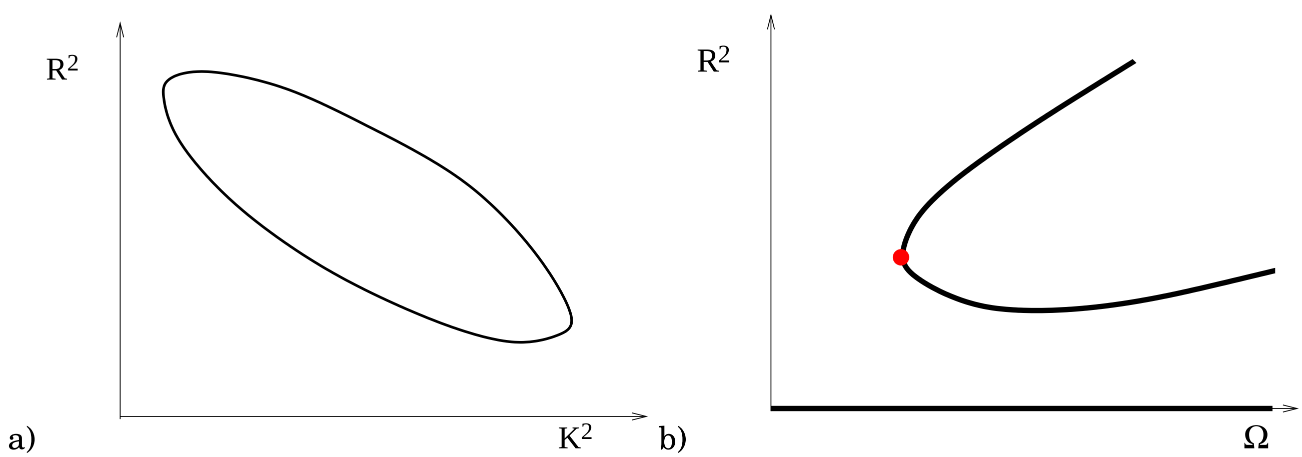

Figure 34: Bifurcation diagrams arising from cuts at fixed Ω (a) and at fixed K (b) in the phase diagram shown in Fig.33.

Note:

- in all the examples here the amplitude equation arose at $\mathcal{O}(\epsilon)$, i.e. at the first order in the perturbation. In general, the amplitude equation can also arise at higher order, with the lower orders being solvable without implying any condition on the amplitude.

Slightly different version for $I(x) = \int_0^{\pi/2} e^{-x\sin^2 t} dt$:
Since $\sin t = t - \frac{1}{6}t^3 + \frac{1}{120}t^5 \ldots$ one might expect

$$t - \frac{1}{6}t^3 < \sin t < t$$

This can be shown to be true for all $t > 0$. Estimate now

$$\left| x\sin^2 t - xt^2 \right| = x\left|\sin t + t\right| \left|\sin t - t\right| < x\, 2t\, \frac{1}{6}t^3 = \frac{1}{3}xt^4$$

Thus, for $t = x^{-\alpha}$ with $\alpha > \frac{1}{4}$ one has

$$\left| x\sin^2 t - xt^2 \right| < \frac{1}{3}x^{1-4\alpha} \to 0 \qquad x \to \infty$$

To leading order one then has

$$e^{-x\sin^2 t} \approx e^{-xt^2} \qquad t \leq x^{-\alpha}, \quad x \to \infty$$

and also

$$\int_0^{x^{-\alpha}} e^{-x\sin^2 t} dt \sim \int_0^{x^{-\alpha}} e^{-xt^2} dt \qquad x \to \infty$$

What about the integrals from $x^{-\alpha}$ to ϵ? They are both exponentially small compared to $I(x)$:

$$\int_{x^{-\alpha}}^{\epsilon} e^{-x\sin^2 t} dt < e^{-x\sin^2 x^{-\alpha}} \epsilon$$

$$\int_{x^{-\alpha}}^{\epsilon} e^{-xt^2} dt < e^{-x^{1-2\alpha}} \epsilon$$

For these estimates to be meaningful we need to choose $\alpha < \frac{1}{2}$.