Lecture Note Sketches

Interdisciplinary Nonlinear Dynamics 438

Hermann Riecke

Engineering Sciences and Applied Mathematics

h-riecke@northwestern.edu

Fall 2003

Contents

1 Introduction

Dynamics arise in many systems

- Mechanics: vibrations, coupled structural elements

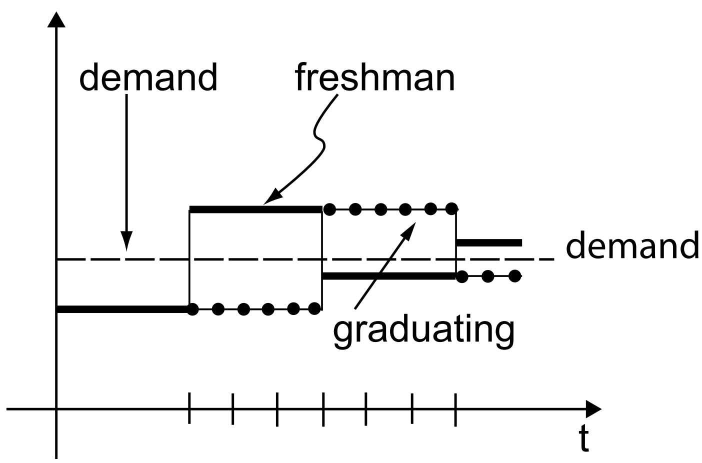
  driven by external force ⇒ complex behavior even in simple driven pendulum

  planetary motion: n-body problem
  planetary system stable?

- Fluids: Rayleigh-Bénard convection → WWW
  transitions between different spatially periodic or disordered states
  transition to turbulence in pipe flow: sudden change from laminar to turbulent flow

- Chemical systems: Belousov-Zhabotinsky reaction: spiral waves → WWW
  flames → WWW

- Economics:
  pig cycle, delay between price and investment
  job market: delay by education
  increase demand: more freshmen
  → 4 years later too many job applicants, reduced chances
  → fewer freshmen



- Dynamics in the heart:

  contraction = excitation of muscle (wave at ball game, excitable)
  propagates like a wave
  defects in waves: spirals, diseases
  fibrillations

  WWW: rabbit heart picture

Nonlinear Systems:

- changes in qualitative behavior

  non-smooth dependence on parameters:

  WWW: Taylor vortex flow: torque & flow pictures

- multiplicity of solutions: hysteresis

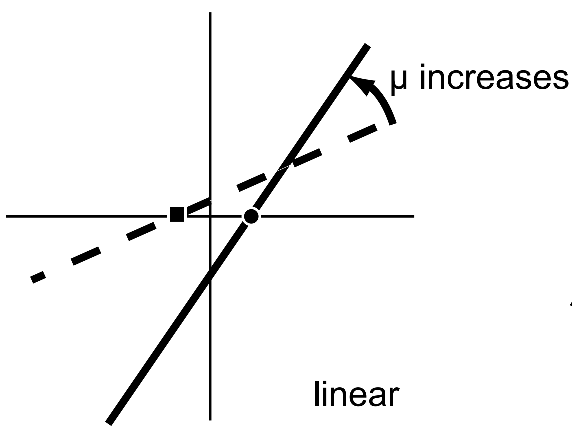
  rolls vs. spiral-defect chaos convection

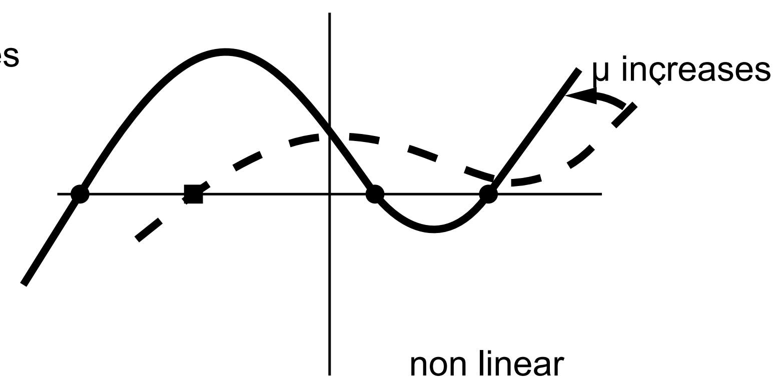
- chaotic dynamics
  many frequencies, coexisting (unstable) periodic solutions

Simple illustration: **linear vs. nonlinear**

Consider

$$f(x, \mu) = 0$$





..(essentially) always
1 unique solution
quantitative but
no qualitative change

..# of solutions can change with μ.
solutions appear and disappear
quantitative and **qualitative** changes

Example: TVF, torque & flow pictures, multiplicity, phase diagram (WWW)

Mathematical Description of Dynamics:

Differential Equations & Maps

ordinary differential equations (ODE):	coupled oscillators
partial differential equations (PDE):	heat eqn., reaction diffusion eqn. Navier-Stokes
Maps:	stroboscopic description of (near) oscillatory behavior

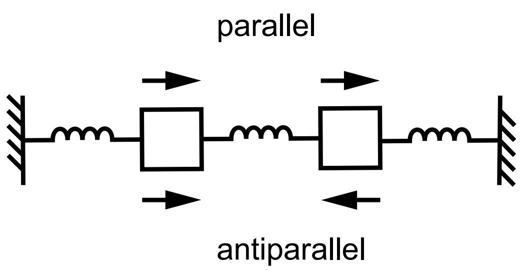
Linear:

$$\ddot{\mathbf{u}} = \mathbf{A}\mathbf{u}$$

Superposition of solutions to get **general solution** for **all** initial conditions

Notation: $\dot{u} \equiv \frac{du}{dt}$, temporal derivative, $\mathbf{u} = (u_1, u_2, ..., u_n)$

Example: 2 masses with springs, u = longitudinal motion of masses
parallel and anti-parallel mode (lower and higher frequency)
any motion = parallel + anti-parallel



PDE for continuous string:

$$\ddot{u} \;=\; u'' \quad 0 < x < L$$

Fourier expansion

$$u(x,t) \;=\; \sum_{n=-\infty}^{\infty} u_n(t)\, e^{i\frac{2\pi}{L}nx}$$

Notation: $u' = \frac{du}{dx}$, spatial derivative

Eigenmodes: each u_n satisfies:

$$\ddot{u}_n = -(\frac{2\pi n}{L})^2 u_n \rightarrow u_n(t) = u_n(0)e^{i\frac{2\pi n}{L}t}$$

Different modes do **not** interact

Nonlinear: no superposition, different modes **do interact!**

$$\ddot{u} = u'' + \underbrace{u^2}_{\sum u_n u_m e^{i\frac{2\pi}{L}(n+m)x}}$$

u^2 generates new wave numbers: couples n & m to $n+m$ and to $n-m$

Any interaction between different objects (A and B) implies nonlinearity:
evolution of A depends on state of A and that of B

→ Cannot build general solution from a set of basic solutions by simply adding them

→ in general: cannot find exact solutions: **HARD**.

Numerical Solution:

- confirms the model/basic equations:
  of great interest if model has not been established, e.g., chemical oscillations, heart muscle
- gives quantitative details for **specific** values of system parameters:
  these details may not be accessible in experiments: 3d fluid flow, turbulent, chemical concentrations of each species

We will also use numerical methods

Overview and Insight: Qualitative Analysis

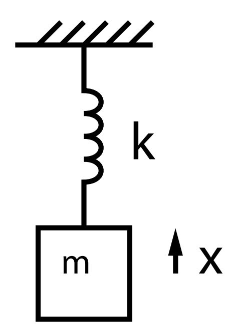
- change in behavior as system parameters are changed
  transitions between **qualitatively** different states
- analytical techniques for transitions
  approximations near transition points
- visualization: **geometry of dynamics**, phase space
- overview of **all** possible behaviors

Example: mass-spring-system

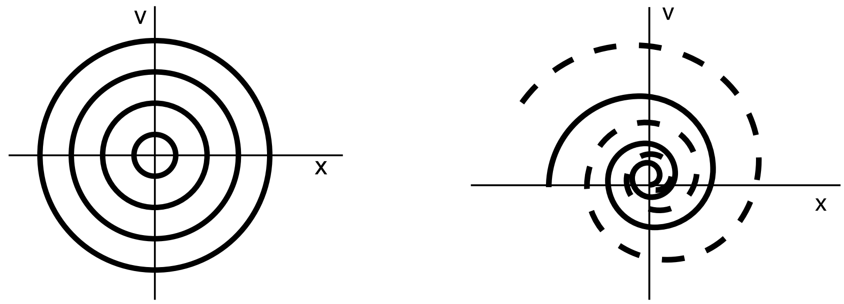
$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

we will write all differential equations as first-order systems:

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= -\frac{k}{m}x \end{aligned}$$



without friction with friction



for **any** initial condition periodic motion

with friction relaxation to fixed point

complete overview of all possible solutions
(here linear $\rightarrow$ not much going on.)

Conservative Systems

- almost all different initial condition lead to different states

Dissipative Systems

- range of initial conditions leads to same state: **attractors**
- **transitions**: **qualitative** change in attractors

We will mostly focus on dissipative systems.

2 1-d Flow

2.1 Flow on the Line

Any first-order differential equation with constant coefficients,

$$\dot{x} = f(x),$$

can be solved exactly for any $f(x)$ (by separation of variables).

$$\int_{x_0}^{x} \frac{dx}{f(x)} = t - t_0$$

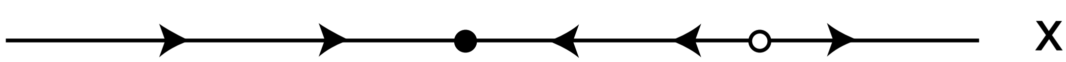
Example: $\dot{x} = \sin x$

$$t = \int \frac{dx}{\sin x} = \int \csc x dx$$
$$t = -\ln|\csc x + \cot x| + C$$

Now what? What have we learned?

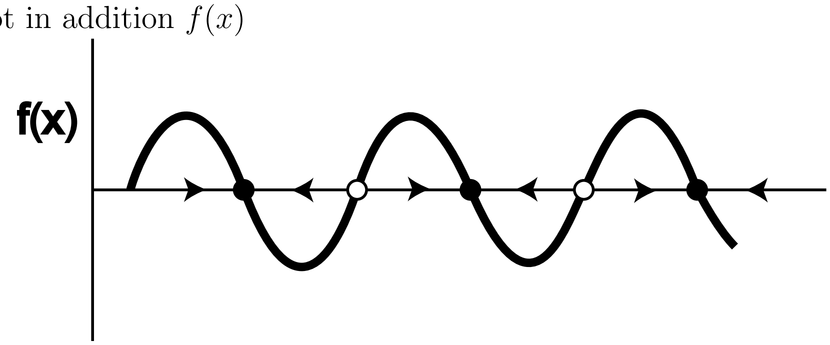
Even if we could solve for x, would we have an overview of the behavior of system for arbitrary initial conditions?

Geometrical picture: **phase space** (or phase line in 1 dimension)



$\dot{x} = f(x)$ defines a *flow* in phase space or a *vector field*

For 1d: plot in addition $f(x)$



Conclude: any i.c. ends up in one of the **fixed points** at $x_n = (2n+1)\pi$.
Fixed points are **stagnation point** of the flow

Stability:

- flow *into* $x_n = (2n+1)\pi$: *stable*
- flow *out of* $x_n = 2n\pi$: *unstable*

Of course: for **quantitative results** ('numbers') we need the detailed solution

Example: Population Growth with Limited Resources

$$\begin{aligned} N &= \#\text{ of animals} \\ \dot{N} &= g(N)N \\ g(N) &= \text{ net birth/death rate} \end{aligned}$$

Limited food/space:
births decrease, deaths increase with increasing N

$$g = \alpha - \beta N$$

Logistic growth model

$$\dot{N} = \alpha N - \beta N^2$$

Make dimensionless:

$$[\alpha] = \tfrac{1}{s} \qquad\qquad [\beta] = \frac{1}{s}\frac{1}{\#}$$

$\frac{1}{\alpha}$ characteristic growth time $\qquad \dfrac{\alpha}{\beta}$ characteristic population site

Introduce

$$t = \frac{1}{\alpha}\tau \qquad\qquad N = \frac{\alpha}{\beta}n$$

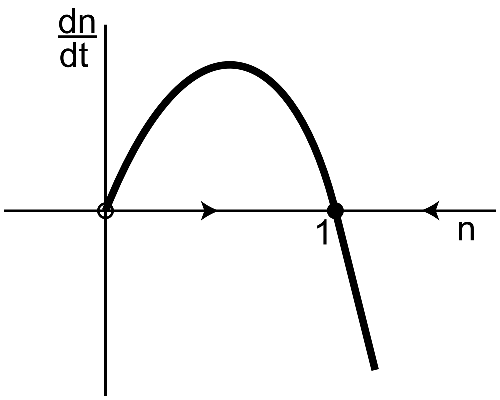
Question: If population goes to some equilibrium, what size would you expect?

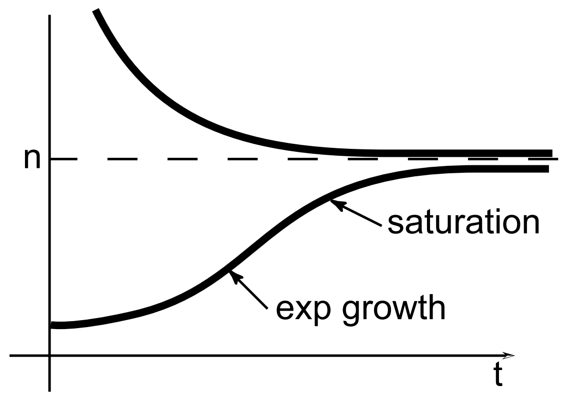
$\frac{\alpha}{\beta}$ is the only characteristic size after initial condition is forgotten $\Rightarrow$ expect N $\rightarrow \frac{\alpha}{\beta}$

$$\partial_\tau n = n - n^2$$

Could solve by partial fraction

Instead, consider phase space:





fixed points: $n = 0$, $n = 1$

flow indicates: $n = 0$ unstable, $n = 1$ stable

indeed: **all i.c.** go to $N = \frac{\alpha}{\beta}$.

Linear Stability:

Study effect of small perturbation away from fixed point

Linearize around fixed points:

$$\begin{aligned}
& n &=& \; n_0 + \epsilon n_1(\tau) \qquad \epsilon \ll 1 \\
\mathcal{O}(\epsilon^0): \quad & 0 &=& \; n_0 - n_0^2 \\
& &\Rightarrow& \; n_0 = 1 \qquad \text{or} \qquad n_0 = 0 \\
\mathcal{O}(\epsilon^1): \quad & \partial_\tau n_1 &=& \; n_1 - 2n_0 n_1 = (1 - 2n_0) n_1 \\
& &\Rightarrow& \; n_1 \propto e^{-(1-2n_0)\tau}
\end{aligned}$$

$$\begin{aligned}
n_0 = 1 \Rightarrow \quad & 1 - 2n_0 < 0 \quad \text{stable} \\
n_0 = 0 \Rightarrow \quad & 1 - 2n_0 > 0 \quad \text{unstable}
\end{aligned}$$

more generally

$$\dot{x} = f(x)$$

stability:

$$\begin{aligned} x &= x_0 + \epsilon x_1 \\ \dot{x}_1 &= f'(x_0) x_1 \end{aligned}$$

stable: $f'(x_0) < 0$
unstable: $f'(x_0) > 0$

Note: for coupled systems $f'(x)$ is replaced by Jacobian matrix:
eigenvalues determine stability.

Discussion of Logistic Growth Model

Examples:
(cf. figures on WWW)

growth of yeast: model seems quite good

beetles: early times O.K., no stable saturation

Assumptions made:

- N and birth/death processes are continuous

  $\Rightarrow$ o.k. for population with large N

  smaller N: expect jumps, fluctuations

- density not too large: ignored N^3 etc.

  - if all neglected terms are saturating: no qualitative change if terms are included
  - if low-order terms are destabilizing:
    need to include higher-order terms to avoid blow-up
    could get *bistability* between 2 populations if terms are included

- growth rate depends only on N at the **same time**: **no delay**

  not satisfied for animals with more complex life cycle

  eggs: hatching and laying new eggs ('new births') much later

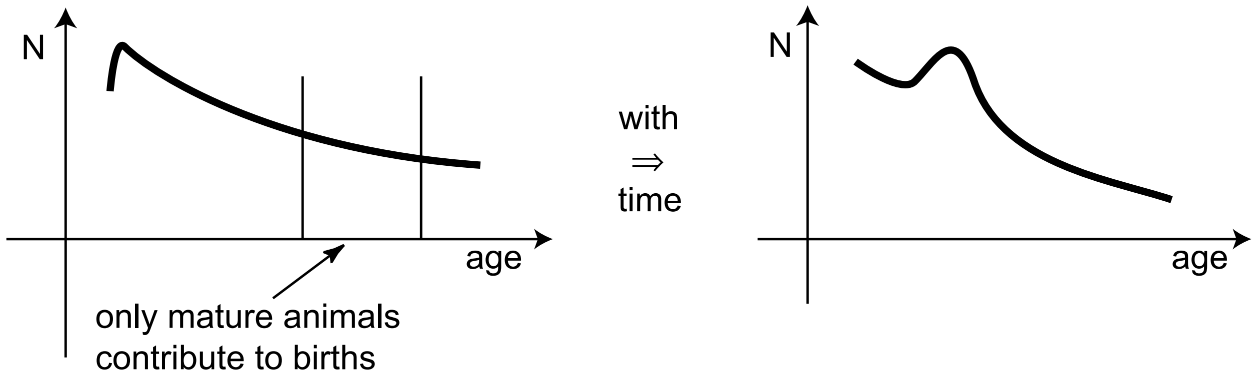
  $\Rightarrow$ **overshoot** possible:
  although little food/space many births resulting from **earlier** high-supply times

  expect oscillations? (HW)

- stable age distribution

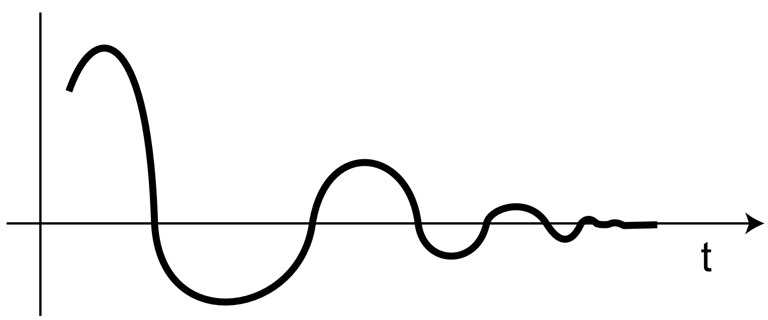
Effect of age distribution:

increased birthrate increases number of *young* animals
→ peak in distribution travels through age distribution



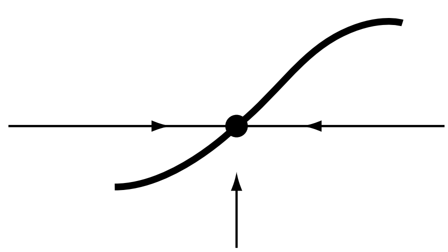
2.1.1 Impossibility of Oscillations:

Can the solution approach fixed point via **oscillations**?



No!

graphically:



to get to other side need to **cross** fixed point

$\rightarrow$ system evolves monotonically between fixed points

More general concept: potential

$$\dot{x} = f(x) = -\frac{dV}{dx} \quad \text{with } V = -\int f(x)dx$$

Consider:

$$\frac{dV}{dt} = \frac{dV}{dx}\dot{x} = -(\frac{dV}{dx})^2 \leq 0$$

V is non-increasing $\Rightarrow V$ cannot return to previous value

$$\frac{dV}{dt} = 0 \qquad \Rightarrow \qquad \frac{dV}{dx} = 0 \qquad \Rightarrow \qquad \dot{x} = 0 \qquad \text{fixed point}$$

x either goes to fixed point or diverges to $-\infty$ (if V is not bounded from below).

Compare: mechanical system is overdamped limit

$$m\ddot{x} = -\beta\dot{x} + F(x)$$

for very small mass (no inertia)

$$\dot{x} = \frac{1}{\beta}F(x)$$

Overshoot requires inertia, 2nd derivative.

Note: The concept of the potential can be extended to higher-dimensional systems

2.2 Existence and Uniqueness

So far we assumed we **always** get a **unique** solution for all times:

- at any time 'we know where to go'
- we can continue this forever

Solutions to

$$\dot{x} = f(x)$$

1. do not have to exist for all times:
   for given initial condition solution may cease to exist beyond some time
2. do not have to be unique:
   same initial condition can lead to different states later.

1. Existence

solution can disappear by becoming infinite

if this happens in *finite* time then there is no solution beyond that time

Example:

$$\dot{x} = +x^{\alpha} \qquad \text{with} \qquad x(0) = x_0 > 0$$

$$\begin{aligned} \int x^{-\alpha}\,dx = \int \frac{dx}{x^{\alpha}} &= t + C \\ \frac{1}{1-\alpha} x^{1-\alpha} &= t + C \end{aligned}$$

initial conditions:

$$\begin{aligned} C &= \frac{1}{1-\alpha} x_0^{1-\alpha} \\ x &= \left((1-\alpha)t + x_0^{1-\alpha}\right)^{\frac{1}{1-\alpha}} \end{aligned}$$

Solution diverges at

$$t^* = \frac{x_0^{1-\alpha}}{\alpha - 1} \qquad \text{if} \qquad \alpha > 1$$

i.e. for $\alpha > 1$ divergence in **finite time**.

Note: divergence in infinite time no problem: $x(t) = e^t$

2. Uniqueness

Consider previous example for $0 < \alpha < 1$

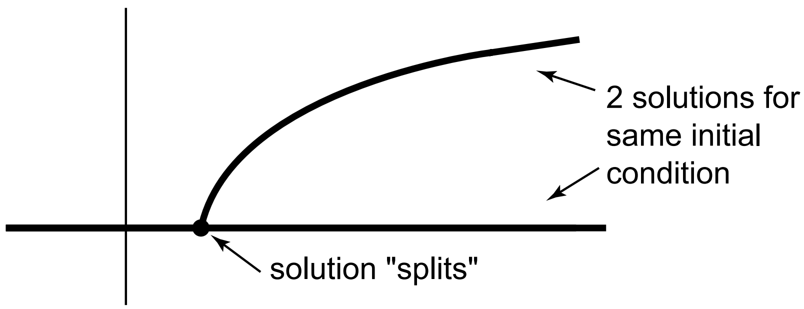
$$\Rightarrow x = 0 \qquad \text{for} \qquad t^* = \frac{x_0^{1-\alpha}}{\alpha - 1} < 0$$

Solution can start at t^* with $x(t^*) = 0$ and grow from there.

But: $\tilde{x}(t) \equiv 0$ is a solution for all times

$\Rightarrow$ can start with $\tilde{x}(t) = 0$ for $t < t^*$ and 'switch' to $x(t) > 0$ beyond t^*. The combined solution is continuous and satisfies the differential equation.

Thus: two *different* solutions satisfy the *same* initial condition (at t^*).



Worse: t^* depends on x_0

$\Rightarrow$ can pick any t^* and patch the solutions at that t^*

$\Rightarrow$ infinitely many solutions with identical i.c. $x = 0$.

Note: in order to get "across the splitting" need to reach 0 in finite time
(splitting has to be crossed in finite time)

Theorem[1]**:**
If for

$$\dot{x} = f(x,t)$$

- $f(x,t)$ is continuous in $|t - t_0| < \Delta t$ in $|x - x_0| \le \Delta x$ and has maximum M there, and
- $f(x,t)$ satisfies Lipschitz condition within Δx and Δt:

  $$|f(x_1,t) - f(x_2,t)| \le K\,|x_1 - x_2| \qquad \forall x_1, x_2 \in |x - x_0| \le \Delta x$$

  with some constant K

then the solution exists for a finite time interval $|t - t_0| \le \Delta T$ and is unique. The interval is given by

$$\Delta T = \min\left(\Delta t, \frac{\Delta x}{M}\right)$$

Discussion:

[1]see, e.g., Lin & Segel, *Mathematics applied to deterministic problems in the natural sciences*, p.57

$f(x) = |x|^\alpha$ does not satisfy Lipschitz condition at $x = 0$ for $0 < \alpha < 1$:
would need

$$|x|^\alpha \leq Kx \quad \forall x \text{ near } x = 0$$

i.e. $K \geq |x|^{\alpha-1} \to \infty$ for $x \to 0$ and $\alpha < 1$

Thus: uniqueness of solution is not guaranteed.

Note: If $f'(x)$ is continuous then $f(x)$ satisfies the Lipschitz condition and the solution is unique.

2.3 Bifurcations in 1 Dimension

We had: in 1d final state always fixed point (if dynamics are bounded)

How many fixed points? How can number of fixed points change?

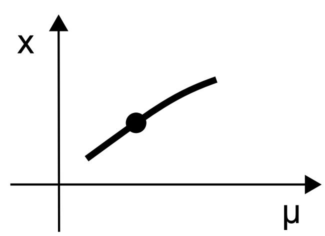
$\Rightarrow$ Introduce parameter μ

$$f(x, \mu) = 0$$

Creation of fixed point: small change in μ

$\Rightarrow$ analysis in **neighborhood** of some special value of μ

Question: Does the solution persist when the parameter is changed? Is it unique?



2.3.1 Implicit Function Theorem

Local analysis near fixed point for small changes in μ:

Taylor expansion

$$f(x, \mu) = \underbrace{f(x_0, \mu_0)}_{=0} + \frac{\partial f}{\partial x}(x - x_0) + \frac{\partial f}{\partial \mu}(\mu - \mu_0) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x - x_0)^2 + \cdots$$

(All derivatives evaluated at x_0, μ_0)

fixed point: $f(x_0, \mu_0) = 0$

If $\frac{\partial f}{\partial x}|_{x_0,\mu_0} \neq 0 \Rightarrow$ solve uniquely for x

$$x - x_0 = -(\mu - \mu_0)\frac{\frac{\partial f}{\partial u}}{\frac{\partial f}{\partial x}} + O(\mu^2)$$

Thus, in this case there is a *branch* of solutions.

More generally for higher dimensions: *Implicit function theorem*

Consider Solutions of

$$\mathbf{f}(\mathbf{x}, \mu) = 0 \qquad \mathbf{x} \in \mathcal{R}^n \qquad \mathbf{f} \text{ smooth in } \mathbf{x} \text{ and } \mu$$

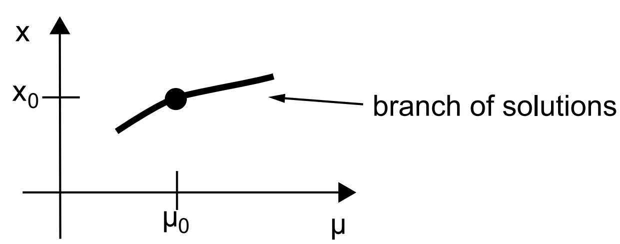
If

$$\mathbf{f}(\mathbf{x} = \mathbf{x}_0, \mu = \mu_0) = 0 \text{ and } \det\left(\frac{\partial f_i}{\partial x_j}\right) \neq 0 \text{ at } \mu = \mu_0 \text{ and } \mathbf{x} = \mathbf{x}_0,$$

then there is a **unique** differentiable $\mathbf{X}(\mu)$ that satisfies

$$\mathbf{f}(\mathbf{X}(\mu), \mu) = 0 \text{ and } \mathbf{X}(\mu = \mu_0) = \mathbf{x}_0.$$

Thus: if $\det\left(\frac{\partial f_i}{\partial xj}\right) \neq 0$ there is a **branch** of solutions going through $\mathbf{x} = \mathbf{x}_0$ as μ is varied.



Notes:

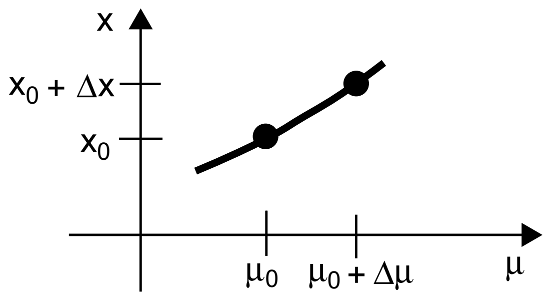
- In 1d: $\det \frac{\partial f_i}{\partial xj} \rightarrow \frac{df}{dx} = f'(x)$
  $\Rightarrow$ as seen in explicit calculation: if $f'(x) \neq 0$ branch persists uniquely
- We have: stability changes if $f'(x) = 0$
  $\Rightarrow$ **change in number of fixed points** requires **change in (linear) stability** of fixed point.
- **generic properties** are those properties that do not require **any tuning** of the parameters

  When picking parameters randomly one expects $\frac{\partial f}{\partial x}|_{x_0,\mu_0} \neq 0$,
  i.e. need to *tune* μ to get $\frac{\partial f}{\partial x}|_{x_0,\mu_0} = 0$

  $\Rightarrow$ **generically** there is a smooth branch

- change in x is smooth in μ if $\frac{\partial f}{\partial x} \neq 0$

$$\Delta x \sim \Delta \mu$$



2.3.2 Saddle-Node Bifurcation

What happens when $\frac{\partial f}{\partial x} = 0$?

Need to go to higher order in Taylor expansion (choose $x_0 = 0, \mu_0 = 0$)

$$f(x,\mu) = \underbrace{f(0,0)}_{=0} + \underbrace{\frac{\partial f}{\partial x}}_{=0} x + \frac{\partial f}{\partial \mu}\mu + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}x^2 + \frac{\partial^2 f}{\partial x \partial \mu}x\mu + \frac{1}{2}\frac{\partial^2 f}{\partial \mu^2}\mu^2 + \ldots$$

Solve again:

$$x^2 = -\frac{2}{\frac{\partial^2 f}{\partial x^2}}\left\{\underbrace{\frac{\partial f}{\partial \mu}\mu}_{x=\mathcal{O}(\mu^{1/2})} + \frac{\partial^2 f}{\partial x \partial \mu}\underbrace{x\mu}_{\mathcal{O}(\mu^{3/2})} + \frac{1}{2}\frac{\partial^2 f}{\partial \mu^2}\mu^2 + \cdots\right\}$$

Try different balances of x and μ

$$\begin{array}{rcl} x \sim \mu & \Rightarrow & \text{contradiction} \\ x \sim \mu^{1/2} & \Rightarrow & \text{consistent} \end{array}$$

Thus

$$x_{1,2} = \pm\sqrt{-2\frac{\frac{\partial f}{\partial \mu}}{\frac{\partial^2 f}{\partial x^2}}\mu} + \mathcal{O}(\mu)$$

Notes:

- If implicit function theorem fails one gets higher-order equation with multiple solutions (depending on parameters)

- change in x is **not smooth** in μ

Dynamics:

$$\dot{x} = f(x, \mu) = a\mu + bx^2 + \text{h.o.t.}$$

with

$$a = \frac{\partial f}{\partial \mu} \equiv \partial_\mu f \qquad b = \frac{\partial^2 f}{\partial x^2} \equiv \partial_x^2 f$$

Note:

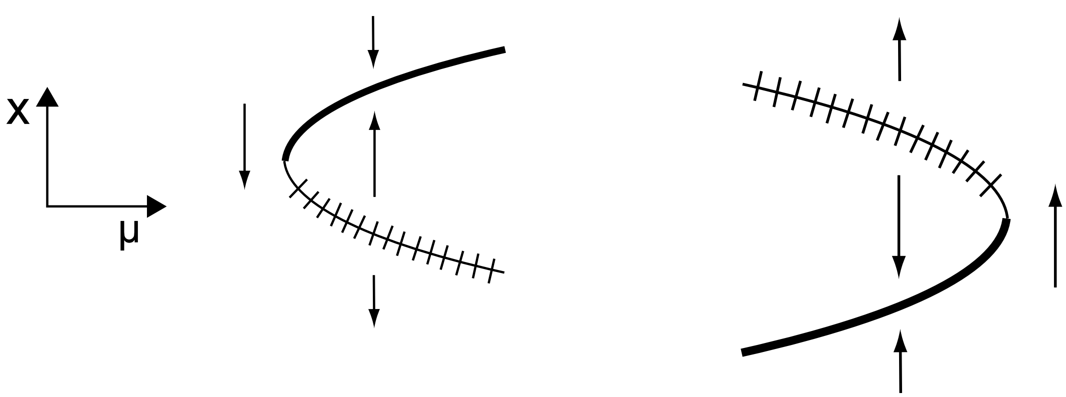
- this is the universal form of equation near saddle-node bifurcation

Bifurcation diagrams: plot all solution branches as function of μ

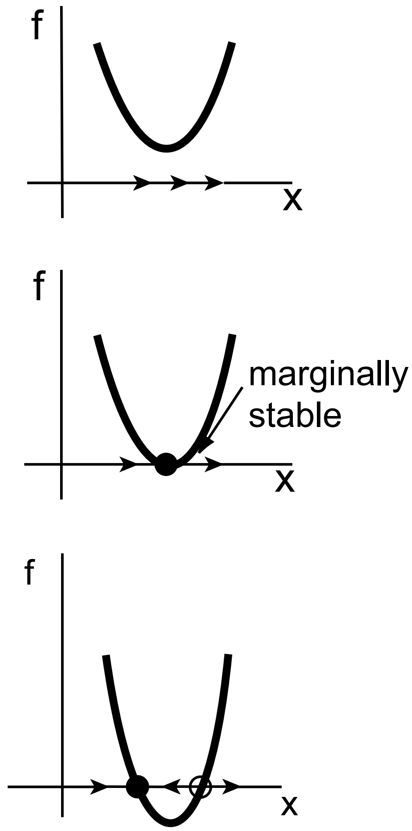
Relevant parameters:

$$\frac{a}{b} = \frac{\partial_\mu f}{\partial_x^2 f} \equiv \frac{\frac{\partial f}{\partial \mu}}{\frac{\partial^2 f}{\partial x^2}}$$

$\frac{a}{b} < 0$ and $a > 0$ $\qquad$ $\frac{a}{b} > 0$ and $a > 0$



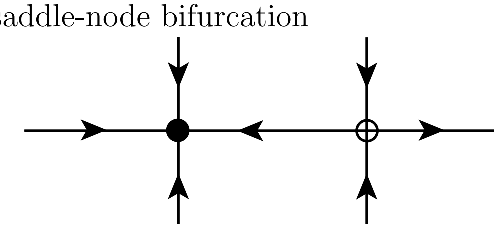
x-direction $\sim$ phase line. Arrows indicate flow on phase line



Minimum of f generically quadratic $\Rightarrow$ universal form of equation

Notes:

- 2 fixed points are created/destroyed. Single solutions cannot simply pop up or disappear: merging and annihilation of 2 solutions
- coinciding fixed points at $\mu = 0$ are marginally stable:
  $\partial_x f$ changes sign going along solution branch: **change in stability**
- flow changes direction **only locally**:
  only when μ goes through 0 and only near bifurcation point $x = 0$ flow changes direction.
  Away from bifurcation point flow qualitatively unchanged when μ changes (arrows far away remain the same).
- in higher dimensions: saddle-node bifurcation



stable $\sim$ node   unstable $\sim$ saddle

- saddle-node bifurcation is the **generic** bifurcation when a real eigenvalue goes through 0.
  The only condition is $\partial_x f = 0$: this is the condition for **any** bifurcation to occur.
  'Expect' saddle-node bifurcation, if any.

Example:
Convection of a layer heated from below (Nu measures heat transport):

Here saddle-node bifurcation part of a larger bifurcation scenario

Saddle-node bifurcation sometimes also called "blue-sky bifurcation"

2.3.3 Transcritical Bifurcation

Consider system for which 1 solution ($x = 0$) exists for all μ (special condition):

$$f(0, \mu) = 0 \quad \text{for all} \;\; \mu$$

Taylor expansion:

$$\Rightarrow f(x, \mu) = \underbrace{f(0,0)}_{=0} + \underbrace{\partial_x f}_{=0}\, x + \underbrace{\partial_\mu f}_{=0}\, \mu + \frac{1}{2}\partial_x^2 f\, x^2 + \partial^2_{x\mu} f\, x\mu + \underbrace{\frac{1}{2}\partial^2_\mu f}_{=0}\; \mu^2 + \ldots$$

First three terms and last term vanish because:
$x = 0$ fixed point, a bifurcation occurs, symmetry of $f(x,\, u)$

Universal evolution equation

$$\dot{x} = x\,(a\,\mu + b\,x) + \cdots$$

with

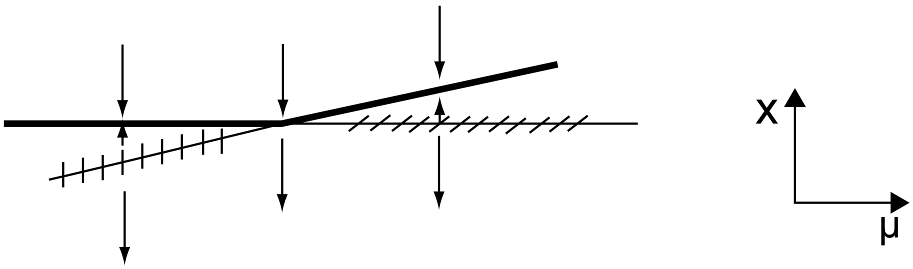
$$a = \partial^2_{x\mu} f \qquad b = \frac{1}{2}\partial_x^2 f$$

Fixed points:

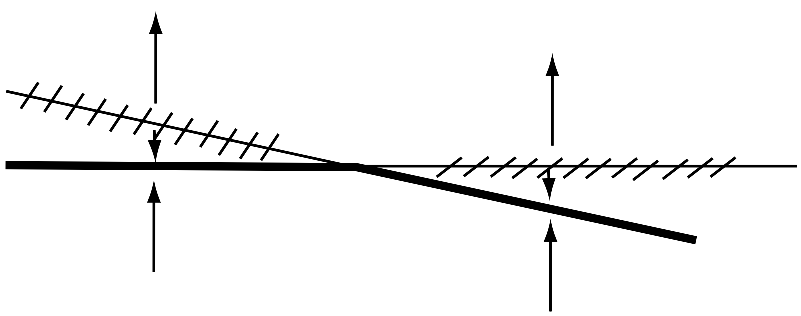
$$x_1 = 0 \qquad x_2 = -\frac{a}{b}\mu \equiv -\frac{\partial^2_{x\mu} f}{\frac{1}{2}\partial_x^2 f}\,\mu$$

Two cases:

$$\frac{a}{b} < 0 \qquad a > 0$$



$$\frac{a}{b} > 0 \qquad a > 0$$

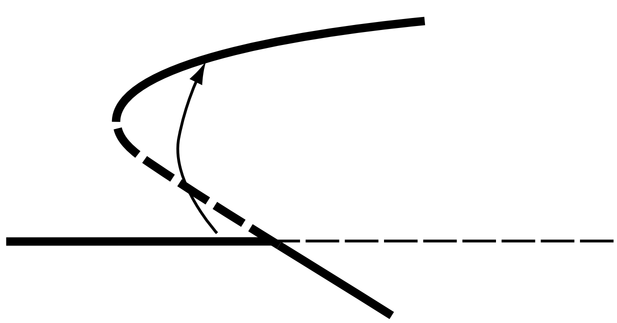


Notes:

- "exchange of stability"
- subcritical branch:
  Sufficiently large perturbation can lead away from (linearly) stable fixed point.

Examples:

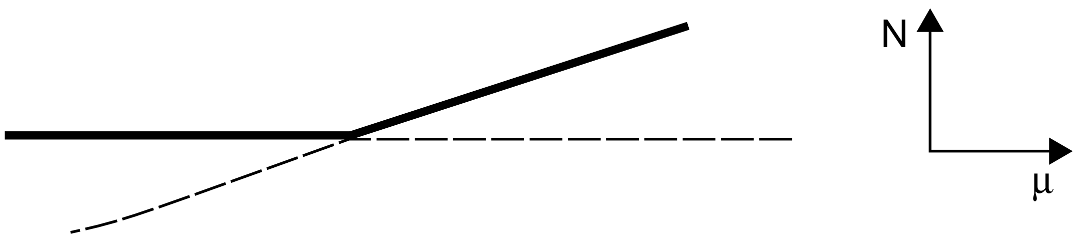
a) Hexagon convection



- large perturbation kicks solution above unstable branch of transcritical bifurcation.
- For $\mu > 0$ the lower branch is unstable in a different way (instability not contained in the single equation)

b) Logistic equation

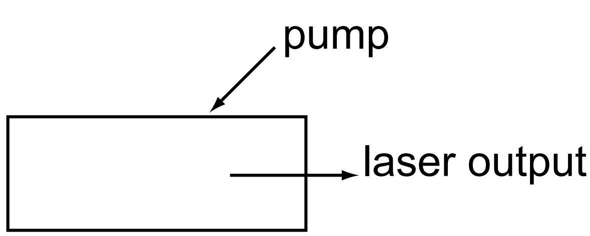
$$\dot{N} = \mu N - N^2$$



for $\mu < 0$ lower branch unphysical since $N > 0$ required

c) Simple Model for Laser

Optical cavity with excitable atoms



Dynamics of atoms:

- atoms are excited by pump[2]
- atoms emit photons and go to ground state
  - spontaneously: spontaneous emission
  - due to other photon: **stimulated emission**
    a photon triggers the emission of a photon from excited atom

$$\dot{N} = P - fN - gnN$$

N: excited atoms, P: pump, f: decay through spontaneous emission,
g: "collision" with photon takes atom to ground state (stimulated emission)

Dynamics of photons:

[2]Atoms are also excited by photons already present; effect much smaller than pump (n is small near onset of lasing)

- photons generated by **stimulated emission**
- photons leave through end mirrors

$$\dot{n} = GNn - \kappa n$$

n: photons, G: gain, κ: output/loss

Note: n counts only photons with correct phase (only those generated by spontaneous emission)

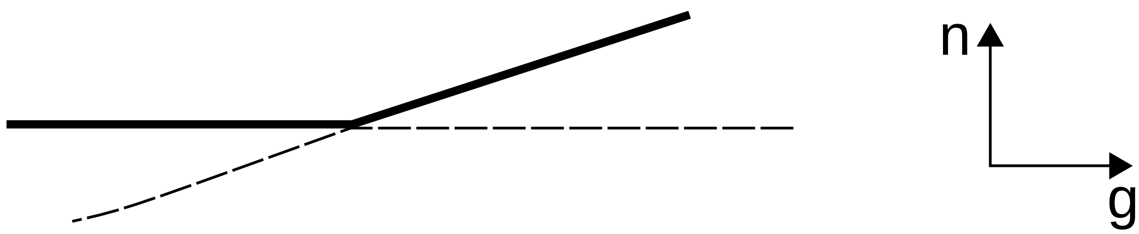
Now: we have 2 equations: *too difficult*

model the N-equation: steady state # of excited atoms will be reduced by photons

$$N = N_0 - \alpha n$$

$$\Rightarrow \dot{n} = G(N_0 - \alpha n)\, n - \kappa n$$

again same equation as for logistic growth



Note:

- we will learn under what conditions the model for N is justified:
  reduction from many ode's to few/single ode by *center-manifold reduction*.

2.3.4 Pitchfork Bifurcation

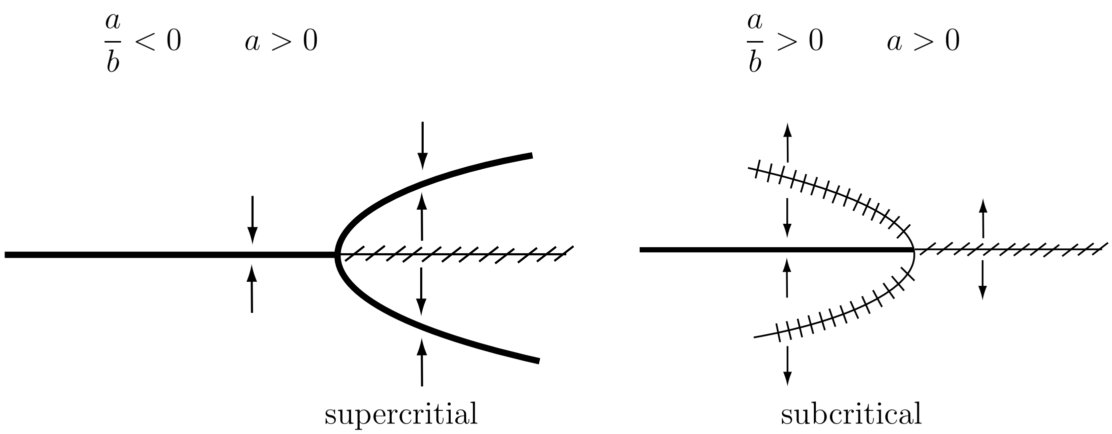
Systems with reflection symmetry $x \rightarrow -x$

$$\dot{x} = f(x,\mu) \qquad \text{with} \qquad f(x,\mu) \text{ odd in } x$$

$$\Rightarrow x = 0 \text{ solution for all } \mu.$$

Taylor expansion:

$$\begin{aligned} f(x,\mu) &= \underbrace{a}_{\partial^2_{x\mu} f} x\mu + \underbrace{b}_{\frac{1}{6}\partial^3_x f} x^3 + \ldots \\ x_0 &= 0 \\ x_{2,3} &= \pm\sqrt{-\frac{a}{b}\mu} \end{aligned}$$



Notes:

- supercritical ⇒ saturation of instability
- subcritical ⇒ no saturation to cubic order
  ⇒ need higher-order terms
- system has reflection symmetry $x \to -x$

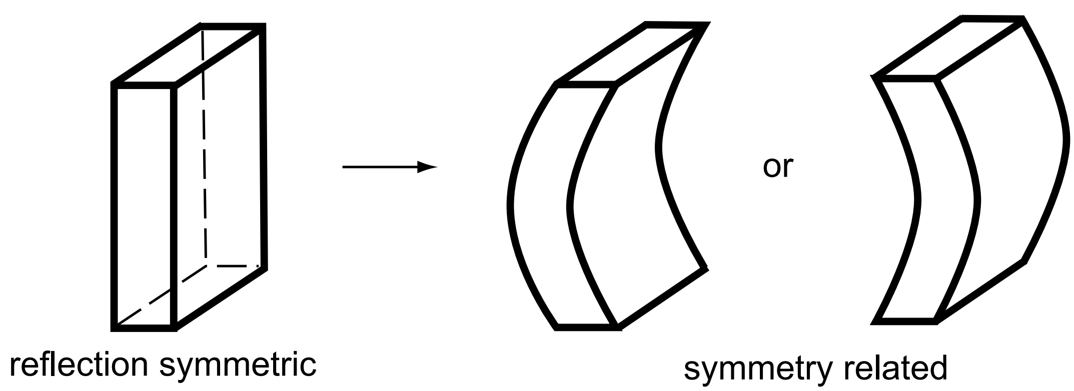
  $x_0 = 0$ solution has that symmetry as well

  $x_{2,3} = \pm\sqrt{\ldots}$ do not have reflection symmetry:
  instead two symmetrically related solutions

  ⇒ pitchform bifurcation = *symmetry-breaking* bifurcation

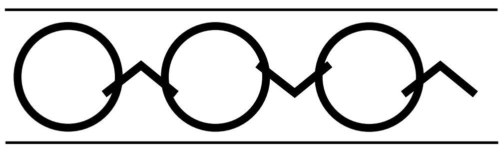
Examples:

a) buckling of a beam

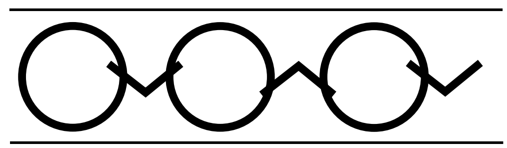


b) Rayleigh-Bénard roll convection:

up-flow



down-flow



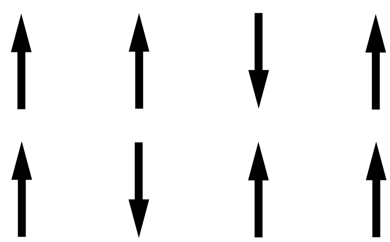
Note:

- up $\Rightarrow$ down corresponds to translations by half a wavelength
  intermediate positions also possible $\Rightarrow$ larger symmetry

c) Ferromagnets

Phase transition as temperature T increased beyond T_c: ferromagnetic $\Rightarrow$ paramagnetic

Each atom carries a magnetic moment (spin): $s_i = \pm 1$



Overall magnetization if the spins align on average: *spontaneous symmetry breaking*

Interactions:

- energy of spins in external magnetic field:

$$E_H = -Hs_i \qquad \text{want to be parallel to field}$$

- energy of spin - spin interaction:

$$E_S = -\sum_{i,j} J_{ij} s_i s_j \qquad J_{ij} > 0, \qquad \text{want to be parallel to each other}$$

$\sum$ is sum over neighbors

Total energy:

$$\begin{aligned} E(s_1 \ldots, s_N) &= -\sum_i H s_i - \sum_{i,j} J_{ij}\, s_i s_j \\ &= -\sum_i \underbrace{\left(H + \sum_j J_{ij} s_j \right)}_{H_i^{eff}} s_i \end{aligned}$$

each spin s_i feels a field that depends on neighbors

$$H_i^{eff} = H + \sum_j J_{ij} s_j$$

Probability of spin i with energy E_i to have value s_i

$$P(s_i) \quad \propto \quad e^{-E_i/kT} = e^{H_i^{eff} s_i/kT} \qquad \text{Boltzmann factor}$$

k Boltzmann constant

Average value of s_i

$$\bar{s}_i = \sum_{s_i=\pm 1} s_i P(s_i) = P(1) - P(-1)$$

However: H_i^{eff} still contains configuration of all the other spins
$\Rightarrow P(s_i)$ very difficult to calculate

Mean field approximation: replace local spin value by average

$$\begin{aligned} H_i^{eff} \to \bar{H} &= H + \sum_j J_{ij} \bar{s}_j \\ &= H + \underbrace{\bar{s}_j}_{\text{independent of } j} \underbrace{\sum_j J_{ij}}_{\bar{J}} \end{aligned}$$

Then

$$P(s_i) = \frac{1}{\mathcal{N}} e^{\bar{H} s_i/kT} \qquad \text{with} \qquad \bar{H} = H + \bar{s}\bar{J}$$

Normalization of probability:

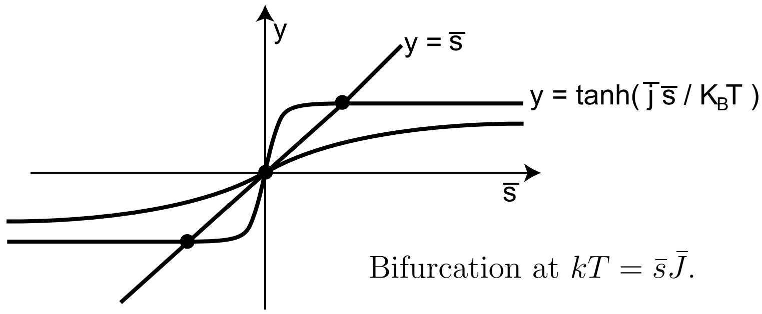
$$1 = P(+1) + P(-1) \Rightarrow \mathcal{N} = e^{\bar{H}/kT} + e^{-\bar{H}/kT}$$

Average magnetization satisfies:

$$\bar{s} = \frac{(+1)\, e^{\bar{H}/kT} + (-1)\, e^{-\bar{H}/kT}}{e^{\bar{H}/kT} + e^{-\bar{H}/kT}} = \tanh\left\{ \frac{(H + \bar{s}\bar{J})}{kT} \right\}$$

Consider H = 0:

$$\bar{s} = \tanh(\frac{\bar{s}\bar{J}}{kT})$$



Bifurcation at $kT = \bar{s}\bar{J}$.

Notes:

- pitchfork bifurcation since symmetry $\bar{s} \to -\bar{s}$
- supercritical pitchfork bifurcation $\Leftrightarrow$ phase transition of 2^{nd} order.
- $H \neq 0$ breaks reflection symmetry $\Rightarrow$ pitchfork bifurcation perturbed $\Rightarrow$ later.

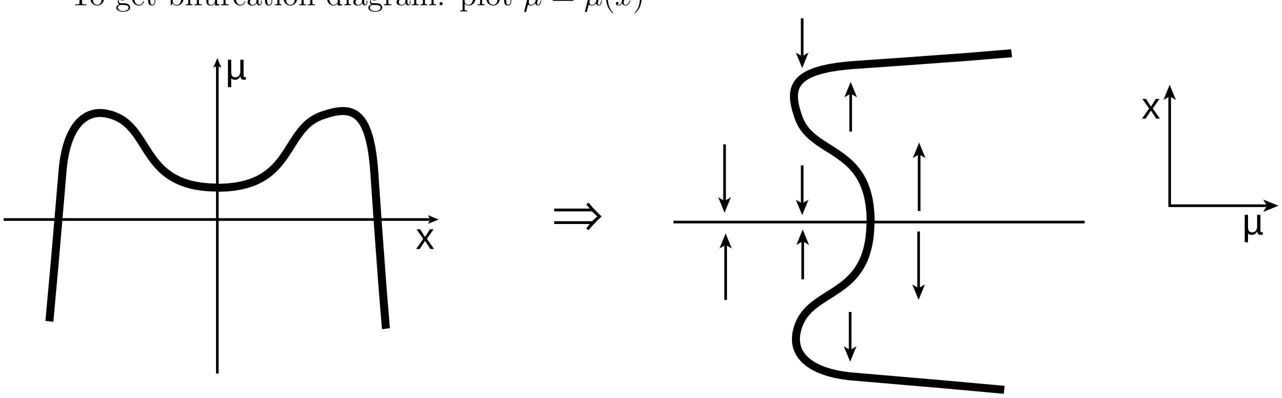
Subcritical Pitchfork Bifurcation:

For $b > 0$ need to include quintic term:

$$\dot{x} = \mu x + \underbrace{bx^2}_{destabilizing} - \underbrace{cx^5}_{stabilizing}$$

Assume $c > 0$. In general need not saturate at quintic order

To get bifurcation diagram: plot $\mu = \mu(x)$

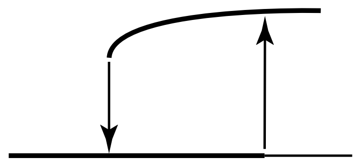


plot of $\mu(x)$ is upside down   flip plot for bifurcation diagram

Note:

- 2 saddle-node bifurcations for $\mu < 0$

- hysteresis loop & bistability



- we performed an expansion in x: analysis strictly only valid if "x small" on upper branch: $b \to 0$, *weakly subcritical*
- $b = 0$: 'tricritical' point

2.4 Imperfect Bifurcations

For transcritical and for pitch-fork bifurcation to occur we needed 2 conditions

- bifurcation occurs: $\partial_x f|_{x_0,\mu_0} = 0$
- additional coefficients "happen to vanish," e.g., because of some symmetry

Only the saddle-node bifurcation requires only 1 condition

Saddle-node bifurcation is a **codimension-1 bifurcation**

Question: What happens when the additional conditions are **weakly broken** in the other cases?

Consider perturbed pitchfork bifurcation

$$\dot{x} = \mu x - x^3 + h$$

Example: Ferromagnet with external field

we had:

$$\bar{s} = \tanh(\beta(H + \bar{J}\bar{s})\}$$

$$\text{with} \qquad \tanh\theta = \theta - \frac{1}{3}\theta^3 + O(\theta^5)$$

$$\text{one gets} \qquad \bar{s} = 1 + b\bar{s} + c\bar{s}^2 - d\bar{s}^3 + \mathcal{O}(\bar{s}^4)$$

To get the equation above for x:

introduce shift: $\tilde{x} = \bar{s} - \bar{s}_0$

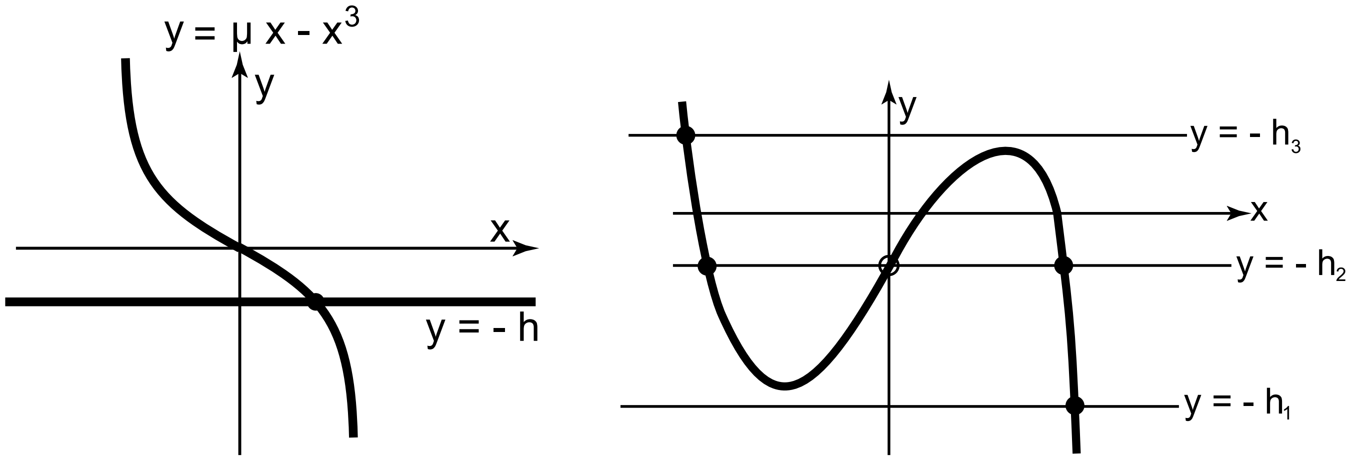
choose $\bar{s}_0$ to eliminate quadratic term $c\tilde{x}^2$

rescale $x = \tilde{x}/\tilde{x}_0$ to set cubic coefficient to -1.

Solving directly for fixed point is cumbersome (although possible).

Graphic solution:

$\mu < 0$ $\qquad\qquad\qquad$ $\mu > 0$



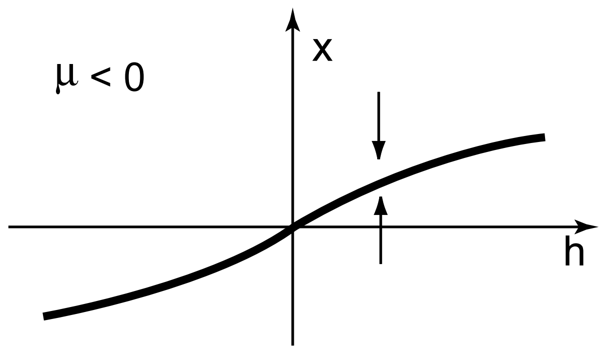
Creation/annihilation of 2 fixed points;

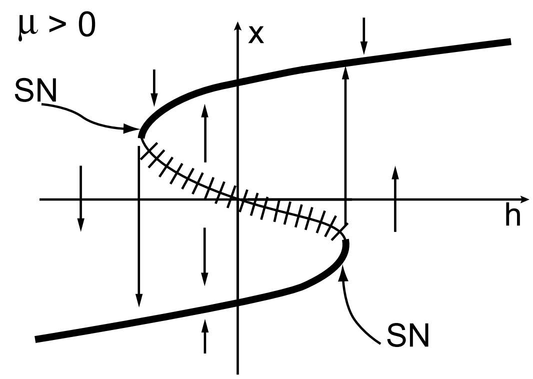
Saddle-node bifurcation at extrema of $\mu x - x^3$

$$x_{SN} = \pm\sqrt{\frac{1}{3}\mu} \qquad h_{SN} = \pm\sqrt{\frac{1}{3}\mu}\,, \frac{2}{3}\mu$$

Bifurcation diagrams: 2 parameters

Vary h**:**

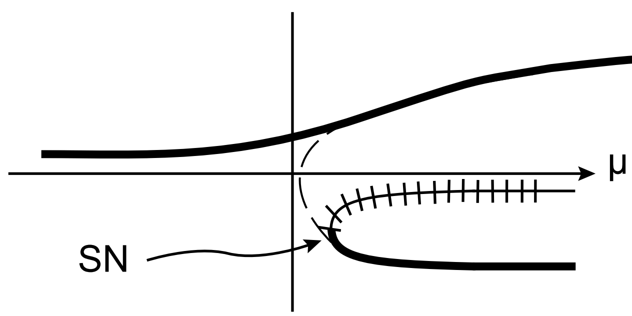




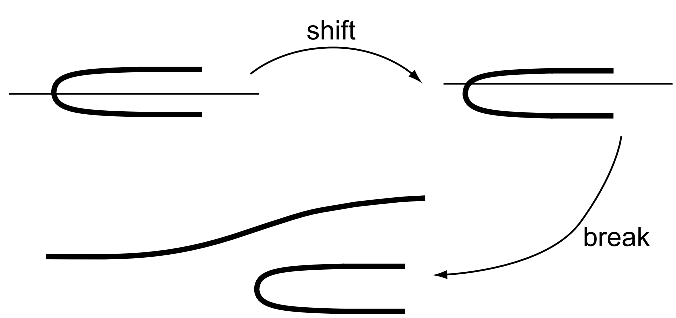
Note:

- vary h up and down beyond saddle-node bifurcations: hysteresis loop

Vary μ:



To make the pitchfork bifurcation generic:
break symmetry $x \to -x \Rightarrow$ transcritical bifurcation
break transcritical $\Rightarrow$ only saddle-node bifurcation



Note:

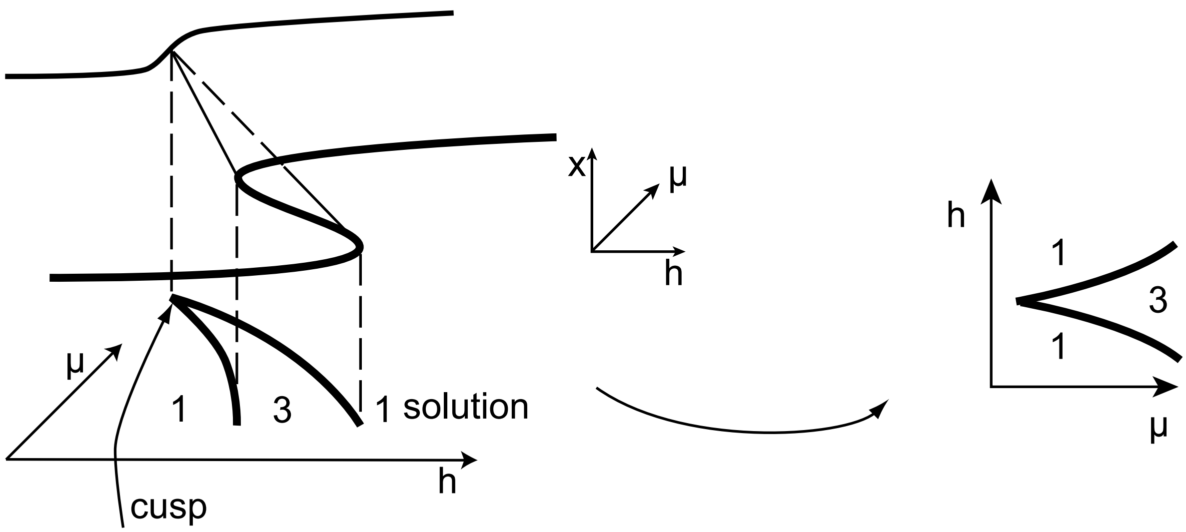
- to get original unperturbed pitch-fork bifurcation have to tune 2 parameters

$$\mu = 0 \qquad \& \qquad h = 0$$

codimension-2 bifurcation

- symmetries of the system may render pitch-fork bifurcation a codimension-1 pheomenon (here reflection symmetry)

Solution surface:



surface of cusp catastrophe:

- catastrophes occur as saddle-node bifurcations are crossed:
  jump to other branch
  minute changes lead to **large** results.

Notes:

- A system is called **structurally stable** if small perturbations of the equations do not qualitatively change its behavior

  i) $\dot{x} = \mu + x$ structurally stable

  fixed point $x = -\mu$

  ii) $\dot{x} = -\mu + x^2$ not structurally stable

  $x = \pm\sqrt{\mu}$ for $\mu > 0$

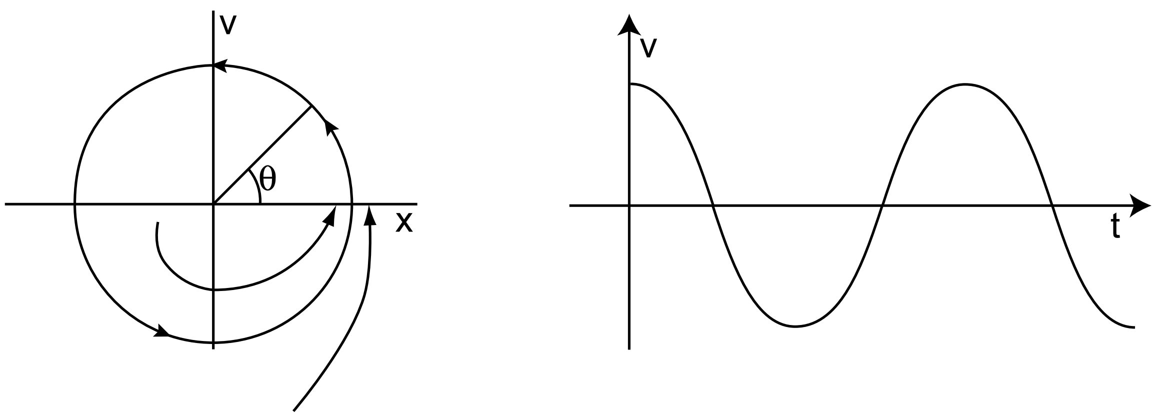
  $x = 0$ for $\mu < 0$

- A bifurcation is called *degenerate* if additional conditions "happen" to be satisfied

- *Unfolding* of degenerate bifurcation:
  introduce sufficiently many parameters that no degeneracy is left.

2.5 Flow on a Circle

For oscillations need return: two dimensions needed



If oscillatory motion (circle) is sufficiently attractive consider only motion along closed orbit:

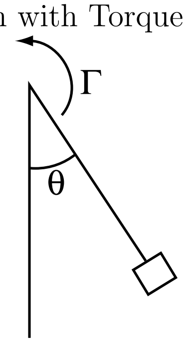
Flow on a circle

$$\dot{\theta} = f(\theta) \qquad \theta \in [0, 2\pi]$$

Notes:

- $f(\theta)$ cannot be arbitrary: has to be single-valued, i.e. 2π-periodic
- $f(\theta)$ gives the instantaneous frequency

Example: Overdamped Pendulum with Torque



$$m\ell^2\ddot{\theta} + \beta\dot{\theta} = -mg\ell\sin\theta + \tilde{\Gamma}$$

consider large damping

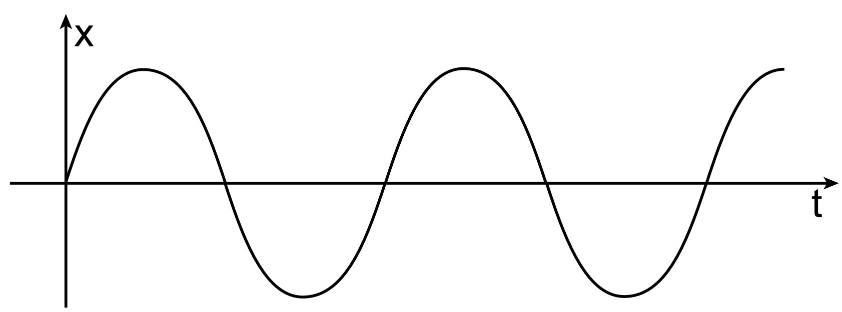
$$\dot{\theta} = \Gamma - a \sin\theta$$

i) $a = 0$ (no gravity)

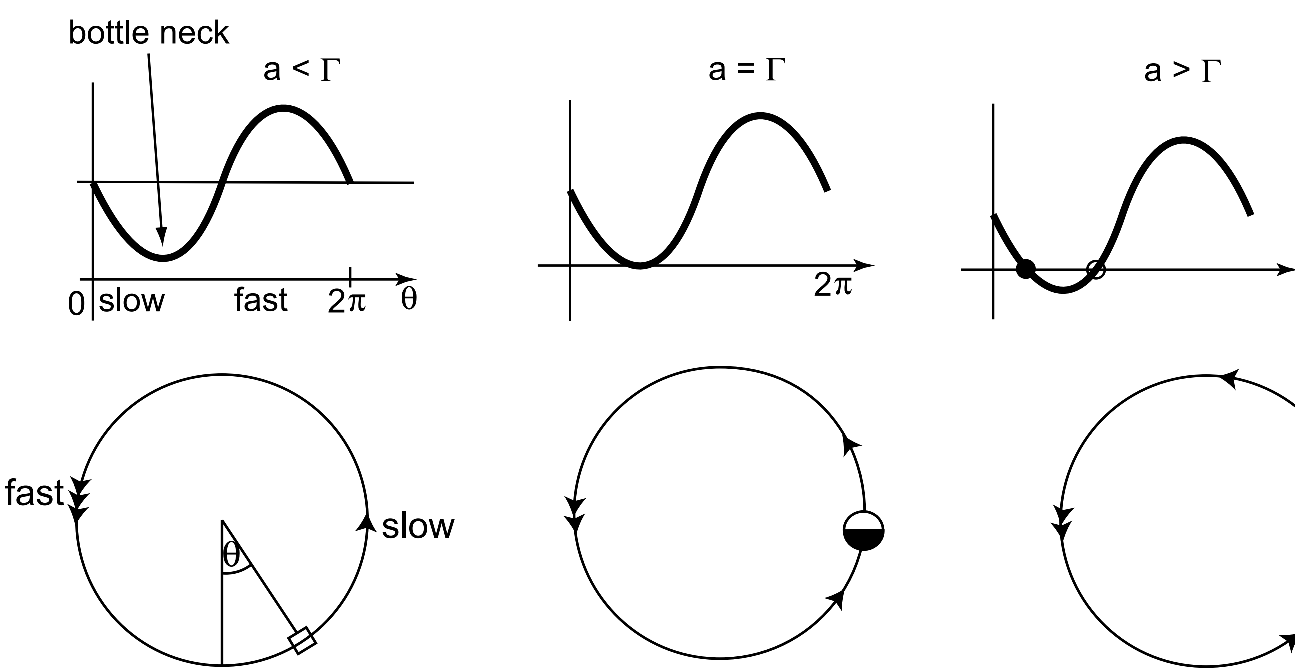
$$\theta = \theta_0 + \Gamma t \qquad \text{whirling motion}$$

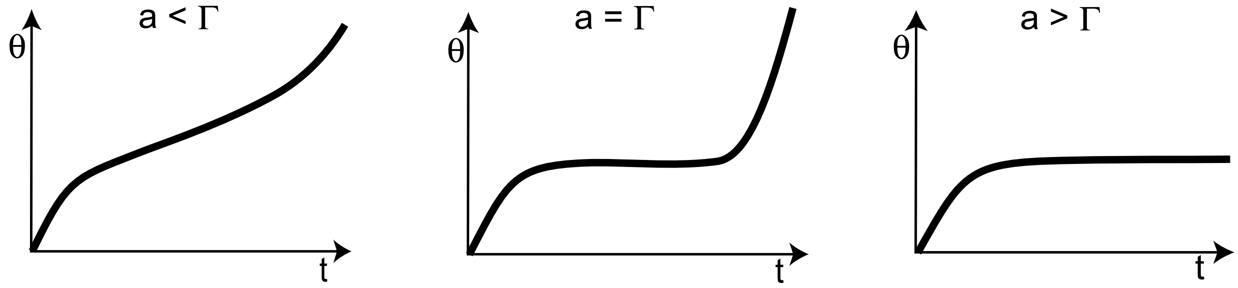
oscillation in horizontal coordinate:

$$x = \ell \sin\theta = \ell \sin(\theta_0 + \Gamma t)$$



ii) $\alpha > 0$ (with gravity)





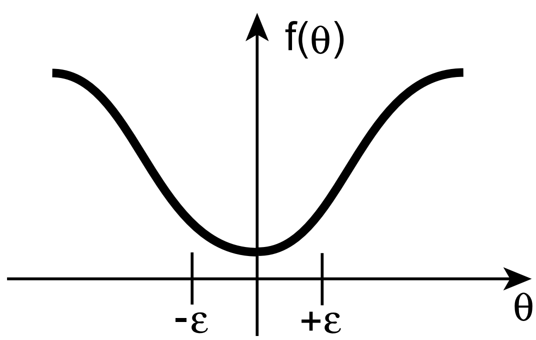
'ghost' of saddle-node bifurcation
$\Rightarrow$ extremely slow motion

Note:

- quite generally: near a steady bifurcation the dynamics become slow: growth/decay rates go to 0 ('critical slowing down').

Estimate period near bifurcation point:

$$T = \int dt = \int_0^{2\pi} \frac{d\theta}{\dot{\theta}} = \int_0^{2\pi} \frac{d\theta}{\omega - a\sin\theta}$$



Consider general case near saddle-node bifurcation

$$\dot{\theta} = f(\theta)$$

with

$$f(0) = \mu, \quad f'(0) = 0$$

$$\Rightarrow f(\theta) = \mu + \underbrace{\frac{1}{2} f''(0)}_{a}\, \theta^2 + \mathcal{O}(\theta^3)$$

$$T = \int_0^{2\pi} \frac{d\theta}{f(\theta)} \quad = \quad \underbrace{\int_{-\epsilon}^{+\epsilon} \frac{d\theta}{\mu + a\theta^2 + \mathcal{O}(\theta^3)}}_{\text{diverges as } \mu \to 0} + \underbrace{\int_{\epsilon}^{2\pi-\epsilon} \frac{d\theta}{f(\theta)}}_{\text{finite as } \mu \to 0}$$

$$\to \int_{-\epsilon}^{+\epsilon} \frac{d\theta}{\mu + a\theta^2} + T_0 \qquad \text{for } \mu \to 0$$

extract μ-dependence for $\mu \to 0$ (at fixed ϵ) using $\psi = \frac{\theta}{\sqrt{\mu}}$

$$\frac{1}{\mu} \int_{-\frac{\epsilon}{\mu^{1/2}}}^{\frac{\epsilon}{\mu^{1/2}}} \frac{\mu^{1/2}\, d\psi}{1 + a\psi^2} + T_0 \;\to\; \frac{1}{\mu^{1/2}} \int_{-\infty}^{\infty} \frac{d\psi}{1 + a\psi^2} + T_0 \;\propto\; \mu^{-1/2}$$

Notes:

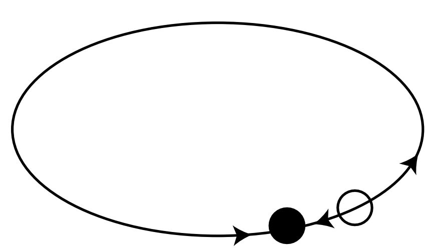
- Saddle-node bifurcation on a circle is **one** way to generate oscillations.
  Generically one has then

$$T \propto \mu^{-1/2}$$

- other types of bifurcations to oscillatory behavior lead to different $T(\mu)$,
  e.g. Hopf bifurcation

$$T(\mu = 0) = T_0 \text{ finite.}$$

- the fact that the saddle-node bifurcation leads to oscillations is a **global** feature of the system:
  need **global connection** between the generated fixed points



Examples:

i) Synchronization of fireflies

- light up periodically
- respond to neighboring fire flies

Consider single firefly with periodic light source

Light source: $\dot{\psi} = \Omega$

Firefly: $\dot{\phi} = \omega + a\sin(\psi - \phi)$

$a > 0 : \psi > \theta \Rightarrow$ firefly speeds up

rewrite: $\theta = \phi - \psi$

$$\dot{\theta} = \underbrace{\omega - \Omega}_{\Gamma} - a\sin(\theta)$$

Γ: frequency mismatch = detuning

- *Fixed point:* fly flashes entrained by light source

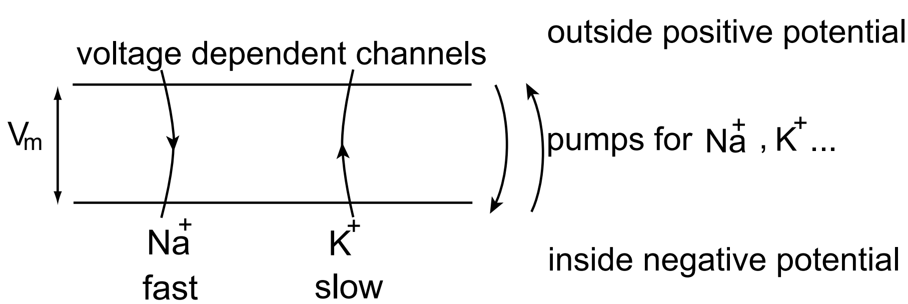
  $$\underbrace{|\omega - \Omega|}_{\text{range of entrainment}} < a \qquad \text{and} \qquad \theta_0 = \arcsin\frac{\omega - \Omega}{a} \neq 0$$

  Fly flashes lag behind/pull ahead, but phase difference fixed: **phase-locked** state
- *"Whirling" motion:* $|\omega - \Omega| > a$
  flashes not synchronized with light source.

Notes:

- entrainment is a common feature of coupled oscillators
- in general coupling bidirectional.

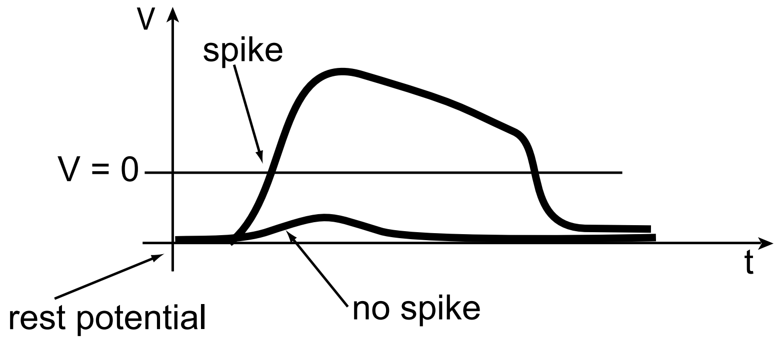
ii) Excitability in neurons



$$\frac{dV}{dt} = \frac{1}{C}(I_{k+} + I_{Ca^{2+}} + +I_{leak} + \cdots)$$
$$\frac{dI_j}{dt} = F_j(V, I_j) \qquad \text{voltage gated channels}$$

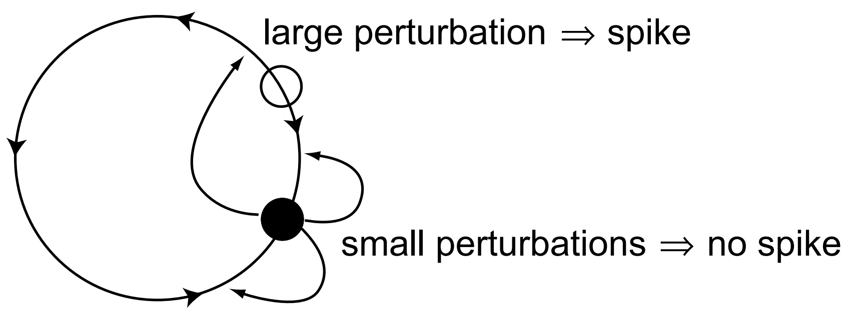
- sufficiently large depolarization (V less negative) $\Rightarrow Na^+$ channels open, V becomes positive rapidly
- positive $V \Rightarrow K^+$ channels open, V becomes negative again

Response:



Very simple model

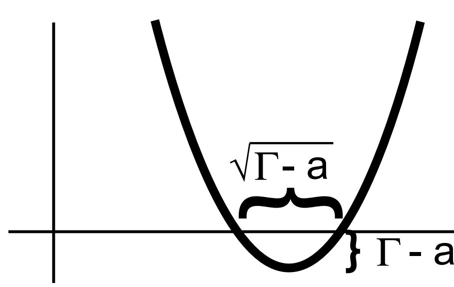
$$\dot{\theta} = \Gamma - a \sin\theta \qquad \Gamma \leq a$$



Note:

- for Γ close to a even small perturbations can be sufficient to excite a spike

$$\Delta\theta_{\text{large}} \approx \theta_{0,a} - \theta_{0,s} \propto |\Gamma - a|^{1/2}$$



3 Two-dimensional Systems

New aspects:

- 'true' oscillations without periodic "boundary" conditions
- reduction of dynamics to lower dimension

3.1 Classification of Linear Systems

General linear system

$$\underline{\dot{x}} = \underline{\underline{A}}\,\underline{x} \qquad \underline{x}(0) = \underline{x}_0$$

Formal solution

$$\underline{x}(t) = e^{\underline{\underline{A}}\,t}\underline{x}_0$$

with

$$e^{\underline{\underline{A}}\,t} = 1 + \underline{\underline{A}}\,t + \frac{1}{2}\underline{\underline{A}}^2 t^2 + \ldots$$

Simplify $\underline{\underline{A}}$ by similarity transformation:

In general can find $\underline{\underline{S}}$ such that

$$\underline{\underline{S}}^{-1}\underline{\underline{A}}\,\underline{\underline{S}} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

or

$$\underline{\underline{S}}^{-1}\underline{\underline{A}}\,\underline{\underline{S}} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \qquad \text{Jordan normal form}$$

Notes:

- $\lambda_{1,2}$ are the eigenvalues of $\underline{\underline{A}}$:

$$\begin{aligned} \underline{\underline{A}}\,\underline{v}_{1,2} &= \lambda_{1,2}\,\underline{v}_{1,2} \\ \underline{\underline{S}}^{-1}\underline{\underline{A}}\,\underline{\underline{S}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} &= \lambda_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \Rightarrow \underline{\underline{A}}\,\underbrace{\underline{\underline{S}}\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\underline{v}_1} &= \lambda_1 \underbrace{\underline{\underline{S}}\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\underline{v}_1} \end{aligned}$$

- all eigenvalues different

$$\Rightarrow \underline{\underline{S}}^{-1}\underline{\underline{A}}\,\underline{\underline{S}} \quad \text{diagonal}$$

- dynamics in eigendirections simple

$$\begin{aligned} e^{\underline{\underline{A}}\,t}\,\underline{v}_i &= \{1+\underline{\underline{A}}\,t+\frac{1}{2}(\underline{\underline{A}}\,t)^2+\cdots\}\,\underline{v}_i = \\ &= \{1+\lambda_i t+\frac{1}{2}\lambda_i^2 t^2+\cdots\}\,\underline{v}_i = \\ &= e^{\lambda_i t}\,\underline{v}_i \end{aligned}$$

  along eigendirections simple exponential time dependence

- general solution

$$\underline{x}(t) = e^{\lambda_1 t}\underline{v}_1 A_1 + e^{\lambda_2 t}\underline{v}_2 A_2$$

  with $\underline{x}_0 = A_1\underline{v}_1 + A_2\underline{v}_2$

- eigenvalues can be complex

$$\begin{aligned} \lambda_{1,2} &= \sigma \pm i\omega \\ \underline{x}(t) &= e^{\sigma t}(A_1 e^{i\omega t}\underline{v}_1 + A_2 e^{-i\omega t}\underline{v}_2) \end{aligned}$$

- degenerate eigenvalues $\rightarrow$ modifications, see later

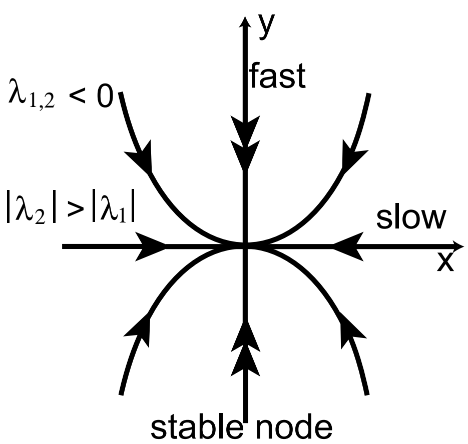
Orbits in phase space (plane):

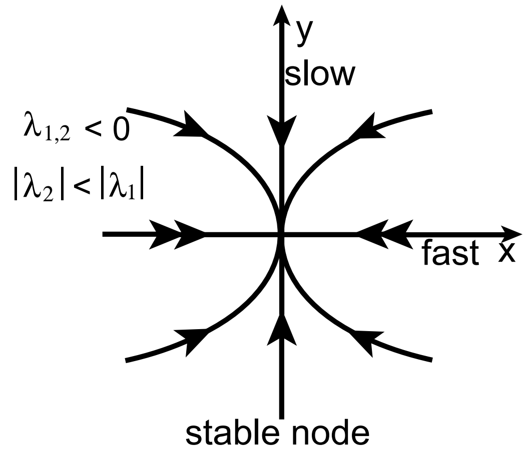
$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{array}{l} x = e^{\lambda_1 t}x_0 \\ y = e^{\lambda_2 t}y_0 \end{array}$$

$$\Rightarrow \qquad e^t = (\frac{x}{x_0})^{1/\lambda_1}$$

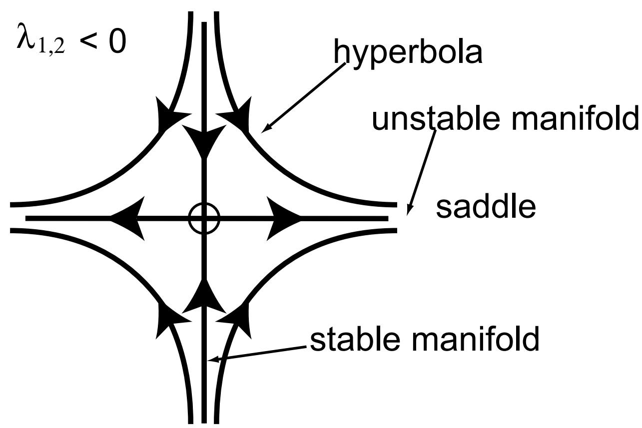
$$y(t) = \left(\left(\frac{x}{x_0}\right)^{1/\lambda_1}\right)^{\lambda_2} y_0 = y_0 \left(\frac{x}{x_0}\right)^{\frac{\lambda_2}{\lambda_1}}$$

Thus $$y(t) = C\,x(t)^{\frac{\lambda_2}{\lambda_1}}$$





$\lambda_1 \, \lambda_2 < 0$



error in labeling: the λ_i have opposite signs in this graph

Definition: Stable/unstable manifold of fixed point x_0:

$$\{\underline{x} \,|\, \underline{x}(0) = \underline{x} \Rightarrow \underline{x}(t) \rightarrow \underline{x}_0 \text{ for } t \rightarrow \pm\infty\}$$

Note: $\cdot$ in general eigenvectors need not be orthogonal

Example

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

eigenvalues

$$\det \begin{vmatrix} +3-\lambda & -2 \\ -1 & 2-\lambda \end{vmatrix} = 6 - 5\lambda + \lambda^2 - 2 = 0$$

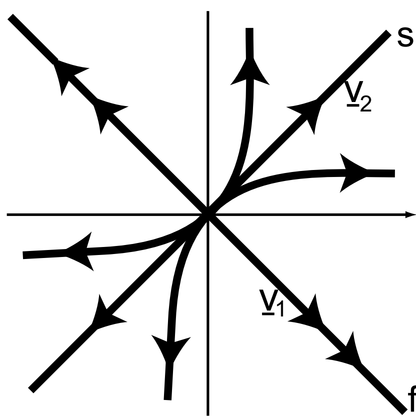
$$\lambda^2 - 5\lambda + 4 = 0$$

$$\lambda = \frac{5 \pm \sqrt{25-16}}{2} = \begin{cases} +4 \\ +1 \end{cases}$$

eigenvectors:

$$\lambda_1 = 4: \qquad 3x - 2y = 4x \ \Rightarrow \ y = -\frac{1}{2}x \qquad \underline{v}_1 = \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix}$$

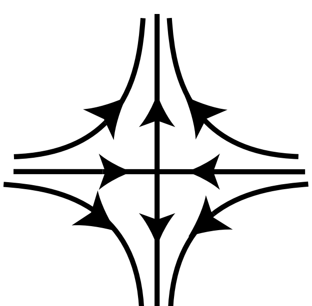
$$\lambda_2 = 1: \qquad 3x - 2y = x \ \Rightarrow \ y = x \qquad \underline{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$



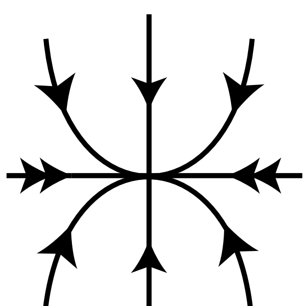
in this graph the (straight) outgoing manifolds should not be orthogonal

Possible Phase Portraits:

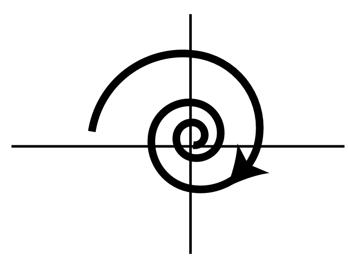
i) generic cases:



saddle

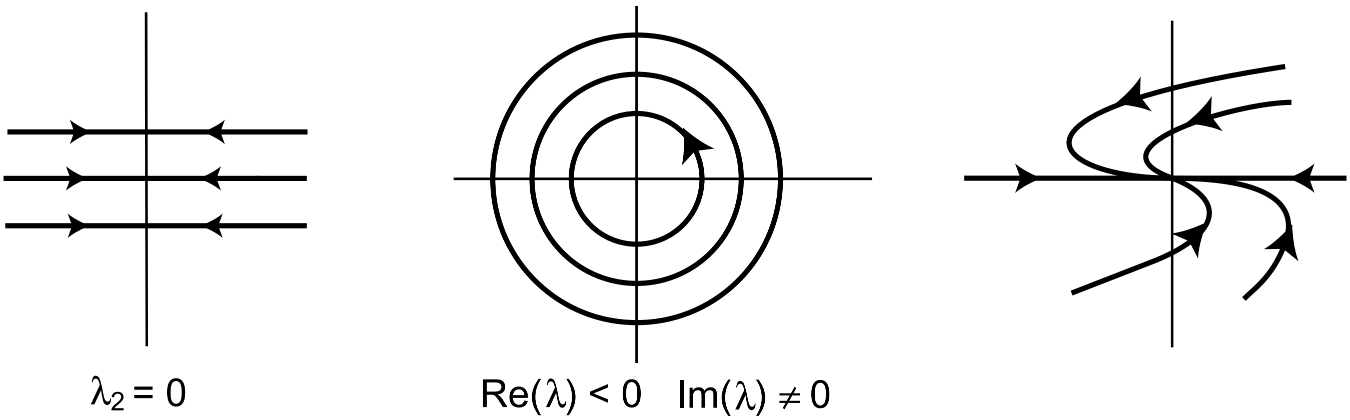


node
(stable or unstable)



spiral (stable)
complex eigenvalue
Re(λ) < 0

ii) special cases:



center has wrong labeling of eigenvalues: $Re(\lambda) = 0$

Last phase plane diagram shows degenerate node: only a single eigenvector

$$\underline{\underline{\mathbf{A}}} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \qquad \lambda < 0$$

the system is *almost* oscillating

- Calculation of eigenvalues in 2d:

$$\det \underline{\underline{A}} = \det(\underline{\underline{S}}^{-1}\underline{\underline{A}}\,\underline{\underline{S}}) = \lambda_1\lambda_2 \qquad tr\underline{\underline{S}}^{-1}\underline{\underline{A}}\,\underline{\underline{S}} = \lambda_1 + \lambda_2$$

quadratic formula

$$\lambda_{1,2} = \frac{+tr\underline{\underline{A}} \pm \sqrt{(tr\underline{\underline{A}})^2 - 4\det\underline{\underline{A}}}}{2}$$

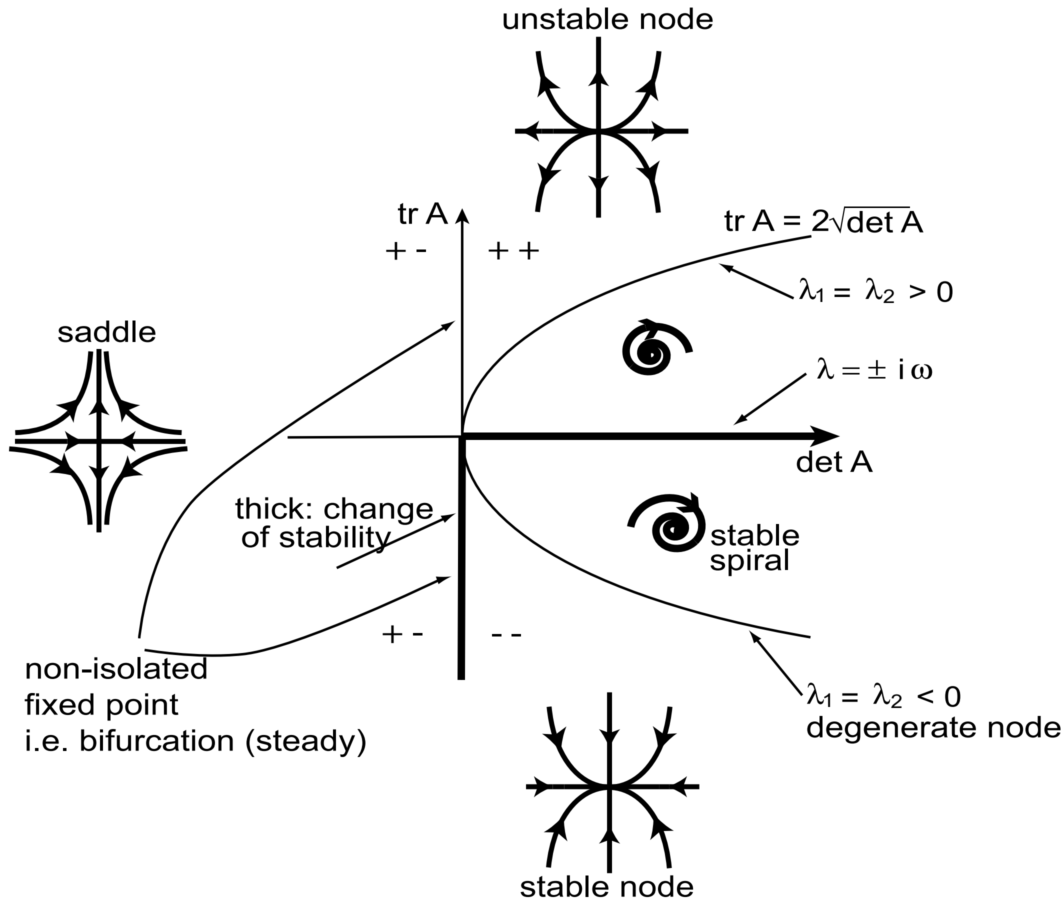
Change in stability: Re $(\lambda_i) = 0$

i) $tr\underline{\underline{A}} = 0$ and $\det\underline{\underline{A}} > 0$ $\quad\Rightarrow\quad$ $\lambda = \pm i\omega$ complex pair crossing imaginary axis

ii) $tr\underline{\underline{A}} < 0$ and $\det\underline{\underline{A}} = 0$ $\quad\Rightarrow\quad$ $\lambda_1 = 0$ $\lambda_2 < 0$ single zero eigenvalue

Change in character:

real $\leftrightarrow$ complex $\qquad \left(tr\underline{\underline{A}}\right)^2 = 4\det\underline{\underline{A}}$



Notes:

- degenerate node ⇒ border between nodes and spirals, does not quite oscillate
- non-isolated fixed points: steady bifurcation, one or more fixed points are created/annihilated (details depend on nonlinearities)

3.2 Stability

So far we had linear stability. In higher dimensions new aspects arise.

Linear Stability

- with respect to infinitesimal perturbations
- determined by linearization

Example:
Damped-driven pendulum

$$m\ell^2\,\ddot{\theta} + \beta\,\dot{\theta} = -mg\ell\,\sin\theta + \Gamma$$

write as first-order system:

$$\begin{aligned}\dot{x} &= y \equiv F_x(x_1 y) \\ \dot{y} &= -\frac{\beta}{m\ell^2}y - \frac{mg\ell}{m\ell^2}\sin x + \Gamma \equiv F_y(x_1 y)\end{aligned}$$

Fixed points:

$$y_0 = 0 \qquad \& \qquad mg\ell \sin x_0 = \Gamma$$

Expand:

$$\begin{aligned}x &= x_0 + \epsilon x_1(t) \qquad \epsilon \ll 1 \\ y &= y_0 + \epsilon y_1(t)\end{aligned}$$

Insert:

$$\begin{aligned}\epsilon \dot{x}_1 &= F_x\left(x_0 + \epsilon x_1(t), y_0 + \epsilon y_1(t)\right) = \\ &= \underbrace{F_x(x_0, y_0)}_{0} + \epsilon x_1\, \partial_x F_x|_{(x_0,y_0)} + \epsilon y_1 \partial_y F_x|_{(x_0,y_0)} + \mathcal{O}(\epsilon^2)\end{aligned}$$

In matrix form:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{y}_1 \end{pmatrix} = \underbrace{\begin{pmatrix} \partial_x F_x & \partial_y F_x \\ \partial_x F_y & \partial_y F_y \end{pmatrix}}_{\text{Jacobian } \underline{\underline{\mathbf{A}}}} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

$\Rightarrow$ linear stability determined by eigenvalues of Jacobian

For pendulum

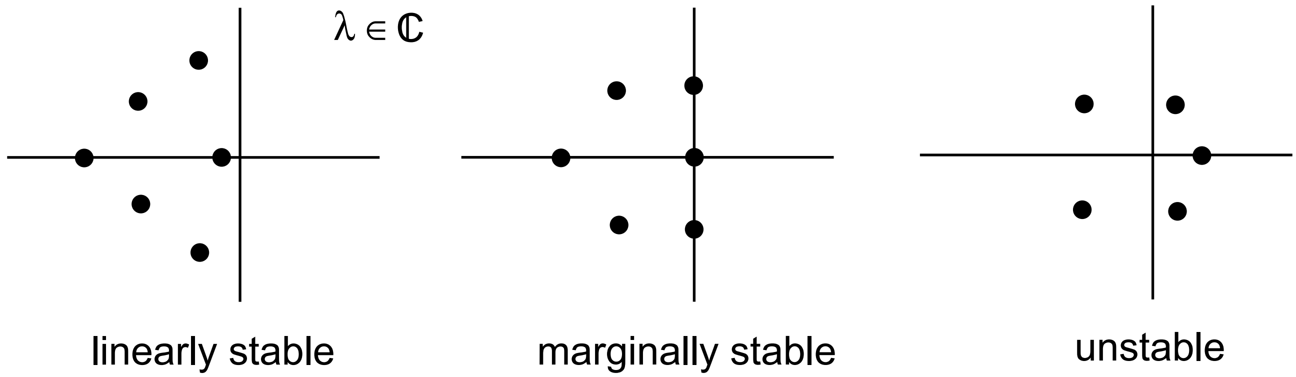
$$\underline{\underline{A}} = \begin{pmatrix} 0 & 1 \\ -\frac{g}{\ell}\cos x_0 & -\frac{\beta}{m\ell^2} \end{pmatrix}$$

eigenvalues:

$$(-\lambda)(-\lambda - \frac{\beta}{m\ell^2}) + \frac{g}{\ell}\cos x_0 = 0$$

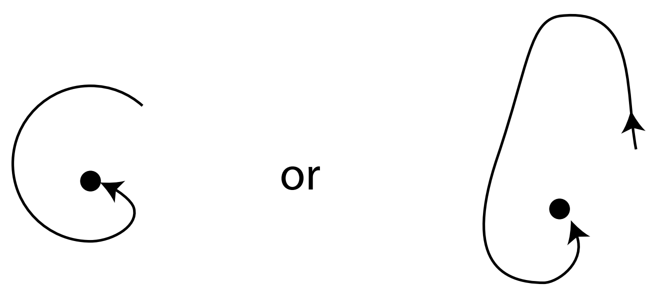
$$\begin{aligned}\lambda^2 &+ \lambda\frac{\beta}{m\ell^2} + \frac{g}{\ell}\cos x_0 = 0 \\ \lambda_{1,2} &= \frac{\beta}{m\ell^2} \pm \frac{1}{2}\sqrt{\left(\frac{\beta}{m\ell^2}\right)^2 - 4\frac{g}{\ell}\cos x_0 = 0}\end{aligned}$$

Eigenvalues in complex plane:



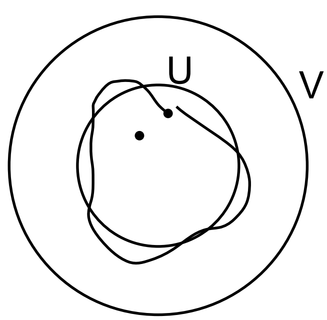
Attractor:
A set of points (e.g. a fixed point) is attracting if all trajectories that start close to it converge to it, i.e.

$$\text{for all } \mathbf{x}(0) \text{ near } \mathbf{x}_{FP} : \mathbf{x}(t) \to \mathbf{x}_{FP} \text{ for } t \to \infty$$



Notes: system need not approach attractor right away

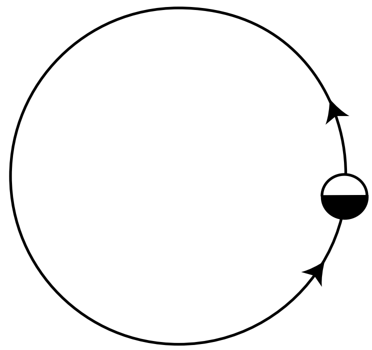
Lyapunov Stability:
A set is (Lyapunov) stable if all orbits that start close to it remain close to it for all times. Technically, for any neighborhood V of $\mathbf{x}_{FP}$ one can find a $U \subseteq V$ such that if $\mathbf{x}(0) \in U$ then $\mathbf{x}(t) \in V$ for all times.



Notes:

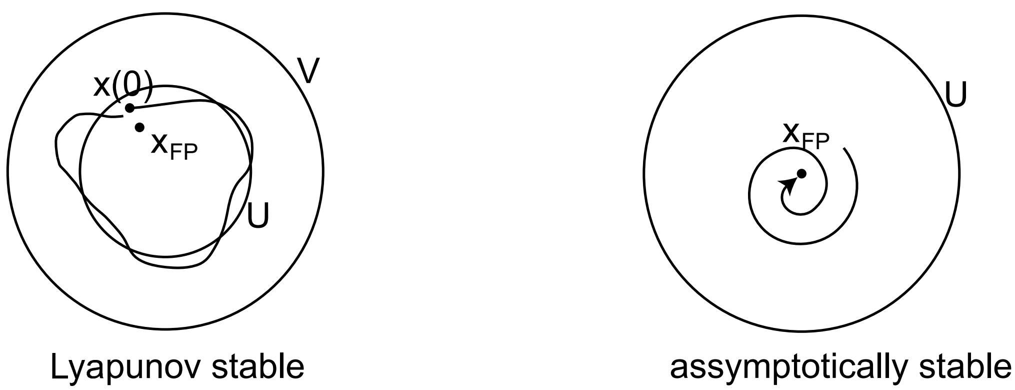
- Lyapunov stability of a set does not imply it is an attractor.

- attractor does not have to be Lyapunov stable



This fixed point is attracting but not Lyapunov stable (cannot find neighborhood that limits excursion)

Asymptotic Stability:
A set is asymptotically stable if it is attracting and Lyapunov stable, i.e. if all orbits that start sufficiently close to a fixed point converge to it as $t \to \infty$.

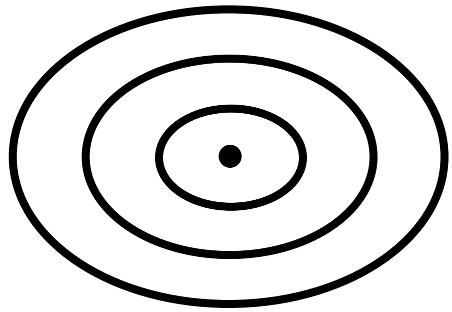


Notes:

- asymptotically stable ⇒ fixed point is attracting, it is an attractor.
- linear stability ⇒ asymptotic stability ⇒ Lyapunov stability
- linear instability ⇒ instability
- **But:** asymptotic or Lyapunov stability **do not imply** linear stability

Examples:

- Center is Lyapunov stable, but linearly neither stable nor unstable (marginally stable)



- Stability can be determined purely by nonlinear terms

$$\dot{x} = \alpha x^3$$

$x = 0$ linearly marginally stable

$$x(t) = \pm\sqrt{\frac{x_0^2}{1 - 2x_0^2\alpha t}}$$

$\Rightarrow$ $\alpha > 0$ (nonlinearly) unstable
$\alpha < 0$ (nonlinearly) asymptotically stable

3.3 General Properties of the Phase Plane

3.3.1 Hartman-Grobman theorem

Linear systems: can be completely understood

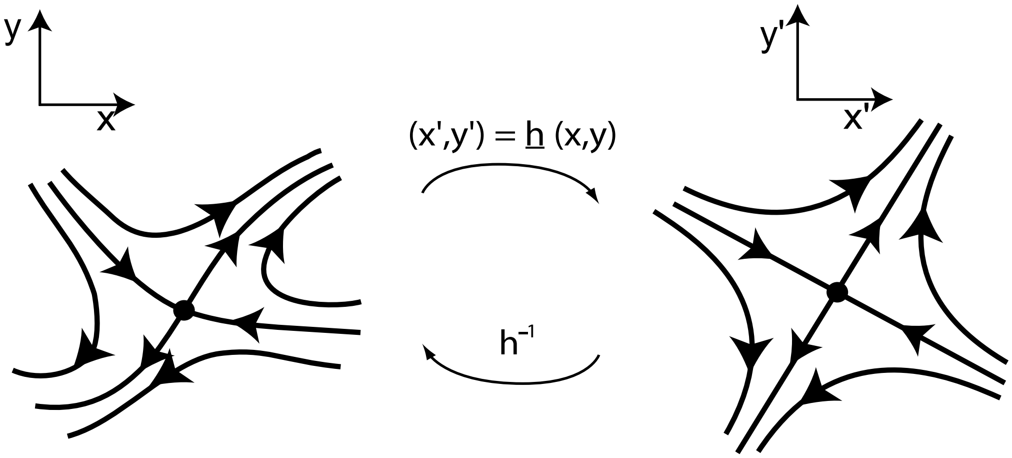
How much of that can be transferred to nonlinear systems?

Definition: A fixed point $\underline{x}_0$ of $\dot{\underline{x}} = \underline{f}(\underline{x})$ is called hyperbolic if all eigenvalues of $\frac{\partial f_i}{\partial x_j}$ have non-zero real parts.

Thus: in all directions a hyperbolic fixed point is either linearly attractive or repulsive. No marginal direction.

Hartman-Grobman Theorem:

If $\underline{x}_0$ is a hyperbolic fixed point of $\dot{\underline{x}} = \underline{f}(\underline{x})$ then there exists a continuous invertible function $\underline{h}(\underline{x})$ that is defined on some neighborhood of $\underline{x}_0$ and maps all orbits of the nonlinear flow into those of the linear flow. The map can be chosen so that the parameterization of orbits by time is preserved.



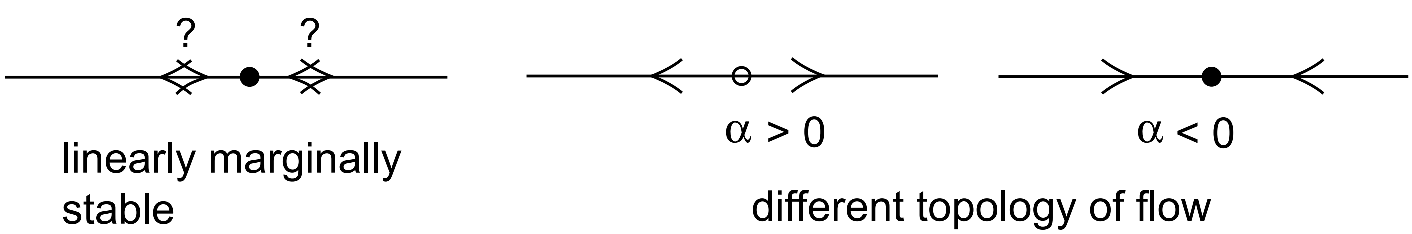
Thus:

- For hyperbolic fixed point $\underline{x}_0$ the linearization of the flow gives the **topology** of the nonlinear flow in a neighborhood of $\underline{x}_0$.

Note:

- If fixed point is not hyperbolic, linearization does not give sufficient information:

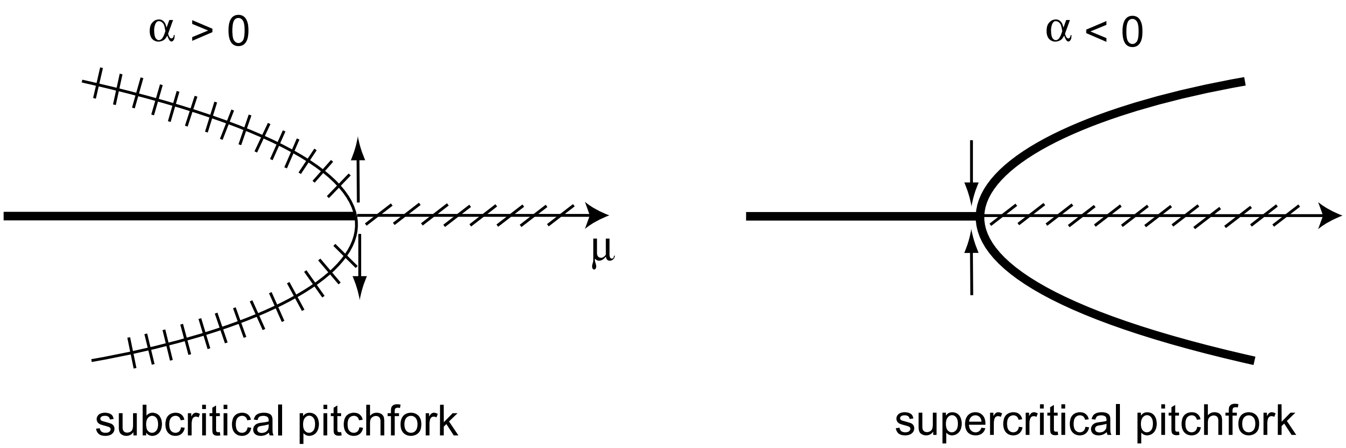
$$\dot{x} = \alpha x^3$$



- At any bifurcation the fixed point is not hyperbolic.

$$\dot{x} = \mu x + \alpha x^3$$

at $\mu = 0$ linear systems equal for all α.



Note:

- the flow in the vicinity of a hyperbolic fixed point is structurally stable. This is not the case without hyperbolicity, e.g. for centers, fixed points undergoing bifurcations.

3.3.2 Ruling out Persistent Dynamics

For what kind of systems can one rule out persistent dynamics like periodic orbits?

i) Gradient Systems, Potential Systems

If

$$\underline{\dot{x}} = -\nabla V(\underline{x}) \qquad \text{i.e.} \qquad \dot{x}_i = -\frac{\partial V}{\partial x_i}$$

with $V \geq V_0$ for all $\underline{x}$ (bounded from below)
then

$$\frac{dV}{dt} = \sum_i \frac{\partial V}{\partial x_i}\dot{x}_i = -\sum_i \left(\frac{\partial V}{\partial x_i}\right)^2 \leq 0$$

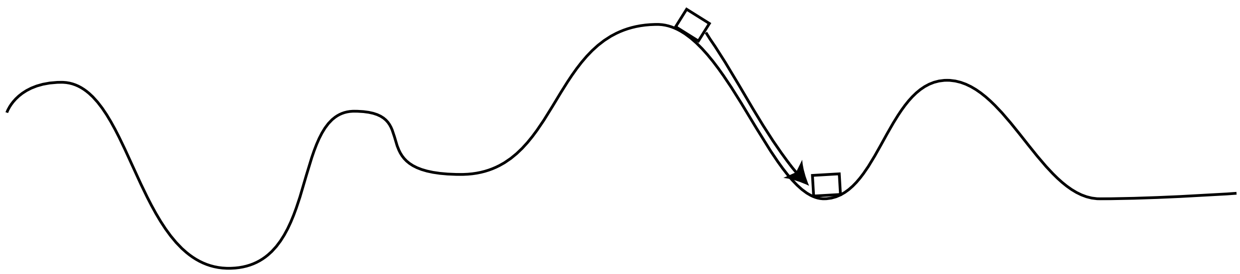
Thus V eventually reaches a (local) minimum.

Then:

$$\frac{dV}{dt} = 0 \Leftrightarrow \frac{\partial V}{\partial x_i} = 0 \Leftrightarrow \dot{x}_i = 0 \qquad \text{for all } i$$

Thus, system goes to a fixed point.

Example: Mechanical overdamped particle in potential



ii) Lyapunov Functional

Need not $\underline{\dot{x}} = -\nabla V$

Sufficient for ruling out periodic orbits:

Assume $V(\underline{x}) > V_0$ for all $\underline{x} \neq \underline{x}_0$ with $\underline{x}_0$ fixed point

- if $\frac{dV}{dt} \leq 0$ for all $\underline{x} \neq \underline{x}_0$ in neighborhood $\mathcal{U}$
  then $\underline{x}_0$ Lyapunov stable
- if $\frac{dV}{dt} < 0$ for all $\underline{x} \neq \underline{x}_0$ in $\mathcal{U}$
  then $\underline{x}_0$ asymptotically stable

Note: Such a V is called a *Lyapunov functional.*

Example:

a) damped particle in bounded potential

$$\ddot{x} + \beta\dot{x} = -\frac{d\mathcal{U}}{dx}$$

i.e.

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= -\beta v - \frac{d\mathcal{U}}{dx} \end{aligned}$$

Try total energy

$$V = \frac{1}{2}\dot{x}^2 + \mathcal{U} = \frac{1}{2}v^2 + \mathcal{U}$$

$$\frac{dV}{dt} = v\dot{v} + \frac{d\mathcal{U}}{dx}\dot{x} = v(-\beta v - \frac{d\mathcal{U}}{dx}) + \frac{d\mathcal{U}}{dx}v = -\beta v^2 < 0$$

$\Rightarrow$ fixed points asymptotically stable, no periodic orbits.

b)

$$\begin{aligned} \dot{x} &= -x + 4y \\ \dot{y} &= -x - y^3 \end{aligned}$$

Simplest attempt: try quadratic function that is bounded from below:

$$V = x^2 + ay^2 \quad \text{with } a > 0.$$

$$\begin{aligned}\frac{dV}{dt} &= 2x(-x+4y) + 2\alpha y(-x-y^3) \\ &= \underbrace{-2x^2}_{\leq 0} + \underbrace{xy(8-2a)}_{\text{undetermined}} \underbrace{-2ay^4}_{\leq 0}\end{aligned}$$

$\Rightarrow$ choose $a = 4 \Rightarrow \frac{dV}{dt} < 0$ for $x \neq 0 \neq y$

$\Rightarrow$ $(0,0)$ asymptotically stable and no periodic orbits

Note: potentials rule out persistent dynamics in **arbitrary dimensions**.

3.3.3 Poincaré-Bendixson Theorem

- How complex can the dynamics be in 2 dimensions?
- Can we guarantee a periodic orbit without explicitly calculating it?

Poincaré-Bendixson Theorem:

Assume

- R is a closed bounded subset of the plane
- $\dot{\underline{x}} = \underline{f}(\underline{x})$ with $\underline{f}(\underline{x})$ continuously differentiable on an open set containing R

then

> any orbit that remains in R for all t either converges to a fixed point or to a periodic orbit.

Simple Illustration:

- in one dimension we had: no periodic orbits
  fixed point *divides* phase line into *left* and *right*
  $\Rightarrow$ cannot go back and forth
  $\Rightarrow$ no oscillatory approach to fixed point
  $\Rightarrow$ no persistent oscillations

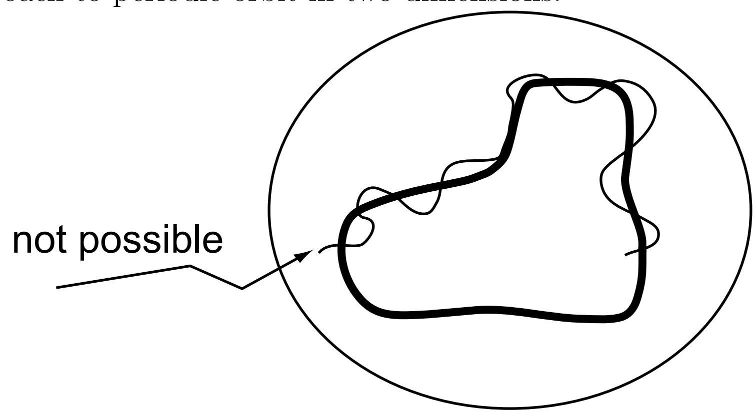
- in two dimensions:
  what is more "complicated" than periodic orbit?
  periodic orbit has single fundamental frequency ω

$$x(t) = A \cos \omega t + B \cos 2\omega t + C \cos 3\omega t + \dots$$

  Can we have 2 incommensurate frequencies? I.e.

$$\frac{\omega_1}{\omega_2} \neq \frac{m}{n} \qquad \text{irrational}$$

  Consider approach to periodic orbit in two dimensions:



  Periodic orbit separates phase plane into *inside* and *outside*.
  Oscillatory approach to periodic orbit not possible $\Rightarrow$ No second frequency.
  System has to go to fixed point or periodic orbit.

Consequence of Poincaré-Bendixson:

- The only attractors of 2d-flows are fixed points or periodic orbits
- **No chaos in 2 dimensions.**

3.3.4 Phase Portraits

A phase portrait captures all relevant features of the phase plane:

- fixed points with their stable/unstable manifolds
- periodic orbits
- separatrices and other additional orbits that visualize the flow

Example 1:

$$
\begin{aligned}
\dot{x} &= f(x,y) = y \\
\dot{y} &= g(x,y) = x(1+y) - 1
\end{aligned}
$$

Nullclines:

$$
\begin{aligned}
f(x,y) = 0 \qquad &\Rightarrow \quad y = 0 \\
g(x,y) = 0 = x(1+y) - 1 \qquad &\Rightarrow \quad y = \frac{1}{x} - 1
\end{aligned}
$$

Fixed Points:

$$y = 0 \qquad x = 1$$

Stability of Fixed Point:

$$x = 1 + \epsilon\, x_1 \qquad y = \epsilon y_1$$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{y}_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

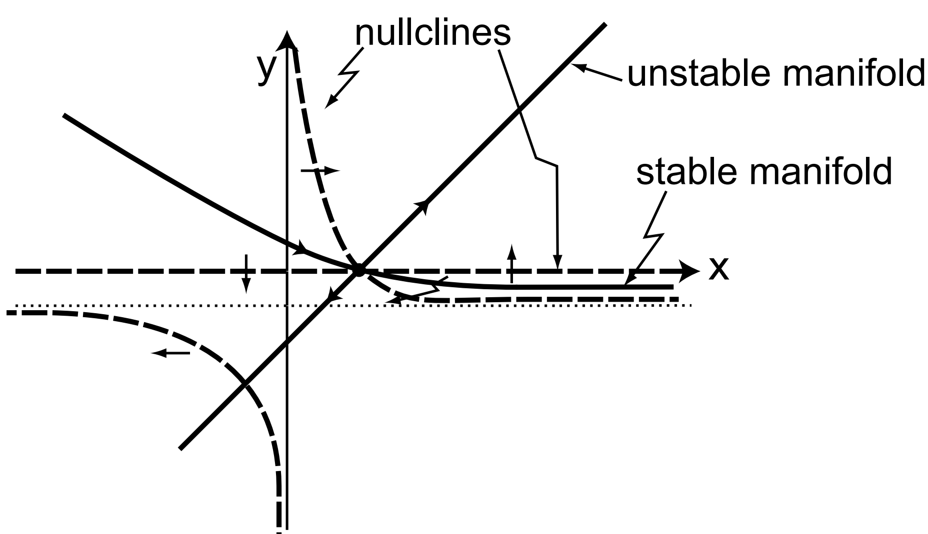
$$\lambda_{1,2} = \frac{1 \pm \sqrt{5}}{2} \qquad \text{saddle point}$$

Eigenvectors:

$$\begin{pmatrix} x_0^{(1,2)} \\ y_0^{(1,2)} \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \pm \sqrt{5} \end{pmatrix}$$

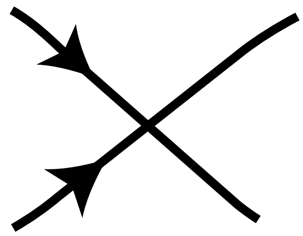
Note:

- Fixed point is hyperbolic $\Rightarrow$ linear eigenvectors give directions of nonlinear stable and unstable manifolds



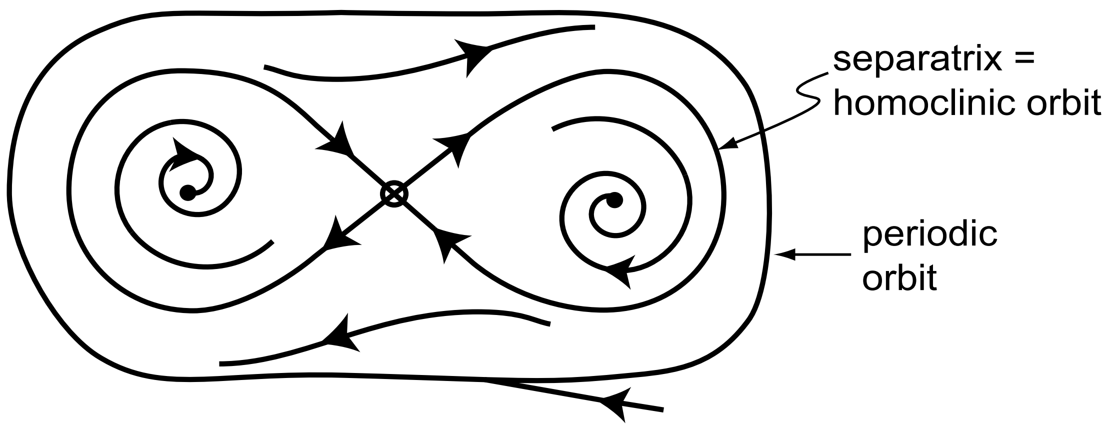
Notes:

- For $\underline{\dot{x}} = \underline{f}(\underline{x})$ solution unique if all $\frac{\partial f_i}{\partial x_j}$ are continuous
  $\Rightarrow$ orbits do not intersect.



  intersection would imply non-unique solution

- more complicated phase portraits can contain
  - separatrix: separates basins of attraction of different attractors
  - heteroclinic orbit: unstable manifold of one fixed point is the stable manifold of another, orbit connects the two fixed points
  - homoclinic orbits: orbit that returns to the same fixed point



Example 2: Glycolysis Oscillations

Yeast cells break down sugar by glycolysis: simple model

$$\begin{aligned} \text{ADP adenosine diphosphate} \qquad \dot{x} &= -x + ay + x^2 y = f(x,y) \\ \text{F6P fructose-6-phosphate} \qquad \dot{y} &= b - ay - x^2 y = g(x,y) \end{aligned}$$

For which parameter ranges can one guarantee the existence of a stable periodic orbit?

Phase portrait:

study **nullclines**: $\dot{x} = 0$ or $\dot{y} = 0$

$$f = 0 \qquad \Rightarrow y = \frac{x}{a + x^2}$$

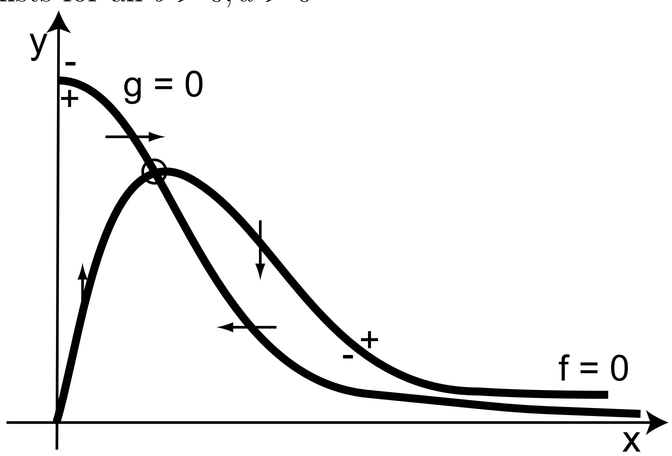
$$g = 0 \qquad \Rightarrow y = \frac{b}{a + x^2}$$

⇒ fixed point at

$$y = \frac{x}{a + x^2} = \frac{b}{a + x^2}$$

$$\Rightarrow \qquad x = b \qquad \text{and} \qquad y = \frac{b}{a + b^2}$$

exists for all $b > 0, a > 0$



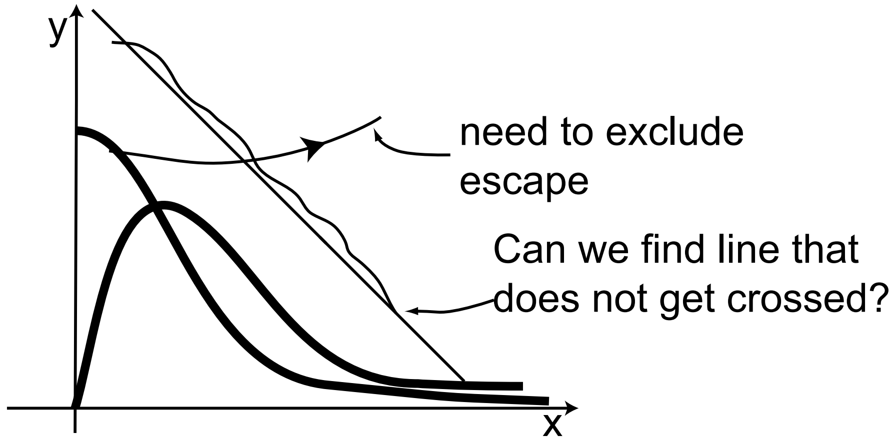
Null clines show: spiraling motion

- to fixed point?
- to periodic orbit? which?
- to infinity?

To use Poincaré-Bendixson:

1. need trapping region $\mathcal{R}$
2. exclude fixed points from trapping region

1) Trapping Region:



Consider large x and y (check possibility of escape)

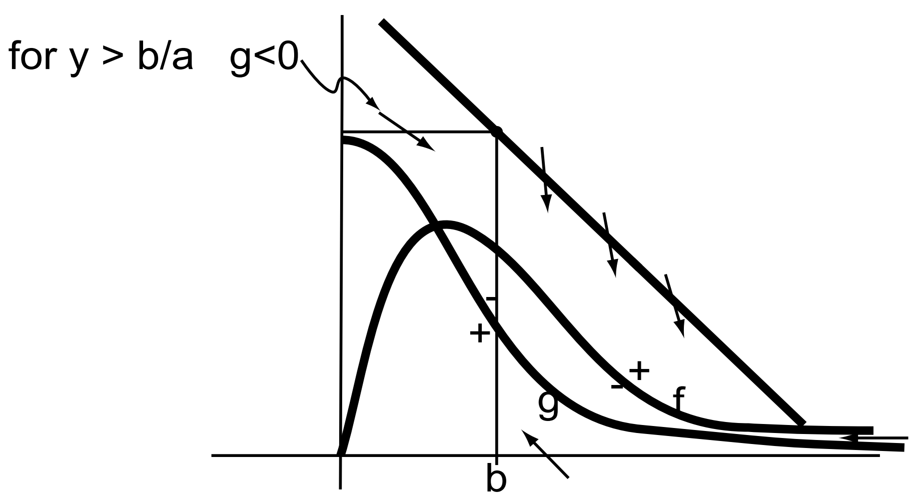
$$\left.\begin{array}{lcl}\dot{x} & \sim & x^2 y \\ \dot{y} & \sim & -x^2 y\end{array}\right\} \text{ along orbit one has: } \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \sim -1$$

Show that slope is steeper than -1
compare $|\dot{x}|$ with $|\dot{y}|$ more precisely

$$\begin{aligned}\dot{x} - (-\dot{y}) &= -x + ay + x^2\, y + b - ay - x^2\, y \\ &= b - x\end{aligned}$$

$\Rightarrow$ for $x > b$ $\quad |\dot{x}| < |\dot{y}|$
$\Rightarrow$ flow inward along $y = -x + C$ for $x > b$ and C large enough

for $y > \frac{b}{a}$ we have $g < 0$
$\Rightarrow$ flow inward for $y > b/a$



2) Fixed Points:

only a single fixed point $\left(b, \frac{b}{a+b^2}\right)$

Stability analysis shows fixed point unstable for

$$1 - 2a - \sqrt{1-8a} < 2b^2 < 1 - 2a + \sqrt{1-8a}$$

$\Rightarrow$ limit cycle guaranteed for this range of b (if $a \leq \frac{1}{8}$)
Instability at $2(b_H^{(1,2)})^2 = 1 - 2a \pm \sqrt{1-8a}$ is Hopf bifurcation. Oscillations occur for $b_H^{(1)} < b < b_H^{(2)}$. No steady bifurcation possible.

Formally: trapping region needs to exclude (small domain around fixed point)
outside this range expect convergence to fixed point.

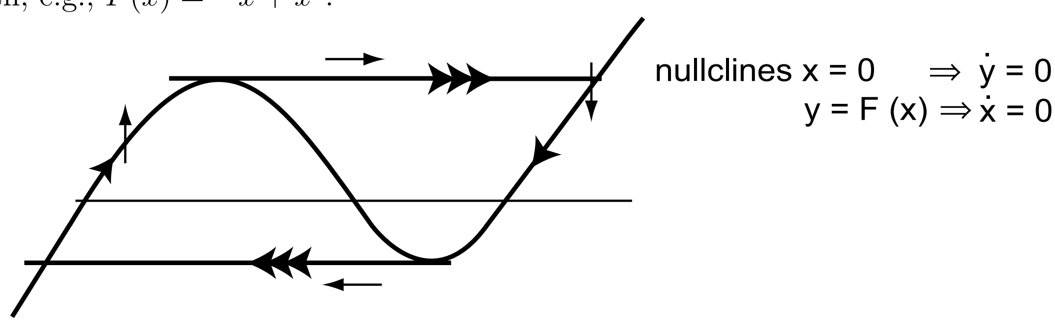
3.4 Relaxation Oscillations

Class of systems for which one can see the periodic orbit relatively easily:
N-shaped nullcline

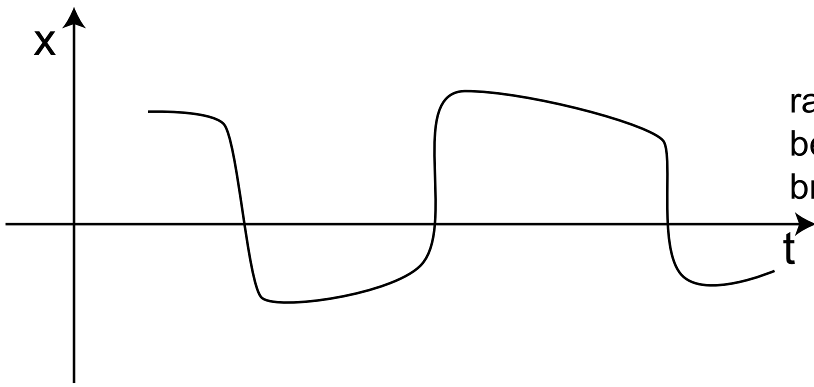
Consider for $\mu \gg 1$:

$$\begin{aligned} \dot{x} &= \mu\left(y - F(x)\right) \\ \dot{y} &= x \end{aligned}$$

with, e.g., $F(x) = -x + x^3$.



For $\mu \gg 1$ horizontal motion much faster than vertical motion, except near the nullcline $y = F(x)$
$\Rightarrow$ *slow* branch and *fast* branch on the periodic orbit



Period of periodic orbit determined mostly by time spent on slow branch.

On slow branch

$$y \sim F(x) \qquad \Rightarrow \qquad \dot{y} \sim \frac{dF}{dx}\,\dot{x}$$
$$\text{use} \qquad \dot{y} = x \quad \Rightarrow \qquad \dot{x} = \frac{x}{\frac{dF}{dx}} \equiv G(x)$$

$$T = \int dt \sim \int \frac{dt}{dx}\,dx = \int \frac{1}{\dot{x}}\,dx = \int \frac{1}{G(x)}\,dx$$

3.5 Weakly Nonlinear Oscillators

Exact nonlinear solutions usually impossible to get.
Develop techniques to calculate *analytically*
- *systematic approximation* to periodic orbits and
- *systematic approximation* to transients approaching periodic orbits.

3.5.1 Failure of Regular Perturbation Theory

Consider simple linear example to demonstrate problem

$$\ddot{x} + 2\epsilon\beta\dot{x} + (1+\epsilon\Omega)^2 x = 0$$

with some initial condition like $x(0) = 0$, $\dot{x}(0) = 1$.

Exact solution:

$$x \sim e^{\lambda t} \qquad \Rightarrow \qquad \lambda^2 + 2\epsilon\beta\lambda + (1+\epsilon\Omega)^2 = 0$$

$$
\begin{aligned}
\lambda_{1,2} &= \frac{-2\epsilon\beta \pm \sqrt{4\epsilon^2\beta^2 - 4(1+\epsilon\Omega)^2}}{2} \\
&= \pm i\sqrt{(1+\epsilon\Omega)^2 - \epsilon^2\beta^2} - \epsilon\beta \\
\Rightarrow & \\
x_{exact} &= e^{-\epsilon\beta t}\left(Ae^{i\omega t} + A^* e^{-i\omega t}\right) \\
\text{with} & \\
\omega &= \sqrt{(1+\epsilon\Omega)^2 - \epsilon^2\beta^2}
\end{aligned}
$$

Attempt perturbation solution using

$$x = x_0 + \epsilon x_1 + h.o.t$$

Insert

$$
\begin{aligned}
\frac{d^2}{dt^2}(x_0 + \epsilon x_1 + \ldots) \quad &+ \quad 2\epsilon\beta\frac{d}{dt}(x_0 + \epsilon x_1 + \ldots) \\
&+ \quad (1+\epsilon\Omega)^2(x_0 + \epsilon x_1 + \ldots) = 0
\end{aligned}
$$

Collect orders in ϵ:
$\mathcal{O}(\epsilon^0)$:

$$
\begin{aligned}
\frac{d^2}{dt^2}x_0 + x_0 &= 0 \\
x_0 = Ae^{it} + A^* e^{-it} &= 2A_r \cos t - 2A_i \sin t
\end{aligned}
$$

$\mathcal{O}(\epsilon^1)$:

$$\frac{d^2}{dt^2}x_1 + 2\beta\frac{d}{dt}x_0 + 2\Omega x_0 + x_1 = 0$$

$$\frac{d^2}{dt^2}x_1 + x_1 = -2\underbrace{\left(-2i\beta Ae^{it} - 2\Omega Ae^{it}\right)}_{\sim \text{ resonant forcing}} + c.c.$$

This is a second-order constant-coefficient *inhomogeneous* differential equation:
General solution:

$$x_1(t) = x_{homo}(t) + x_{particular}(t)$$

with

$$\frac{d^2}{dt^2}x_{homo} + x_{homo} = 0 \qquad \Rightarrow x_{homo} = A_1 e^{it} + c.c.$$

Guess ('ansatz') for particular solution (since inhomogeneity is simple exponential function):

$$x_{particular} = Be^{it} + c.c.$$

However:

$$\frac{d^2}{dt^2}Be^{it} + Be^{it} = 0 \qquad \Rightarrow \text{cannot balance inhomogeneity on r.h.s.}$$

Could use method of *variation of constants* $x_{particular} = B(t)e^{it}$ and reduce the order of the equation and solve the resulting first-order equation by integration.

Here try ansatz:

$$x_{particular} = B\,t\,e^{it} + c.c.$$

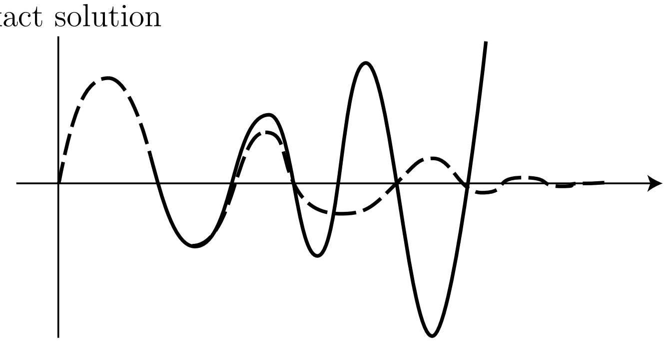
Insert:

$$\begin{aligned} \frac{d^2}{dt^2}x_{particular} + x_{particular} &= \\ B\left(0\,e^{it} + 2ie^{it} + (i)^2\,t\,e^{it} + t\,e^{it}\right) + c.c. &= -2\left(-2i\beta Ae^{it} - 2\Omega Ae^{it}\right) + c.c. \\ \Rightarrow B &= \frac{1}{2i}\left(-2i\beta - 2\Omega\right)A \end{aligned}$$

Notes:

- resonant forcing leads to (linear) growth without bounds: secular terms
- solution breaks down for $t = \mathcal{O}(\epsilon^{-1})$

Compare with exact solution



Approximation (solid line) grows (instead of decay) and has wrong frequency.

But: approximation is expansion of exact solution in ϵ:

$$x_{exact} = \underbrace{e^{-\epsilon\beta t}}_{1-\epsilon\beta t+\mathcal{O}(\epsilon^2)} \left(Ae^{i\omega t} + c.c.\right)$$

with

$$\omega = \underbrace{\sqrt{(1+\epsilon\Omega)^2 - \epsilon^2\beta^2}}_{1+\epsilon\Omega+\mathcal{O}(\epsilon^2)}$$

$$x_{exact} = Ae^{it} + \epsilon\left(-\beta t + i\Omega t\right) + \mathcal{O}(\epsilon^2) + c.c.$$

Thus:

- Straightforward perturbation expansion misses
  - slow growth/decay
  - small change in frequency
- secular terms suggest what true solution is doing.

3.5.2 Multiple Scales

Exact solution suggests that there are multiple time scales

$$t \qquad \text{and} \qquad T_1 = \epsilon t \qquad \text{and} \qquad T_2 = \epsilon^2 t \ldots$$

$$x_{exact} = A\sin(t + \Omega T_1 + \ldots)e^{-\beta T_1}$$

$\Rightarrow$ try $\qquad x = x_0(\hat{t}, T_1, T_2 \ldots) + \epsilon x_1(\hat{t}, T_1, T_2 \ldots) + \ldots$

Note:

- in this approach the two (or more times) are treated as essentially independent variables ($T \equiv T_1$):

$$\frac{d}{dt}x = \partial_{\hat{t}} x \frac{d\hat{t}}{dt} + \partial_T x \frac{dT}{dt} + \cdots = \left(\partial_{\hat{t}} + \epsilon\partial_T + \mathcal{O}(\epsilon^2)\right)x$$

Redo same linear problem:

$$\begin{aligned}(\partial_{\hat{t}} + \epsilon\partial_T + \cdots)^2 (x_0 + \epsilon x_1 + \cdots) &+ 2\epsilon\left(\partial_t + \epsilon\partial_T + \cdots\right)(x_0 + \epsilon x_1 + \cdots) \\ &+ (1+\Omega\epsilon)(x_0 + \epsilon x_1 + \cdots) = 0\end{aligned}$$

$\mathcal{O}(\epsilon^0)$:

$$\frac{d^2}{dt^2}x_0 + x_0 = 0$$

$$x_0 = Ae^{it} + A^* e^{-it} = 2A_r \cos t - 2A_i \sin t$$

Note:

- now A is not constant: $A = A(T, T_2, \ldots)$

$\mathcal{O}(\epsilon^1)$:

$$2\partial_{\hat{t}}\partial_T x_0 + \partial_{\hat{t}}^2 x_1 + 2\beta\partial_{\hat{t}} x_0 + 2\Omega x_0 + x_1 = 0$$
$$\partial_{\hat{t}}^2 x_1 + x_1 = -2\left(i\frac{d}{dT}Ae^{it} - 2i\beta A - 2\Omega A e^{it}\right) + c.c.$$

Need to avoid secular terms $\Rightarrow$ require

$$\frac{d}{dT}A = -\beta A + i\Omega A$$

then no secular terms arise that grow linearly in time.

Solution

$$A = \mathcal{A}e^{-\beta T + i\Omega T}$$
$$x_0 = \mathcal{A}e^{-\beta T}e^{i\hat{t} + i\Omega T} + c.c. = \mathcal{A}e^{-\epsilon\beta t}e^{i(1+\epsilon\Omega)t} + c.c.$$

Thus:

- Two-timing (multiple scales) avoids secular terms and gets frequency shift and slow damping correct to the order considered
- calculation easier in complex exponentials

Example: Duffing oscillator

$$\ddot{x} + x + \epsilon x^3 = 0$$

Ansatz:

$$x = x_0(t, T) + \epsilon x_1(t, T) + \cdots$$
$$\left(\frac{d}{dt}\right)^2 \to \partial_t^2 + 2\epsilon\partial_t\partial_T + O(\epsilon^2)$$

$\mathcal{O}(\epsilon^0)$:

$$\partial_x^2 x_0 + x_0 = 0 \qquad x_0 = Ae^{it} + A^* e^{-it}$$

$\mathcal{O}(\epsilon^1)$:

$$\partial_t^2 x_1 + x_1 + \underbrace{2\partial_t \partial_T x_0}_{2i\frac{dA}{dT}e^{it}+c.c.} + \underbrace{x_0^3}_{A^3 e^{3it}+3|A|^2 A e^{it}+3|A|^2 A^* e^{-it}+A^{*3}e^{-3it}} = 0$$

thus

$$\frac{d^2}{dt^2}x_1 + x_1 = \underbrace{e^{it}}_{\text{secular resonance term}} \left\{2i\frac{dA}{dT} + 3|A|^2 A\right\} + e^{3it}A^3 + c.c.$$

require:

$$\frac{dA}{dT} = +\frac{3}{2}i|A|^2 A \qquad \Rightarrow \qquad A = \mathcal{A}e^{\frac{3}{2}i\mathcal{A}^2 t}$$

$$x_0 = \mathcal{A}\, exp\left(i(1 + \frac{3}{2}\epsilon\mathcal{A}^2)t\right) + c.c.$$

Notes:

- nonlinearity induces frequency shift

  $\rightarrow$ soft and hard spring ($\epsilon \lessgtr 0$)

$$\ddot{x} + (1 + \epsilon x^2)x = 0$$

- at higher orders in ϵ additional frequency shifts

  $\Rightarrow$ approximate and exact solution get out of sync for $t \sim \mathcal{O}(\epsilon^{-2})$:

$$\cos\left((\omega + \epsilon\omega_1 + \underbrace{\epsilon^2\omega_2)t}_{\epsilon^2\omega_2 t\sim 2\pi \;\Rightarrow\; t=\mathcal{O}(\frac{1}{\epsilon^2})}\right)$$

- two-timing also very useful near bifurcation, where one time scale becomes very slow.

More general formulation:

At $\mathcal{O}(\epsilon)$ one obtains

$$Lx_1 = I(x_0)$$

with linear operator L singular: $Lx_0 = 0$

Operators correspond to matrices:

if $\underline{\underline{M}}\,\underline{x}_0 = 0$ then

- $\det(\underline{\underline{M}}) = 0$ and
- $\underline{\underline{M}}\,\underline{x} = \underline{b}$ has only a solution for special values of $\underline{b}$:

Solvability condition to remove secular terms ('Fredholm alternative' see later).

3.5.3 Hopf Bifurcation

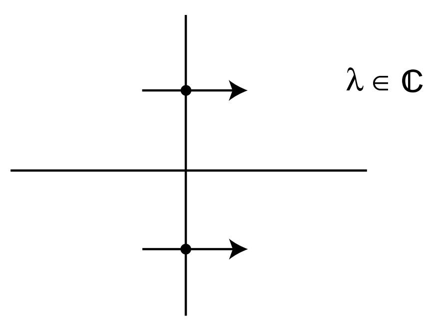
Consider example

$$\begin{aligned} \dot{x} &= \mu x - \omega y + \gamma x(x^2 + y^2) - \delta y(x^2 + y^2) \\ \dot{y} &= \omega x + \mu y + \delta x(x^2 + y^2) + \gamma y(x^2 + y^2) \end{aligned}$$

Linear stability of $(0, 0)$:

Eigenvalues of

$$\begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix} \Rightarrow \lambda_{1,2} = \mu \pm i\omega$$



Rewrite in terms of complex amplitude

$$A = x + iy$$

$$\Rightarrow \dot{A} = (\mu + i\omega)A + (\gamma + i\delta)|A|^2 A$$

rewrite again

$$A \;=\; Re^{i\Theta}$$

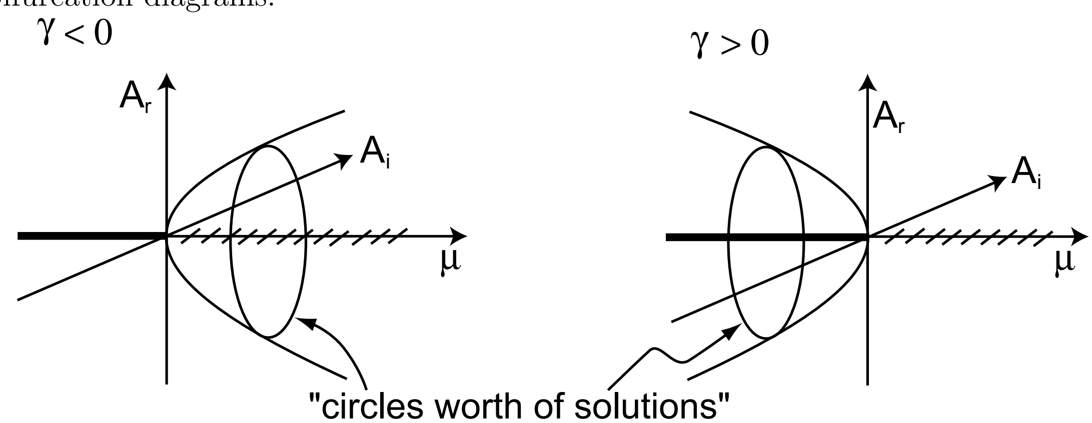
$$\Rightarrow \dot{R} = \mu R + \gamma R^3 \qquad \dot{\Theta} = \omega + \delta R^2$$

$\Rightarrow$ steady solution

$$R_0 \;=\; \sqrt{-\frac{\mu}{\gamma}} \qquad \Theta = (\omega + \delta R_0^2)t + \Theta_0$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \;=\; R_0 \begin{pmatrix} \cos\left[(\omega + \delta R_0^2)t + \Theta_0\right] \\ \sin\left[(\omega + \delta R_0^2)t + \Theta_0\right] \end{pmatrix} \qquad \text{periodic orbit}$$

Bifurcation diagrams:



Note:

- Solutions exist for any phase Θ_0: continuous family of solutions
- Although this example looks very special, it is the **normal form** for the Hopf bifurcation and also for weakly nonlinear oscillators.
  cf. Duffing result
  $$\frac{dA}{dT} = \frac{3}{2}|A|^2 A$$
  corresponds to $\mu = 0, \gamma = 0, \delta = \frac{3}{2}$,
- determinant of linearization does not vanish
  $\Rightarrow$ *via* implicit function theorem the number of fixed points does not change.

Example:

$$\dot{u} = \mu u - v + u^2$$
$$\dot{v} = u + \mu v + u^2$$

Linear stability of $(0,0)$ again

$$\begin{pmatrix} \mu & -1 \\ 1 & \mu \end{pmatrix} \Rightarrow \lambda = \mu \pm i$$

Eigenvectors at the bifurcation point

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \pm i \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$$

Compare Duffing oscillator: $\quad \ddot{x} + x = -\epsilon x^3$
There we expanded in ϵ: assumed nonlinear term weak

Here: assume u and v small

Based on previous example guess u, $v \sim \mu^{1/2}$

Expansion

$$\mu = \mu^2 \epsilon^2 \qquad T = \epsilon^2 t$$

$$\begin{pmatrix} u \\ v \end{pmatrix} = \epsilon \begin{pmatrix} u_1(t,T) \\ v_1(t,T) \end{pmatrix} + \epsilon^2 \begin{pmatrix} u_2(t,T) \\ v_2(t,T) \end{pmatrix} + \epsilon^3 \begin{pmatrix} u_3(t,T) \\ v_3(t,T) \end{pmatrix} + c.c.$$

insert:

$\mathcal{O}(\epsilon)$:

$$\left. \begin{array}{l} \frac{d}{dt} u_1 = -v_1 \\ \frac{d}{dt} v_1 = u_1 \end{array} \right\} \qquad \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \underbrace{A}_{A=A(T)} e^{it} \begin{pmatrix} 1 \\ -i \end{pmatrix} + \underbrace{A^* e^{-it} \begin{pmatrix} 1 \\ +i \end{pmatrix}}_{\text{c.c.}}$$

$\mathcal{O}(\epsilon^2)$:

$$\begin{aligned} \frac{d}{dt} u_2 &= -v_2 + u_1^2 \\ \frac{d}{dt} v_2 &= +u_2 + u_1^2 \end{aligned}$$

need u_1^2

$$u_1^2 = e^{2it} A_1^2 + 2|A|^2 + e^{-2it} A_1^{*2}$$

Ansatz

$$
\begin{aligned}
u_2 &= B_1 e^{2it} + B_1^* e^{-2it} + C_1 \\
v_2 &= B_2 e^{2it} + B_2^* e^{-2it} + C_2
\end{aligned}
$$

$\propto e^{2it}$:

$$
\begin{aligned}
2iB_1 &= -B_2 + A_1^2 \\
2iB_2 &= B_1 + A_1^2 \qquad \Rightarrow \qquad B_1 = 2iB_2 - A_1^2 \\
-4B_2 - 2iA_1^2 &= -B_2 + A_1^2
\end{aligned}
$$

$$
\underline{B_2 = \frac{1}{3}(-2i-1)A_1^2}
$$

$$
B_1 = A_1^2\left(\frac{2}{3}(+2-i)-1\right) = A_1^2\left(+\frac{1}{3}-\frac{2}{3}i\right) = \underline{+\frac{1}{3}(1-2i)A_1^2}
$$

$\propto e^{0it}$:

$$
\begin{aligned}
0 &= -C_2 + 2|A_1|^2 \qquad C_2 = 2|A_1|^2 \\
0 &= C_1 + a|A|^2 \qquad \underline{C_1 = -2|A_1|^2}
\end{aligned}
$$

$\mathcal{O}(\epsilon^3)$:

$$
\begin{aligned}
\partial_T u_1 &+ \partial_t u_3 = \mu_2 u_1 - v_3 + 2u_1 u_2 \\
\partial_T v_1 &+ \partial_t v_3 = u_3 + \mu_2 v_1 + 2u_1 u_2
\end{aligned}
$$

$u_1 u_2$ generates $e^{\pm 3it}$ and $e^{\pm it}$.

reorder:

$$
\begin{aligned}
\partial_t u_3 + v_3 &= -\partial_T u_1 + \mu_2 u_1 + 2u_1 u_2 \equiv I_1 e^{it} + J_1 e^{3it} + c.c. \\
\partial_t v_3 + u_3 &= -\partial_T v_1 + \mu_2 v_1 + 2u_1 u_2 \equiv I_2 e^{it} + J_2 e^{3it} + c.c.
\end{aligned}
$$

$$
\Rightarrow \begin{pmatrix} u_3 \\ v_3 \end{pmatrix} = \begin{pmatrix} D_1 \\ D_2 \end{pmatrix} e^{it} + \begin{pmatrix} E_1 \\ E_2 \end{pmatrix} e^{3it} + c.c.
$$

need to solve

$$
\underbrace{\begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix}}_{\underline{\underline{M}}_1} \begin{pmatrix} D_1 \\ D_2 \end{pmatrix} = \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} \text{ and } \underbrace{\begin{pmatrix} 3i & 1 \\ -1 & 3i \end{pmatrix}}_{\underline{\underline{M}}_3} \begin{pmatrix} E_1 \\ E_2 \end{pmatrix} = \begin{pmatrix} J_1 \\ J_2 \end{pmatrix}
$$

Now:

- $\det \underline{\underline{M}}_3 \neq 0 \qquad \Rightarrow \qquad \underline{\underline{M}}_3$ can be inverted $\Rightarrow \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}$
- $\det \underline{\underline{M}}_1 = 0 \qquad \Rightarrow \qquad \underline{\underline{M}}_1$ **cannot be inverted!**

  Solutions exist only if $\begin{pmatrix} I_1 \\ I_2 \end{pmatrix}$ in range of $\underline{\underline{M}}_1$

Determine eigenvectors associated with 0-eigenvalue of $\underline{\underline{M}}_1$:

$$\begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow$ look for left-eigenvector:

$$\begin{aligned} (u_0^+, v_0^+) \begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix} &= (0,0) \\ \Rightarrow (u_0^+, v_0^+) &= (1, +i) \end{aligned}$$

Multiply equation from the left with (u_0^+, v_0^+)

$$\underbrace{(u_0^+, v_x^+)\underline{\underline{M}}_1}_{= 0 \text{ for any} \begin{pmatrix} D_1 \\ D_2 \end{pmatrix}} \begin{pmatrix} D_1 \\ D_2 \end{pmatrix} = (u_0^+, v_0^+) \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}$$

$\Rightarrow$ **Fredholm Alternative**: there is only a solution if

$$u_0^+ I_1 + v_0^+ I_2 = 0 \qquad \text{Solvability Condition}$$

Specifically:

$$\begin{aligned} I_1 &= -\frac{d}{dT} A_1 + \mu_2 A_1 + 2\{-2A_i|A_1|^2 + A_1^*(+\frac{1}{3}(1-2i)A_1^2)\} \\ I_2 &= -\frac{d}{dT} A_1(-i) + \mu_2(-i) + 2\{-2A_1|A_1|^2 + A_1^*(+\frac{1}{3}(1-2i)A_1^2)\} \end{aligned}$$

Amplitude equation:

$$\frac{d}{dT} A_1 = \mu_2 A_1 - |A_1|^2 A_1 \left\{ +1 + \frac{7}{3} i \right\}$$

Notes:

- The solvability condition arises because linearization around fixed point is singular: always the case for bifurcation problems (steady bifurcation see later)

- Fredholm alternative: either $\begin{pmatrix} I_1 \\ I_2 \end{pmatrix}$ satisfies solvability condition or there is no solution.

Note: For the more general equation

$$\begin{aligned} \dot{u} &= \mu u - v + auv + b_1 u^2 \\ \dot{v} &= u + \mu v - auv + b_2 u^2 \end{aligned}$$

one obtains for the cubic coefficient g

$$g = \frac{1}{2}a(b_1 + b_2) - b_1 b_2 + i\frac{1}{6}\left(-4b_1^2 + ab_2 - 5b_1 a - 10b_2^2 - 2a^2\right)$$

Thus:
for $a = 0 = b_1$ the cubic coefficient is purely imaginary: *vertical* bifurcation $\Rightarrow$ have to go to higher order

Origin of **normal form**: *time-translation symmetry*

look at periodic solution

$$u(t) = \epsilon e^{it} A(T) + \ldots = \epsilon e^{it} R_0 e^{i\frac{7}{3}R_0^2 T + i\Theta_0} + \ldots$$

coefficients in system are time-independent

$$\Rightarrow \tilde{u}(t) \equiv e^{i(t-t_0)} R_0 e^{i\frac{7}{3}R_0^2 T + i\Theta_0}$$

is also a solution

$\tilde{u}(t)$ can also be written as

$$\tilde{u}(t) = e^{it} R_0 e^{i\frac{7}{3}R_0 T + i(\Theta_0 - t_0)}$$

Time translation by t_0 can be absorbed into a shift $\Theta_0 \Rightarrow \Theta_0 - t_0$

$\Rightarrow A(T)e^{i\Theta_0}$ must be a solution for any Θ_0.

$\Rightarrow$ the only nonlinear terms that are allowed are $|A_1|^{2n} A_1$, n integer.

3.6 1d-Bifurcations in 2d: Reduction of Dynamics

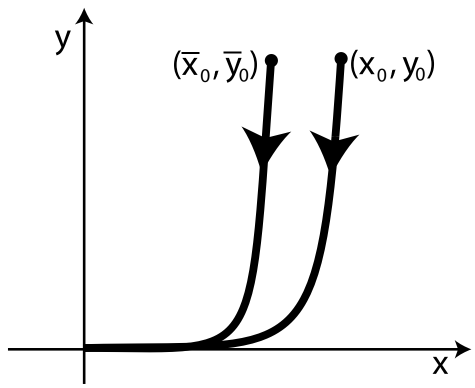
Higher-dimensional systems can undergo the same bifurcations as 1-dimensional systems.

$\Rightarrow$ can reduce dynamics to 1 dimension near the bifurcation.

3.6.1 Center-Manifold Theorem

Consider first linear example of stable node

$$\left.\begin{aligned}\dot{x} &= \mu x \\ \dot{y} &= -y\end{aligned}\right\} y = y_0 \left(\frac{x}{x_0}\right)^{+\frac{1}{|\mu|}} \qquad \mu < 0$$



For small $|\mu|$ $y \to 0$ extremely rapidly as $x \to 0$

$\Rightarrow$ after short time **any initial condition** approaches x-axis

Thus:

- dynamics effectively one-dimensional

Goal:

- obtain description of higher-dimensional system in terms of these one-dimensional dynamics

Note:

- description will be valid at most **after decay** of transients: forget certain details of initial conditions

To get mathematically justified description need $\mu \to 0$: **separation of time scales**.

For $\mu = 0$ there are 3 types of eigenvectors/eigenspaces:

– stable eigenspace $E^{(s)} = \{\underline{x} \,|\, \underline{x} = \sum \alpha_i \underline{v}_i^{(s)}\}$
$\underline{v}_i^{(s)}$ are the eigenvectors of linear system with $Re(\lambda_i^s) < 0$

– center eigenspace $E^{(c)} = \{\underline{x} \,|\, \underline{x} = \sum \alpha_i \underline{v}_i^{(c)}\}, Re(\lambda_i^{(c)}) = 0$

– unstable eigenspace $E^{(u)} = \{\underline{x} \,|\, \underline{x} = \sum \alpha_i \underline{v}_i^{(u)}\}, Re(\lambda_i^{(u)}) > 0$

Thus:

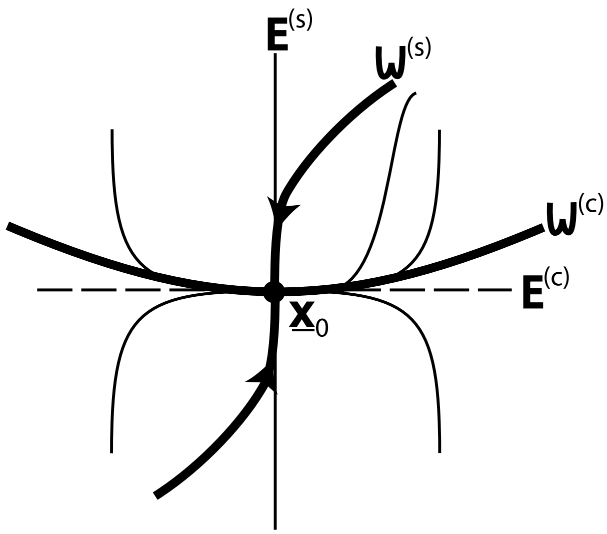
• Need to be at bifurcation point to have center eigenspace

Extension to nonlinear systems:

Center Manifold Theorem:

For a fixed point $\underline{x}_0$ with eigenspaces $E^{(s,u,c)}$ there exist stable, unstable, and center manifolds $W^{(s,u,c)}$ such that $W^{(s)}$ and $W^{(u)}$ are tangent to $E^{(s)}$ and $E^{(u)}$ at $\underline{x}_0$ and $W^{(c)}$ is tangent to $E^{(c)}$ at $\underline{x}_0$.

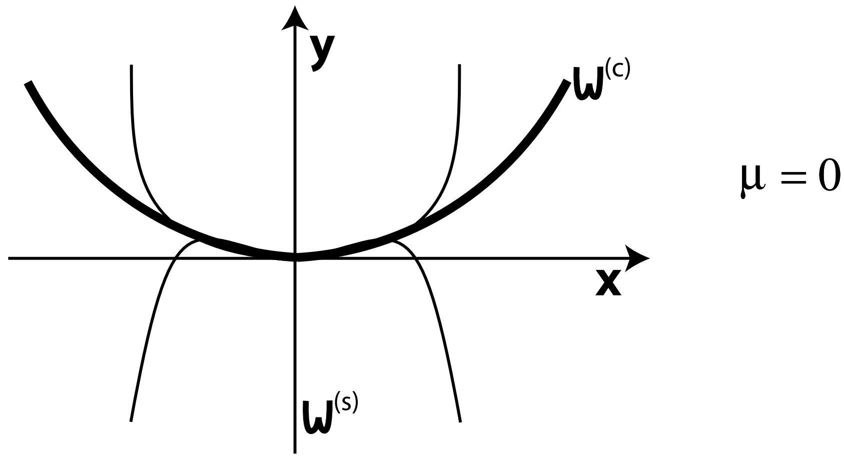
$W^{(s,u,c)}$ are invariant under the flow. $W^{(s)}$ and $W^{(u)}$ are unique. $W^{(c)}$ need not be unique.



Example:

$$\begin{aligned} \dot{x} &= \mu x + xy - \gamma x^3 \\ \dot{y} &= -y + x^2 \end{aligned}$$

$\mu < 0:$	$E^{(s)} = R^2$	$E^{(c)}$ empty	$E^{(u)}$ empty
$\mu = 0:$	$E^{(s)} = y$-axis	$E^{(c)} = x$-axis	$E^{(u)}$ empty
$\mu > 0:$	$E^{(s)} = y$-axis	$E^{(c)}$ empty	$E^{(u)} = x$-axis



Expect:

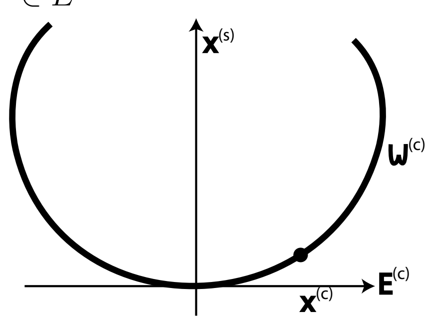
- for $|\mu| \ll 1$ still fast contraction onto a manifold close to $W^{(c)}(\mu = 0)$
- evolution on that manifold may depend strongly on μ since linear growth rate goes through 0.

3.6.2 Reduction to Dynamics on $W^{(c)}$

Want description of dynamics on $W^{(c)}$:

$$\underline{x} = (\underline{x}^{(c)}, \underline{x}^{(s)})$$

with $\underline{x}^{(s)} = \underline{h}(\underline{x}^{(c)})$ and $\underline{x}^{(c)} \in E^{(c)}$



Note:

- description *locally* (near the fixed point) possible since $W^{(c)}$ tangent to $E^{(c)}$ at fixed point
- further away correspondence may become multivalued.

Example:

$$\begin{aligned} \dot{x} &= \mu x + xy - \gamma x^3 \\ \dot{y} &= -y + x^2 \end{aligned}$$

For $W^{(c)}$ to exist need to be at bifurcation point: $\mu = 0$

$$E^{(c)} = \{(x,0)\}, \qquad E^{(s)} = \{(0,y)\}$$

$\Rightarrow$ write $\underline{x} = (x,y)$ with $y = h(x)$

insert into o.d.e.:

$$\dot{y} = \frac{dh}{dx}\dot{x} = \frac{dh}{dx}(xy - \gamma x^3) \overbrace{=}^{!} -y + x^2 = -h(x) + x^2$$

Thus:

- obtain nonlinear differential equation for $h(x)$
- $W^{(c)}$ tangent to $E^{(c)} \Rightarrow h(x)$ is strictly nonlinear
- local analysis $\Rightarrow$ expand $h(x)$ for small x

Expansion

$$h = h_2 x^2 + h_3 x^3 + h_4 x^4 + \cdots$$

inserted

$$(2h_2 x + 3h_3 x^2 + \cdots)\{x(h_2 x^2 + h_3 x^3) - \gamma x^3\} = \\ \overbrace{=}^{!} -h_2 x^2 - h_3 x^3 - h_4 x^4 + x^2$$

collect:

$$\begin{aligned} \mathcal{O}(x^2) &: \;\; 0 = -h_2 + 1 \Rightarrow h_2 = 1 \\ \mathcal{O}(x^3) &: \;\; 0 = h_3 \Rightarrow h_3 = 0 \\ \mathcal{O}(x^4) &: \;\; 2h_2(h_2 - \gamma) = -h_4 \\ & \quad\;\; h_4 = 2(\gamma - 1) \end{aligned}$$

Thus:

$$y = h(x) = x^2 + 2(\gamma - 1)x^4 + \mathcal{O}(x^5)$$

$$\dot{x} = x(x^2 + 2(\gamma - 1)x^4 + \cdots) - \gamma x^3$$

Evolution equation on center manifold:

$$\dot{x} = (1 - \gamma)x^3 + 2(\gamma - 1)x^5 + \cdots$$

More generally: we want also description for $0 \neq |\mu| \ll 1$

To use center manifold theorem consider **suspended system**

$$\begin{aligned}\dot{\mu} &= 0\\ \dot{x} &= \mu x + xy - \gamma x^3\\ \dot{y} &= -y + x^2\end{aligned}$$

Thus:

- μx is now a nonlinear term
- dynamics in μ-direction is trivial:
  value of μ is simply given by initial condiiton

Now:

$$E^{(c)} = \{(\mu, x, 0)\} \qquad E^{(s)} = \{(0, 0, y)\}$$

$$\Rightarrow y = h(\mu, x) \qquad \text{for} \qquad (\mu, x, y) \in W^{(c)}$$

Expand $h(\mu, x)$ in μ and x:
to keep relevant terms in expansion guess relationship $x \Leftrightarrow \mu$ from expected equation on $W^{(c)}$

Symmetries:
Reflections: $(\mu, x, y) \to (\mu, -x, y)$

$\Rightarrow$ expect

$$\begin{aligned}\dot{x} &= f(\mu, x) \qquad \text{with } f \text{ odd in } x\\ &= a\mu x + bx^3 + \cdots\end{aligned}$$

$\Rightarrow$ expect $\mu \sim \mathcal{O}(x^2)$, h even in x

Expand $h(\mu, x) = \underbrace{h_{20}\mu^2}_{\text{higher order}} + \underbrace{h_{11}\mu x}_{\text{wrong symmetry}} + h_{02}x^2 + [h_{12}\mu x^2 + h_{04}x^4] + \ldots$

Inserted:

$$\dot{y} = \frac{dh}{dx}\dot{x} + \frac{dh}{d\mu}\underbrace{\dot{\mu}}_{0} = (h_{11}\mu + 2h_{02}x + 2h_{12}\mu x + 4h_{04}x^3 + \ldots)(\mu x + x(h_{02}x^2 + \ldots) - \gamma x^3)$$
$$= -(h_{20}\mu^2 + h_{11}\mu x + h_{02}x^2 + h_{12}\mu x^2 + \cdots) + x^2$$

$$\begin{aligned}\mathcal{O}(\mu^2 x^0): &\quad -h_{20} &&= 0\\ \mathcal{O}(\mu^1 x^1): &\quad -h_{11} &&= 0\\ \mathcal{O}(\mu^0 x^2): &\quad 0 &&= -h_{02} + 1 \Rightarrow h_{02} = 1\\ \mathcal{O}(\mu^1 x^2): &\quad 2h_{02}(1 + h_{10}) &&= -h_{12}\\ \Rightarrow &\quad h_{12} &&= -2\\ \mathcal{O}(x^4): &\quad -2h_{02}\gamma + 2h_{02}^2 &&= -h_{04}\\ &\quad h_{04} &&= 2(1-\gamma)\end{aligned}$$

$$
\begin{aligned}
y &= x^2 - 2\mu x^2 + 2(1-\gamma)x^4 \\
\dot{x} &= \mu x + x(x^2 - 2\mu x^2 + 2(1-\gamma)x^4) - \gamma x^3
\end{aligned}
$$

Evolution on center manifold

$$
\dot{x} = \mu x - (\gamma - 1 + 2\mu)x^3 + \Big[2(1-\gamma)x^5 + \ldots\Big]
$$

Thus:

- For $\gamma > 1$ supercritical pitchfork bifurcation
  For $\gamma < 1$ subcritical pitchfork bifurcation

Equivalent result by multiple-scale analysis

Consider

$$
\dot{\underline{u}} = \underline{\underline{L}}\,\underline{u} + \underline{N}(\underline{u}, \underline{u})
$$

Analogous to Hopf: expand for small amplitudes A in the 'direction' of the critical eigenvector of the linearized operator

$$
\begin{aligned}
\underline{\underline{L}} &= \underline{\underline{L}}_0 + \epsilon \underline{\underline{L}}_1, \\
\underline{u} &= \epsilon^{\beta} A(T)\,\underline{v}_1 + \epsilon^{2\beta}\underline{u}_2(T) + \ldots
\end{aligned}
$$

with

$$
\begin{aligned}
\underline{\underline{L}}_0\,\underline{v}_1 &= 0 \\
\text{control parameter } \mu &= \mu_1 \epsilon \\
T &= \epsilon^{\alpha} t
\end{aligned}
$$

$\underline{\underline{L}}_0$ singular $\Rightarrow$ solvability condition for

$$
\begin{aligned}
\underline{\underline{L}}_0 \underline{u}_2 &= \underline{N}(A\underline{v}_1, A\underline{v}_1) \\
\underline{\underline{L}}_0 \underline{u}_3 &= \underline{N}(A\underline{v}_1, \underline{u}_2)
\end{aligned}
$$

Pick scaling such that $\partial_T A$ is determined through a solvability condition that also contains μ_1.

Symmetries suggest scaling: e.g. pitchfork:

$$
A^3 \sim \partial_T A \sim \mu_1 A
$$

Solvability condition will arise at $\mathcal{O}(\epsilon^{3/2})$

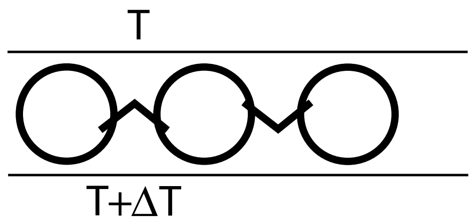
Note:

- Center-Manifold reduction $\sim$ adiabatic elmination of damped modes $\sim$ slaving

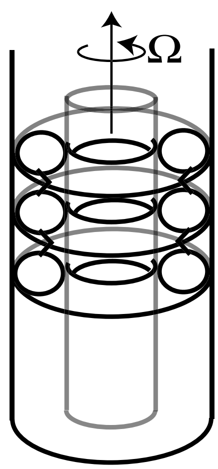
4 Pattern Formation. PDE's

Bifurcation Theory and reduction of dynamics also applicable to high-dimensional systems, PDE's.

Examples: Convection



Taylor vortices



Patterns form through instability:

- growth rate passes through 0
- bifurcation

⇒ separation of time scales
⇒ reduction of dynamics to lower dimension

4.1 Amplitude Equations from PDE

Simple model system: Swift-Hohenberg equation

$$\partial_t \psi = \mu\psi - (\partial_x^2 + 1)^2\psi - \psi^3$$

This model captures many aspects of realistic systems. Was originally derived semi-quantitatively for the temperature at the mid-plane in Rayleigh-Bénard convection.

- Control parameter: $\mu \sim \Delta T, \Omega$
- Basic state: $\psi = 0$ exists for all μ
- Linear stability:

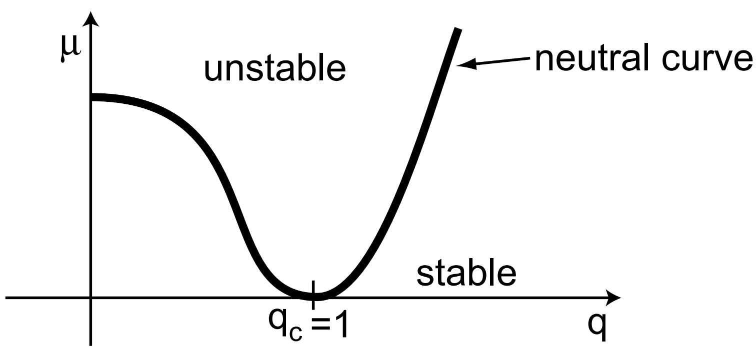
$$\partial_t \psi = \mu\psi - \underbrace{(\partial_x^2 + 1)^2 \psi}_{\partial_x^4 \psi + 2\partial_x^2 \psi + \psi}$$

Constant coefficients: Fourier ansatz

$$\psi = \psi_0 e^{iqx + \sigma t}$$

$$\sigma = \mu - (-q^2 + 1)^2$$

Instability threshold: $\sigma = 0 \quad \mu = (1 - q^2)^2$



Thus:

- Basic state stable for $\mu < 0$
- Basic state first destabilized at $\mu_c = 0$ with $q = q_c \equiv 1$.
- Basic state unstable to modes e^{iqx} for $\mu > 0$ with $q_{\min} \leq q \leq q_{\max}$.
- Consider single wave number $q = q_c \equiv 1$

$$\psi = Ae^{ix} + Be^{2ix} + C + De^{3ix} + Ee^{4ix} + \cdots + c.c.$$

Insert into Swift-Hohenberg equation and sort by Fourier modes

$$\begin{aligned}
\partial_t A &= \mu A - \Big(3|A|^2 A + 3DA^{*2} + \ldots\Big) \\
\partial_t B &= (\mu - 9)B - \Big(6|A|^2 B + 3EA^{*2} + 3A^2 C + \ldots +\Big) \\
\partial_t C &= (\mu - 1)C - \Big(6|A|^2 C + 3BA^{*2} + 3B^* A^2 + \ldots\Big) \\
\partial_t D &= (\mu - 64)D - \Big(6|A|^2 D + A^3 + \cdots\Big)
\end{aligned}$$

Thus:

- Coupled ODE's: infinitely many
- Damping increases strongly for higher harmonics

⇒ weakly nonlinear approach should work
– **center manifold** $\sim e^{ix}$
– *eliminate harmonics adiabatically*

Expect expansion

$$\psi = \epsilon A e^{ix} + \epsilon^3 D e^{3ix} + \epsilon^5 F e^{5ix} + \cdots + c.c.$$

with

$$\begin{aligned} \mu &= \epsilon^2 \mu_2 \\ A &= A(T) \qquad T = \epsilon^2 t \qquad \text{slow time-dependence} \end{aligned}$$

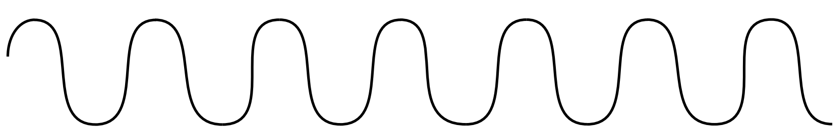
Insert and get solvability condition at $\mathcal{O}(\epsilon^3)$.

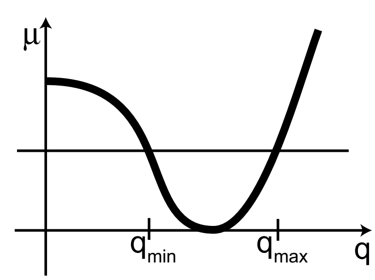
Note:

- expansion could be done for $q \neq q_c$ if $\mu \geq (1-q^2)^2$

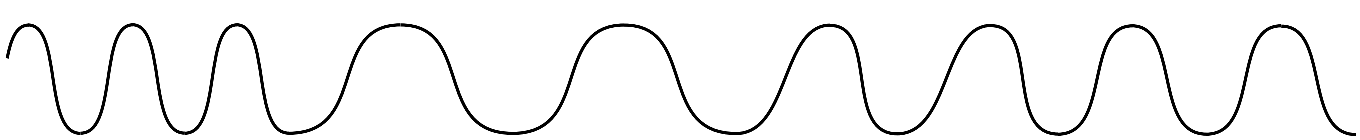
4.2 Ginzburg-Landau Equation

So far ψ is strictly periodic, with $q_{\min} < q < q_{\max}$





Expect: also slight variations in wave number possible with $q_{\min} < q(x) < q_{\max}$



What is their dynamics?

How can we describe them?

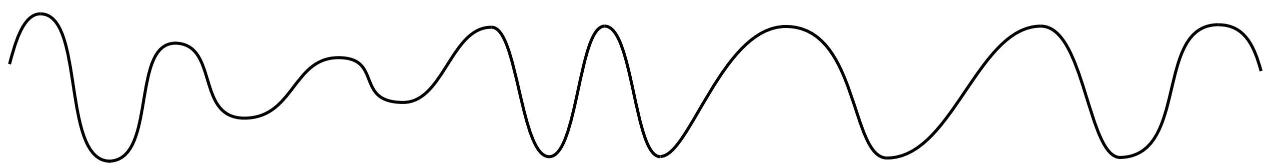
We had

$$\begin{aligned}\psi(x,t) &= \epsilon A(T)e^{iqx} + \mathcal{O}(\epsilon^3) \qquad \text{with } q_{\min} < q < q_{\max} \\ &= \epsilon \underbrace{A(T)e^{i(q-1)x}}_{A(X,T)} e^{ix} + \mathcal{O}(\epsilon^3)\end{aligned}$$

$q-1 = \epsilon\, Q$ small deviation from critical wavenumber $\qquad X = \epsilon x$ slow space variable

Thus:

- Slow spatial variation allows different wavenumbers and solutions that are not quite periodic



Expansion:

$$\psi = \epsilon A(X,T)e^{ix} + \epsilon^3 D(X,T)e^{3ix} + \ldots + c.c.$$

with $T = \epsilon^2 t, X = \epsilon x, \mu = \epsilon^2 \mu_2$

Note:

- scaling can be "guessed" by using symmetry arguments.

Need some expressions:

$$\begin{aligned}\partial_t &\rightarrow \epsilon^2 \partial_T \\ \partial_x &\rightarrow \partial_x + \epsilon \partial_X \\ \partial_x^2 &\rightarrow \partial_x^2 + 2\epsilon\partial_x\partial_X + \epsilon^2\partial_X^2 \\ \partial_x^4 &\rightarrow \partial_x^4 + 4\epsilon\partial_x^3\partial_X + 6\epsilon^2\partial_x^2\partial_X^2 + \mathcal{O}(\epsilon^3)\end{aligned}$$

i) $\mathcal{O}(\epsilon)$:

$$0 = 0$$

formally we have $L_0 = -(\partial_x^2 + 1)^2$ singular since $L_0 e^{ix} = 0$
$\Rightarrow$ expect solvability condition

ii) $\mathcal{O}(\epsilon^2)$

$$0 = -(4(-i)\partial_x A + 2 \cdot 2i\partial_x A)$$

is already satisfied

Note:

- Can check that this condition is automatically satisfied for expansion around **minimum** of neutral curve.

iii) $\mathcal{O}(\epsilon^3)$:

$$\begin{aligned} e^{ix}: \qquad & \partial_T A = \mu_2 A - (6(-1)\partial_x^2 A + 2\partial_x^2 A) - 3|A|^2 A \\ e^{3ix}: \qquad & 0 = 64D - A^3 \qquad \Rightarrow \qquad D = \frac{-A^3}{64} \end{aligned}$$

Thus: Ginzburg-Landau equation

$$\partial_T A = 4\partial_X^2 A + \mu_2 A - 3|A|^2 A$$

Notes:

- Solvability condition through mode e^{ix} since $L_0\, e^{ix} = 0$
  (could have kept term $\epsilon^3 A_3 e^{ix}$ in expansion; it would not have been able to balance inhomogeneity)
- Special form of nonlinear term: spatial translation symmetry

  $$\psi(x + \Delta x, t) = \epsilon \underbrace{A(X,T)e^{i\Delta x}}_{A(X,T)e^{i\phi}} e^{ix} + \cdots$$

  Phase shift symmetry: $x \to x + \Delta x \Leftrightarrow \phi \to \phi + \Delta x \underbrace{q}_{1}$

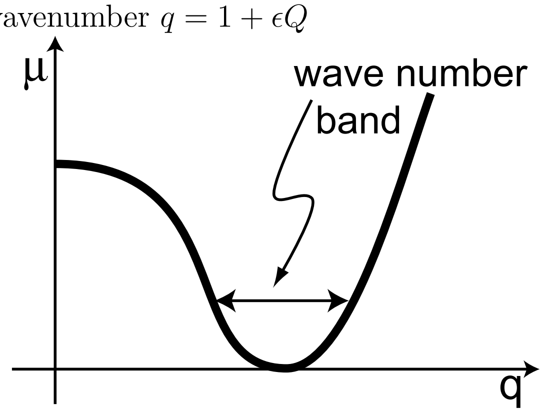
Simple periodic solution:

$$A = Re^{iQX} \qquad \text{with} \qquad R^2 = \frac{1}{3}(\mu_2 - 4Q^2)$$

then

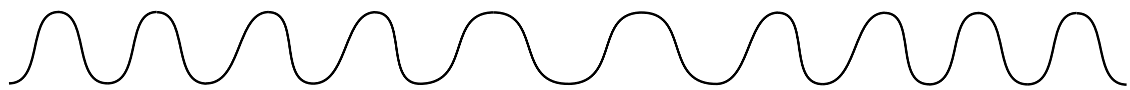
$$\psi = \epsilon Re^{iQX} e^{ix} + \cdots$$

gives solutions with wavenumber $q = 1 + \epsilon Q$



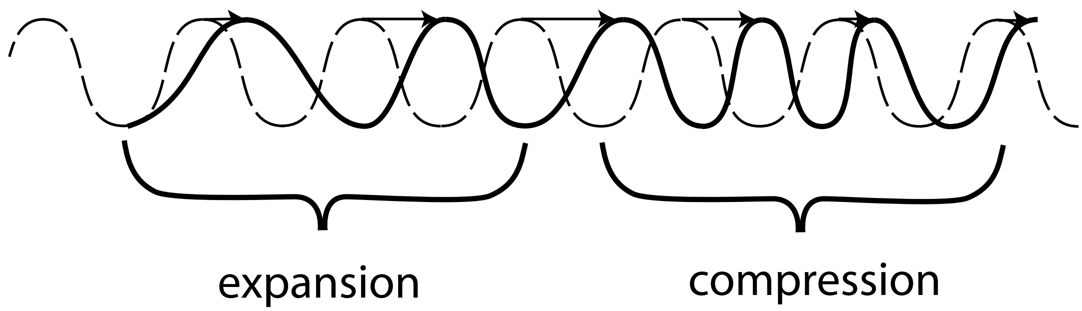
4.3 Slow Dynamics Through Symmetry. Phase Dynamics

Consider pattern in large, translation-invariant system



Wave number can vary slowly in space:
– will pattern relax to constant wave number?
– can dynamics be described in simple terms?

Translation symmetry: can shift pattern by arbitrary amounts and no restoring force
mathematically: linearization has 0 eigenvalue



Expect: Dynamics only from gradients in translation
if expansion/compression occurs on longer and longer space scales, relaxation becomes slower and slower.
Mathematically: for long-wave perturbations the 0-eigenvalue is only perturbated slightly: small eigenvalue = slow dynamics

⇒ Long-wave dynamics slow
⇒ Separation of time scales
⇒ Reduction in dynamics possible

Consider Ginzburg-Landau equation:

$$\partial_t A = \partial_x^2 A + \mu A - |A|^2 A$$

Note:

- rescaled space and amplitude
- write spatial variable in Ginzburg-Landau equation now as fast variables

Rewrite in magnitude and phase: $A = Re^{i\phi}$

$$\partial_t R \;=\; \partial_x^2 R - (\partial_x \phi)^2 R + \mu R - R^3$$

$$\partial_t \phi = \partial_x^2 \phi + 2\partial_x \phi \frac{\partial_x R}{R}$$

Note:

- $\partial_t \phi \to 0$ as $\partial_x \phi \to 0$: long-wave dynamics

Consider pattern with almost constant wavenumber:

$$\begin{aligned}
\phi &= qx + \epsilon \Phi(X,T), \qquad \underbrace{X = \epsilon x, T = \epsilon^2 t}_{\text{superslow scales}} \\
R &= R_0 + \epsilon^2 r(X,T)
\end{aligned}$$

need

$$\partial_x \phi = q + \epsilon^2 \partial_X \Phi \qquad \partial_x^2 \phi = \epsilon^3 \partial_X^2 \Phi \qquad \partial_x R = \epsilon^3 \partial_X r$$

inserted:

$$\begin{aligned}
\mathcal{O}(\epsilon^0) : 0 &= (\mu - q^2) R_0 - R_0^3 \qquad \Rightarrow \qquad R_0 = \sqrt{\mu - q^2} \\
\mathcal{O}(\epsilon^2) : 0 &= -2q \partial_X \Phi R_0 - q^2 r + \mu r - 3 \underbrace{R_0^2}_{\mu - q^2} r \\
&= -2q \partial_X \Phi R_0 - 2(\mu - q^2) r \\
r &= -\frac{q R_0}{\mu - q^2} \partial_X \Phi
\end{aligned}$$

$$\begin{aligned}
\mathcal{O}(\epsilon^3) : \partial_T \Phi &= \partial_X^2 \Phi + 2\frac{q}{R_0} \partial_X r \\
&= \partial_X^2 \Phi + \frac{2q}{R_0} \left(\frac{q R_0}{\mu - q^2} \right) \partial_X^2 \Phi \\
&= \partial_X^2 \Phi \left\{ 1 - \frac{2q^2}{\mu - q^2} \right\}
\end{aligned}$$

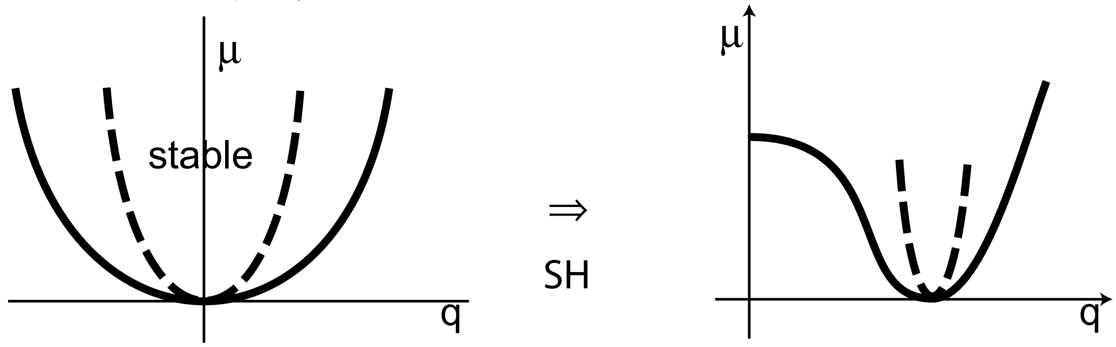
Thus:

$$\begin{aligned}
\partial_T \Phi &= D\, \partial_X^2 \Phi \\
D &= \frac{\mu - 3q^2}{\mu - q^2}
\end{aligned}$$

Notes:

- Relaxation of wavenumber gradients is diffusive
  (symmetry arguments: reflection symmetry in space but not in time)

- Diffusion coefficient can be negative, since neutral curve is given by $\mu = q^2$: Eckhaus instability at $\mu = 3q^2$



- Eckhaus instability is universal instability of steady one-dimensional patterns → e.g. experiments in Taylor vortex flow

Notes:

- Nonlinear evolution of Eckhaus instability:
  – no saturation of instability
  – phase slip ⇒ change in wave number

General Mechanism for Slow Dynamics:

Breaking of *continuous* symmetry

⇒ continuous family of solutions
⇒ slow long-wave dynamics when different members of the family are connected spatially

Further example: oscillation in system with time-translation symmetry

e.g. Hopf bifurcation: complex Ginzburg-Landau equation

$$\partial_t A = \mu A - (1 + ic_3)|A|^2 A + (1 + ic_1)\partial_x^2 A$$

simple traveling-wave solutions:

$$A = Re^{iqx+i\omega t} \qquad \text{with} \qquad \omega = c_3 q^2 + c_1 R^2 \qquad R^2 = \mu - q^2$$

continuous family of solutions $A \Rightarrow Ae^{i\phi}$

allow slow variation of phase $\Rightarrow \phi = \phi(x,t)$ again phase equation

$$\partial_T \phi = v_g \partial_x \phi + D\partial_x^2 \phi$$

Near stability limit $D \sim 0$
⇒ in co-moving frame

$$\partial_T \phi = D\partial_x^2 \phi + g\partial_x^4 \phi + h(\partial_x \phi)^2$$

Kuramoto-Sivashinsky equation

Note:

- Kuramoto-Sivashinsky-equation can display chaotic dynamics

5 Chaos

in 2 dimensions: at most periodic orbits (Poicaré-Bendixson theorem)

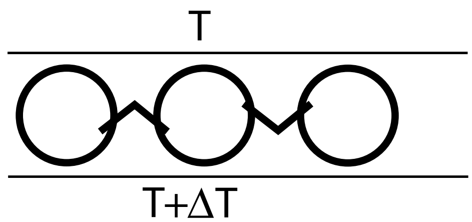
$\Rightarrow$ Consider 3-dimensional systems

Visualization: reduce to maps instead of flows

5.1 Lorenz Model

Convection

Simple Model



Stream function:

$$\psi = 2\sqrt{6}\, X(t)\ \cos \pi z \sin\left(\frac{\pi}{\sqrt{2}}x\right) \qquad \text{with} \qquad (u, w) = (-\partial_t \psi, \partial_x \psi)$$

Temperature:

$$T(x,z,t) = \underbrace{-rz}_{\text{basic profile}} + \underbrace{9\pi^3\sqrt{3}\, Y(t)\ \cos \pi z \cos\left(\frac{\pi}{\sqrt{2}}x\right)}_{\text{critical mode}} + \underbrace{\frac{27\pi^3}{4} Z(t)\ \sin 2\pi z}_{\text{harmonic mode}}$$

Rayleigh number r control parameter

critical wave number $q_c = \frac{\pi}{\sqrt{2}}$

Galerkin projection back on the same types of modes:

$$\begin{aligned}
\dot{X} &= -\sigma(X - Y) \\
\dot{Y} &= rX - Y - ZX \\
\dot{Z} &= b(XY - Z)
\end{aligned}$$

Notes:

- model constitutes severe truncation of Galerkin expansion for free-slip boundary conditions

[3]

Demos: Excellent Java programs by M. Cross (Caltech) at
`http://www.cmp.caltech.edu/%7emcc/Chaos_Course/Lesson1/Demos.html`

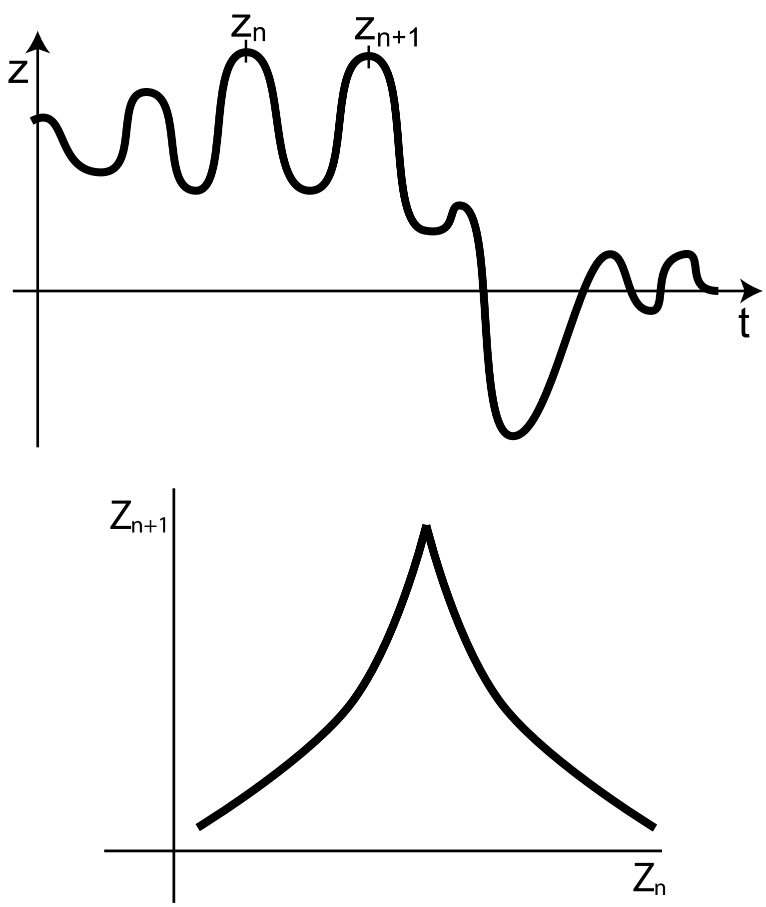
Demo 1: **Lorenz Attractor**

increase r ($= a$ in Cross program): 0.5 1.2 1.8 10 24 24.4 24.5 25.
transitions occur at: r=1 r=24.45

Demo 8: **Sensitive dependence on initial conditions**
simulation with $x_0 = 2$ $y_0 = 5$ $z_0 = 20$ and $z_0 = 20 + \Delta z$
$r(= a) = 28$ $\sigma(= c) = 10$ $b = 8/3$ $\Delta z = 10^{-3}$ 10^{-5} 10^{-7}
$x - z$ plot (top option on web page of demo 8) and
$x - t$ plot (bottom option on web page of demo 8)

Question: Can one get a simpler representation?

Lorenz map:



[3]picture of Lorenz attractor missing

Demo: chemical oscillations (Swinney *et al.*) $\Rightarrow$ WWW

Note:

- the reduction to a map is only approximate:
  original ode's can also be solved backward
  map cannot be iterated backward: $f^{-1}(z)$ multiple valued
- the line is actually not a line, but has finite thickness
  here thickness small $\Rightarrow$ approximation should give a good idea of dynamics of system.

Poincare Section:

Dimension of system can be reduced by monitoring only locations where flow 'pierces' a certain surface (e.g. $x-y$-plane):

- periodic orbit $\Rightarrow$ fixed point
- quasi-periodic orbit (2 frequencies) $\Rightarrow$ closed loop (not periodic)
- chaotic orbit $\Rightarrow$??

5.2 One-Dimensional Maps

Consider maps as dynamical systems

$$x_{n+1} = f(x_n)$$

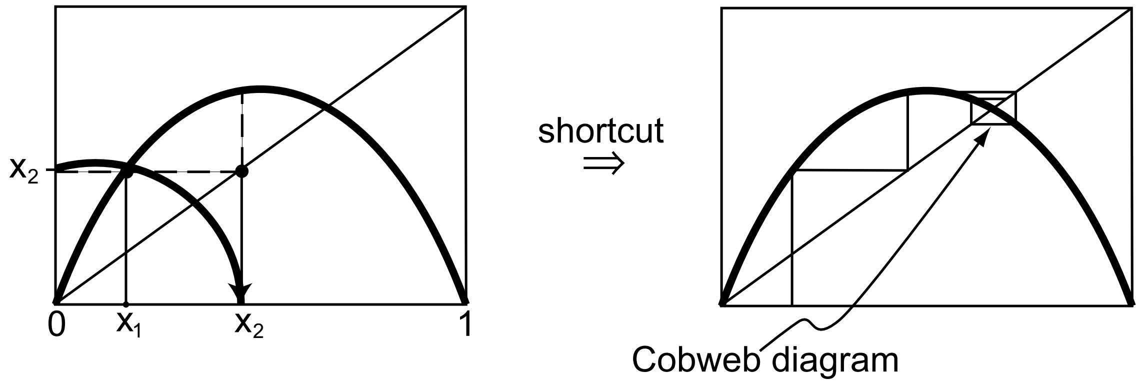
Example: logistic map

$$x_{n+1} = ax_n(1 - x_n)$$

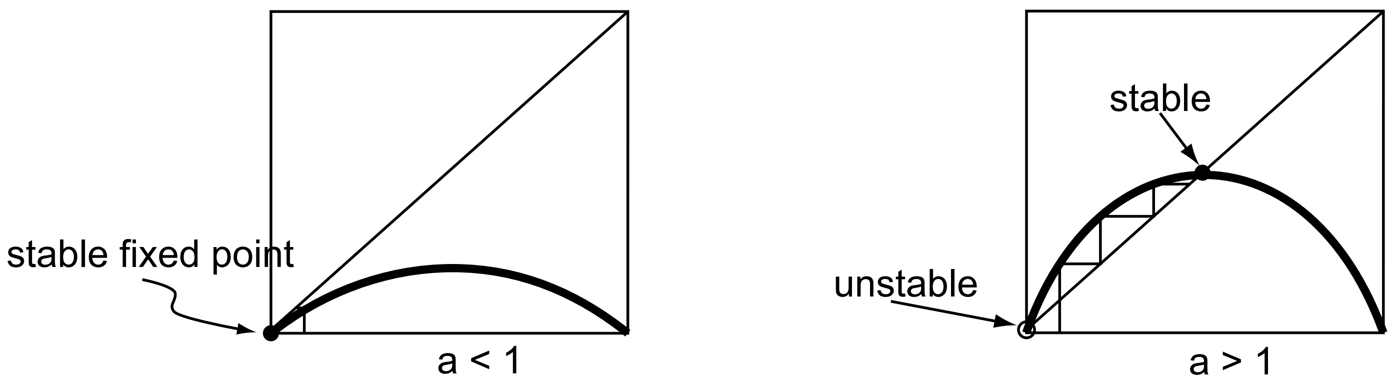
Note:

- this map could be thought of a (very poor) numerical solution of logistic differential equation

Graphical iteration



Vary a:

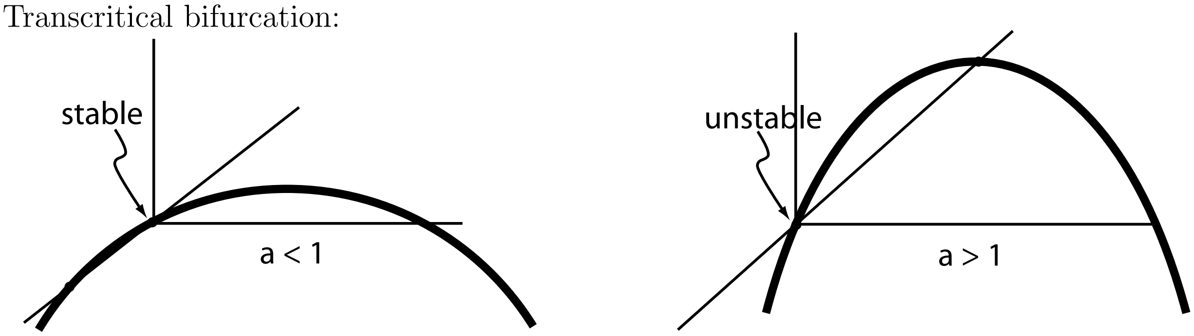


For $a = 1$ the fixed point $x = 0$ becomes unstable

$$x = ax - ax^2 \qquad \Rightarrow \qquad 0 = x\,(a - 1 - ax)$$

$$x^{(1)} = \frac{a-1}{a}$$

Transcritical bifurcation:



Stability Analysis:

linearize around fixed point x_f

$$x_n = x_f + \epsilon \tilde{x}_n$$

$$\begin{aligned} x_f + \epsilon\tilde{x}_{n+1} &= f(x_f + \epsilon\tilde{x}_n) = f(x_f) + \epsilon\tilde{x}_n f'(x_f) \\ \Rightarrow \tilde{x}_{n+1} &= \tilde{x}_n f'(x_f) \end{aligned}$$

$\Rightarrow$ $\tilde{x}_n$ grows for $|f'(x_f)| > 1$ $\tilde{x}_n$ decays for $|f'(x_f)| < 1$

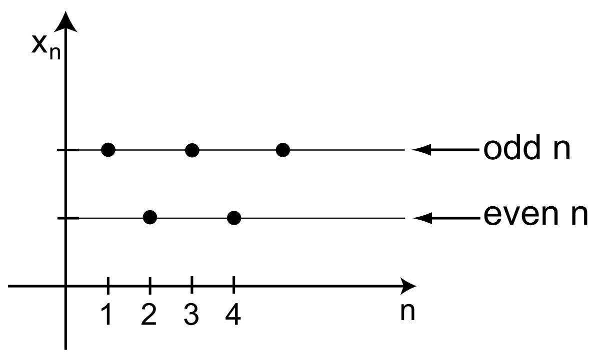
Stability of fixed point $x_1 = \frac{a-1}{a}$:

$$f'(x_1) = a - 2ax_1 = a - 2(a-1) = 2 - a$$

$$|f'(x_1)| < 1 \qquad \text{for} \qquad \underbrace{1}_{\text{transcritical}} < a < 3$$

Demo: what happens at the bifurcation at $a = 3.0$?

$\Rightarrow$ converges to period-2 solution



Determine period-2 solution:
period 2: fixed point under second iterate of $f(x)$

$$\begin{aligned} x_{n+2} &= f(x_{n+1}) = f(f(x_n)) \equiv f^{(2)}(x_n) \\ &= ax_{n+1}(1 - x_{n+1}) = a\ (a\,x_n(1-x_n))\,(1 - a\,x_n(1-x_n)) \end{aligned}$$

Fixed point of $f^{(2)}: x_{n+2} = x_n$

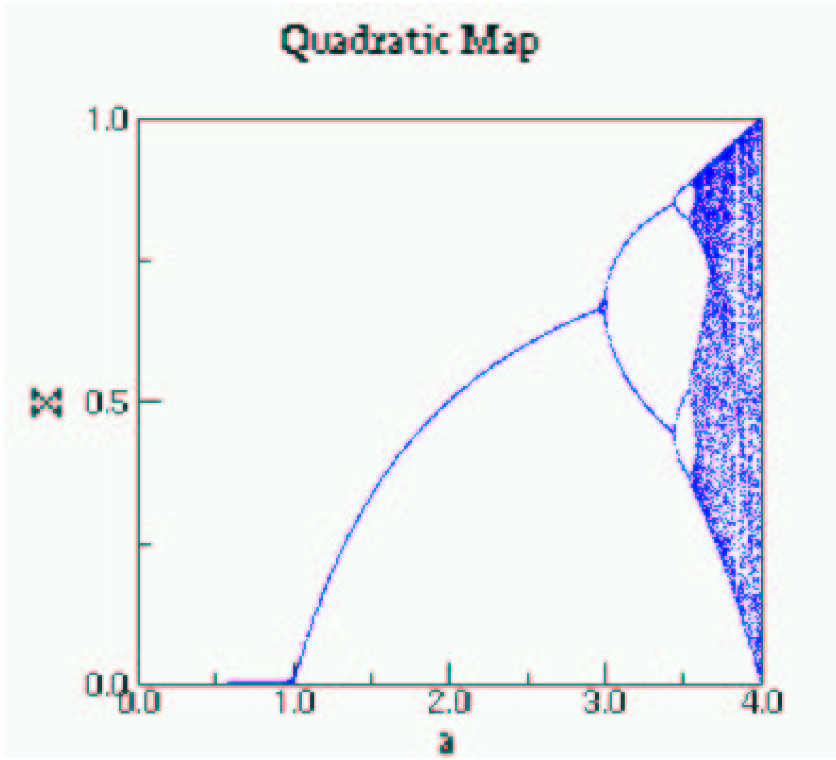
$$x^{(2)} = f^{(2)}(x^{(2)})$$

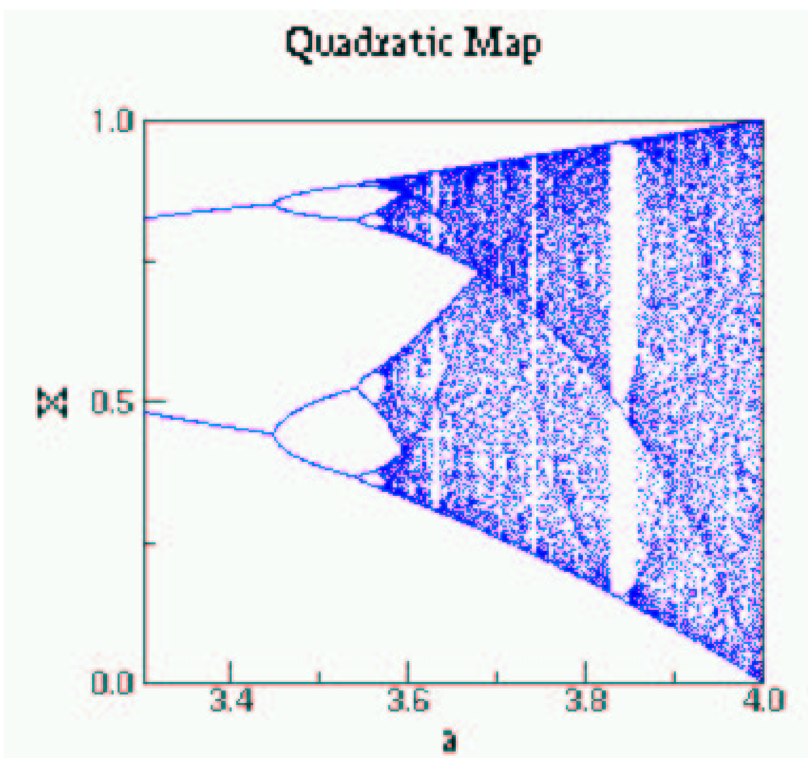
can be factored as

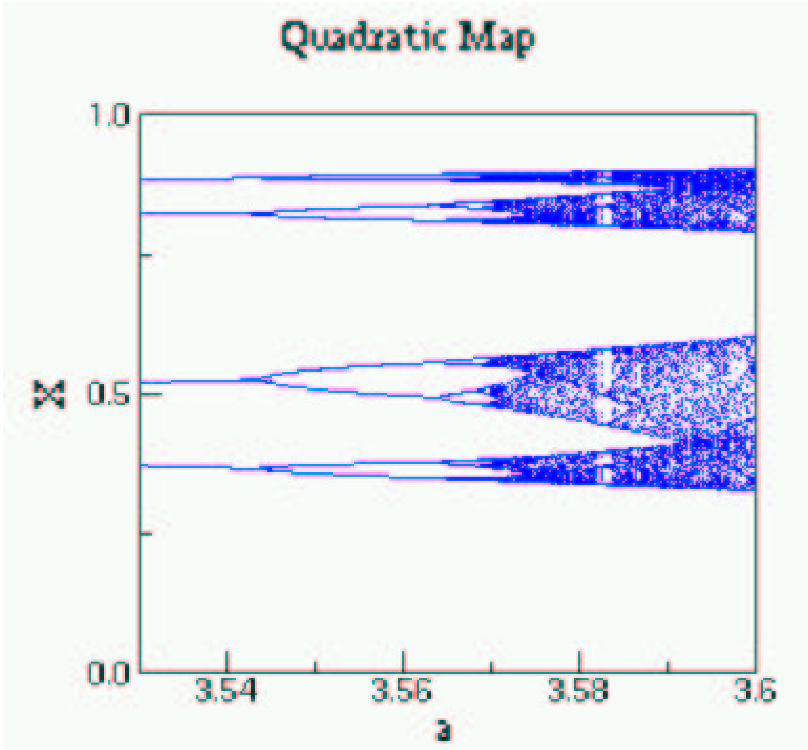
$$\underbrace{-x(xa + 1 - a)}_{\text{known fixed points}}\ (a^2x^2 - a(1+a)x + 1 + a) = 0$$

$$x^{(2)}_{1,2} = \frac{1}{2a}\left\{1 + a \pm \underbrace{\sqrt{a^2 - 2a - 3}}_{x^{(2)}_{1,2} \text{ exist for } a>3}\right\}$$

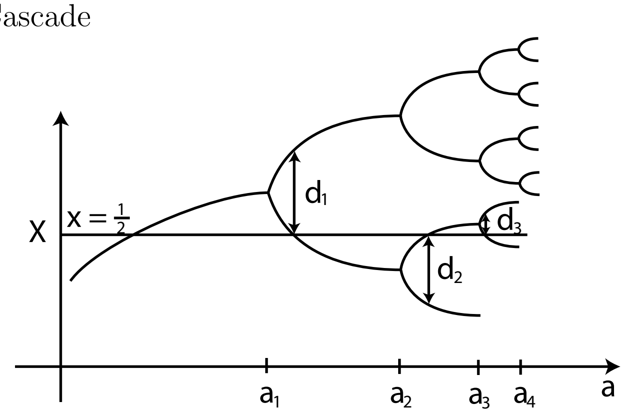
Cross Demo 3


Quadratic Map
1.0
x 0.5
0.0
0.0
1.0
2.0
3.0
4.0
a


Quadratic Map
1.0
x 0.5
0.0
3.4
3.6
3.8
4.0
a



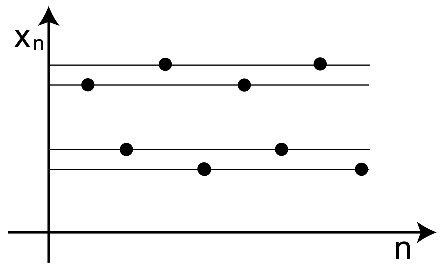
Period-Doubling Cascade



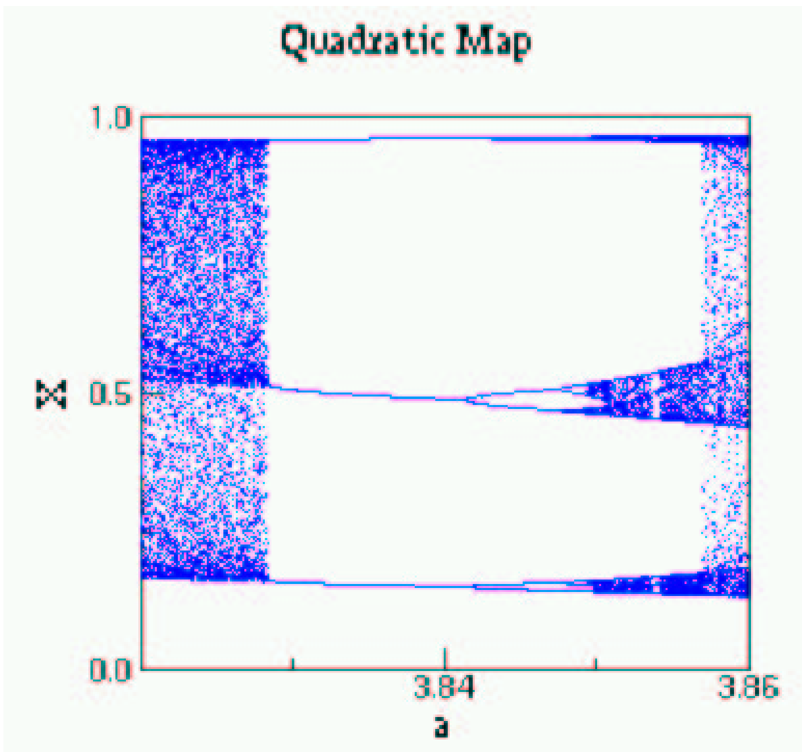
Scaling of bifurcations:

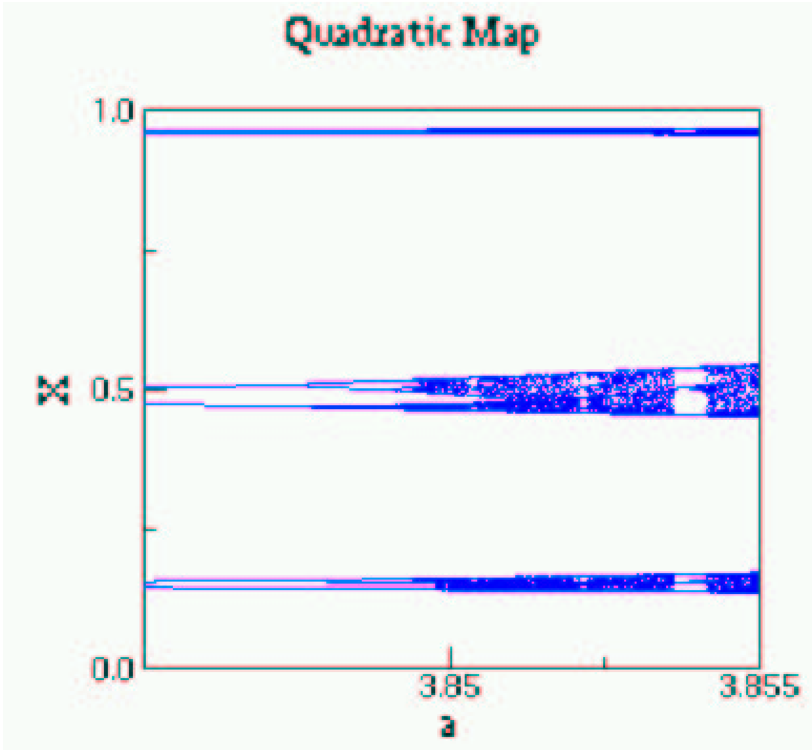
$$\lim_{n\to\infty} \frac{a_n - a_{n-1}}{a_{n+1} - a_n} = \delta = 4.669...$$

$$\lim_{n\to\infty} \frac{d_n}{d_{n+1}} = \alpha = -2.5029...$$



'Chaotic' regime, periodic windows:
$a = 3.83 \rightarrow 3.85 \rightarrow 3.86$
$a = 3.83 \rightarrow 3.82$





Tangent bifurcation (saddle-node bifurcation) at $1 + \sqrt{8} = 3.8284$
Cascade in periodic window

Intermittency near saddle-node bifurcation: $a = 3.82837$

Exact Solution for $a = 4$

$$x_{n+1} = 4x_n(1 - x_n)$$

$$\text{let} \qquad x_n = \sin^2\theta_n \qquad x_{n+1} = \sin^2\theta_{n+1}$$

$$\Rightarrow \qquad \sin^2\theta_{n+1} = 4\sin^2\theta_n\,(\underbrace{1 - \sin^2\theta_n}_{\cos^2\theta_n}) =$$

$$= (2\sin\theta_n\cos\theta_n)^2 = \sin^2(2\theta_n)$$

$\Rightarrow$ dynamics in θ simple

$$\theta_{n+1} = 2\theta_n$$

$$\Rightarrow \qquad \theta_n = 2^n\theta_0$$

Perturb initial condition $\tilde{\theta}_0 = \theta_0 + \epsilon$

$$\begin{aligned} x_n - \tilde{x}_n &= \sin^2(2^n(\theta_0)) - \sin^2(2^n(\theta_0 + \epsilon)) = \\ &= \frac{1}{2}\left(1 - \cos(2^{n+1}\theta_0) - \left\{1 - \cos(2^{n+1}(\theta_0 + \epsilon))\right\}\right) \\ &= \frac{1}{2}\left[\cos(2^{n+1}(\theta_0 + \epsilon)) - \cos(2^{n+1}\theta_0)\right] \end{aligned}$$

The cosines differ substantially if

$$2^{n+1}\epsilon = \pi$$

$$\Rightarrow (n+1) = \frac{\ln \pi - \ln \epsilon}{\ln 2}$$

Example:

ϵ	10^{-5}	10^{-6}	10^{-7}
$n+1$	18	22	25

Thus:
The time over which the two solutions stay close to each other increases **very** slowly (logarithmically) with ϵ:
Sensitive Dependence on Initial Condition.

Experiments in a convection cell by Libchaber, Fauve, Laroche (Physica D 7 (1983) 73) (see WWW):

$$\delta = 4.4 \pm 0.1$$

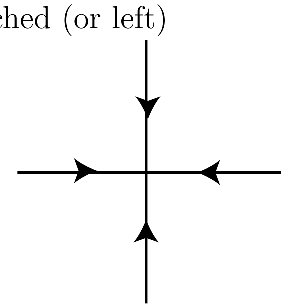
5.3 Lyapunov Exponents

We had:
logistic map: irregular looking behavior
Lorenz model: qualitatively sensitive dependence on i.c.

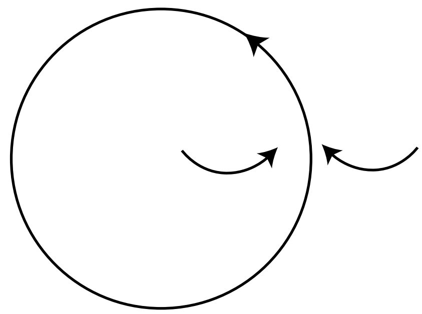
Quantitative measure for sensitivity: Lyapunov exponent

Extension of linear stability
Consider first flows: For fixed points only relevant question:

How fast is fixed point approached (or left)



For period orbits:



- attractivity transverse to orbit
- along orbit marginally stable: *time-translation symmetry* $\Rightarrow$ 0 eigenvalue

Orbits with similar initial conditions do not diverge rapidly, at most they do not get closer (if one is 'ahead' of the other along the orbit)

Measure distance between orbits with nearby i.c.
Focus on behavior of different trajectories on attractor (long-term behavior) rather than on the approach towards attractor (transients).

Consider for simplicity 1-d map:

start with i.c.

$$x_0 \quad \& \quad x_0 + \delta_0 \qquad \Rightarrow \qquad x_n \quad \& \quad x_n + \delta_n$$

$$\left|\frac{\delta_n}{\delta_0}\right| = \left|\frac{f^{(n)}(x_0 + \delta_0) - f^{(n)}(x_0)}{\delta_0}\right| \to \left|f^{(n)\prime}(x_0)\right| \qquad \text{for } \delta_0 \to 0$$

$$f^{(n)\prime}(x_0) = \frac{d}{dx} f(f(\ldots f(x)))|_{x-x_0} = f'(f^{n-1}(x_0)) \cdot f'((f^{n-2}(x_0)) \cdot \ldots \cdot f'(x_0) \qquad (*)$$

$$= \prod_{i=0}^{n-1} f'(x_i) \qquad \text{with } x_i = f^{(i)}(x_0)$$

If $f'(x_i) \sim const.$ expect

$$\left|\frac{\delta_n}{\delta_0}\right| \sim \mu^n = e^{\lambda n} \qquad \text{for large } n$$

Define Lyapunov exponent λ:

$$\lambda = \lim_{n\to\infty} \lim_{\delta_0 \to 0} \frac{1}{n} \ln \left|\frac{\delta_n}{\delta_0}\right|$$

Note:

- For each finite n, δ_0 is taken infinitesimal and only then $n \to \infty$

For one-dimensional map:

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln |f'(x_i)|$$

Note:

- In general Lyapunov depends on initial condition
  $\Rightarrow$ average over different initial conditions
- In ergodic systems λ independent of initial conditions: any point on attractor is visited.

Limitations on predictions for x_n:

$$|\delta_n| = \delta_0 e^{\lambda n}$$

To predict with an error ϵ and an initial precision δ

$$n = \frac{1}{\lambda} \ln \frac{\epsilon}{\delta}$$

As found in demo simulations of Lorenz model:
duration of prediction grows only logarithmically with precision of initial data:
each **10-fold** increase in initial precision increases prediction interval only by a **constant** duration n_{10}:

$$n_{10} = \frac{1}{\lambda} \ln 10$$

Example: λ for periodic orbit

Consider f at a parameter value with a stable p-cycle:

$$f^{(p)}(x_i) = x_i \text{ for } i = 0, \ldots, p-1$$

Thus $f^{(p)}$ has p fixed points, which are stable by assumption of a stable p-cycle of $f(x)$

$$\Rightarrow |f^{(p)\prime}(x_i)| < 1$$

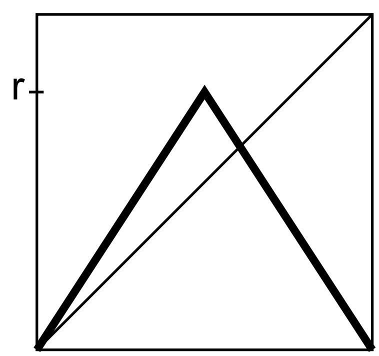
$$
\begin{aligned}
\lambda &= \lim_{n\to\infty}\left\{\frac{1}{n}\sum_{i=0}^{n-1}\ln|f'(x_i)|\right\} \\
&= \frac{1}{p}\sum_{i=0}^{p-1}\ln|f'(x_i)| \quad \text{since the cycle repeats itself at } x_p = x_0 \\
&\underbrace{=}_{\text{using } (*)} \frac{1}{p}\ln\underbrace{|f^{(p)\prime}(x_i)|}_{<1} < 0
\end{aligned}
$$

Note:

- As expected stable periodic orbit has negative Lyapunov exponents.
- 0 eigenvalue has disappeared because of transition from flow to map (the map would formally be the same even if the underlying system was forced periodically in time $\Rightarrow$ no time translation symmetry).
- Superstable orbits have $f'(x_i) = 0$ for at least one x_i of the periodic orbit: $\Rightarrow \lambda \to -\infty$

Example: Tent Map

$$
f(x) = \begin{cases} rx & \text{for} \quad 0 \le x \le \frac{1}{2} \\ r(1-x) & \text{for} \quad \frac{1}{2} \le x \le 1 \end{cases}
$$



Lyapunov exponent:

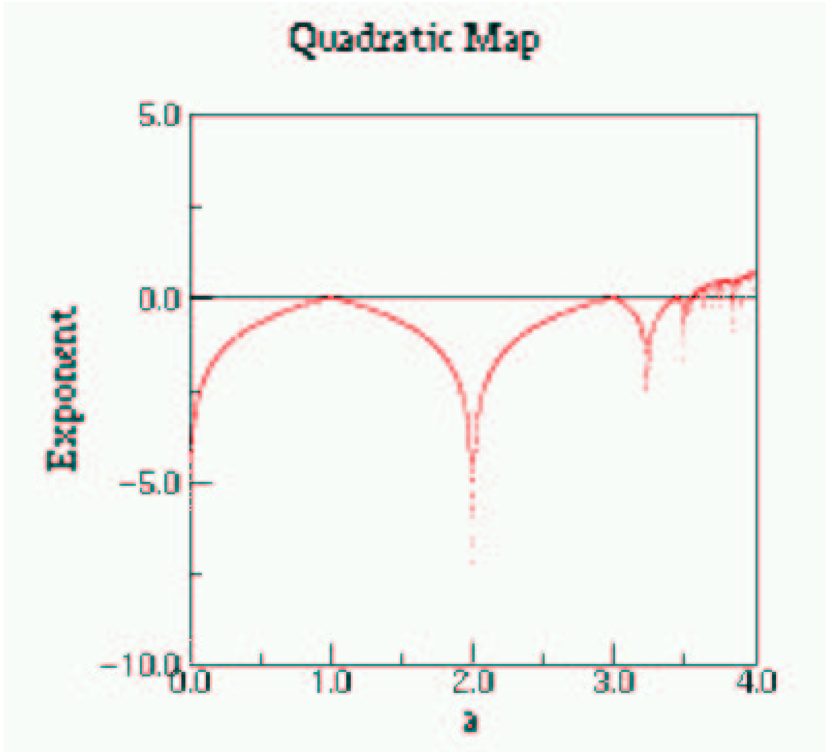
$$
f'(x) = \pm r \qquad \text{for any } x \qquad \Rightarrow \lambda = \ln r
$$

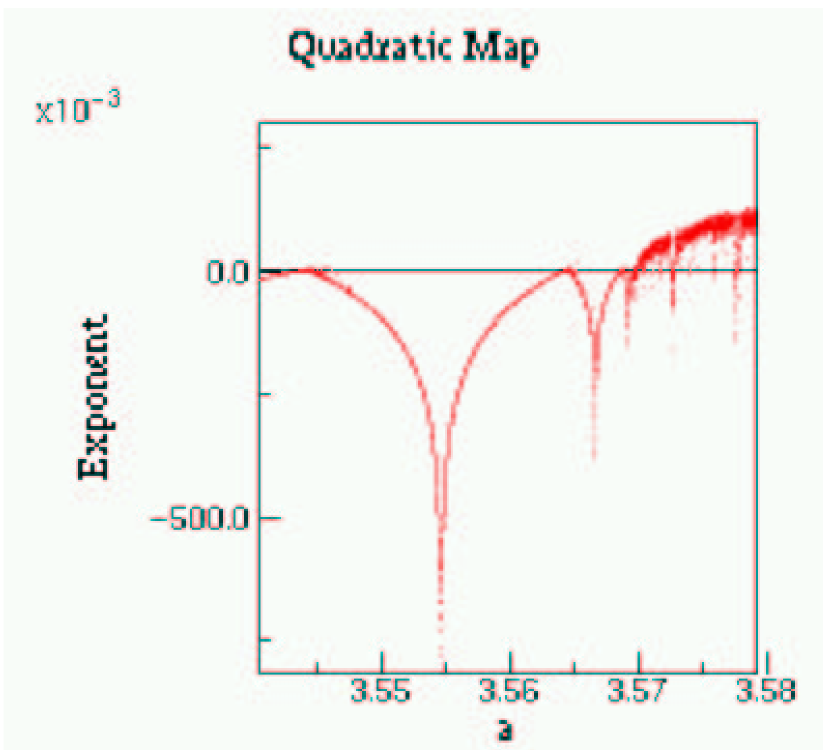
Thus:

- expect sensitive dependence on i.c. for $r > 1$

Example: Logistic Map

Demo 4 by Cross





Note:

- period doubling cascade
  $\lambda = 0$ at period-doubling bifurcation: $f^{(p)'} = 1$ change of stability.
- superstable orbits: $\lambda \to -\infty$

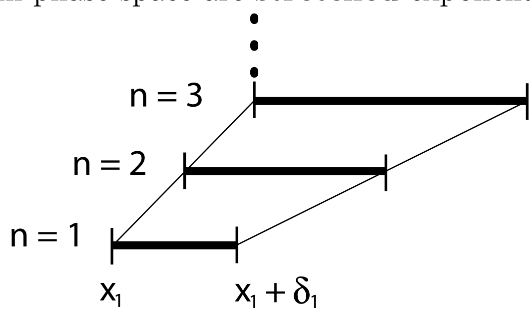
- periodic windows

Thus:

- one defining feature of chaotic dynamic is a positive Lyapunov exponent
- in n-dimensional systems n exponents, some positive, some 0, some negative.

Note:

- Lyapunov exponent positive
  $\Rightarrow$ on average trajectories diverge exponentially
  intervals $[x_n, x_n + \delta_n]$ in phase space are **stretched** exponentially

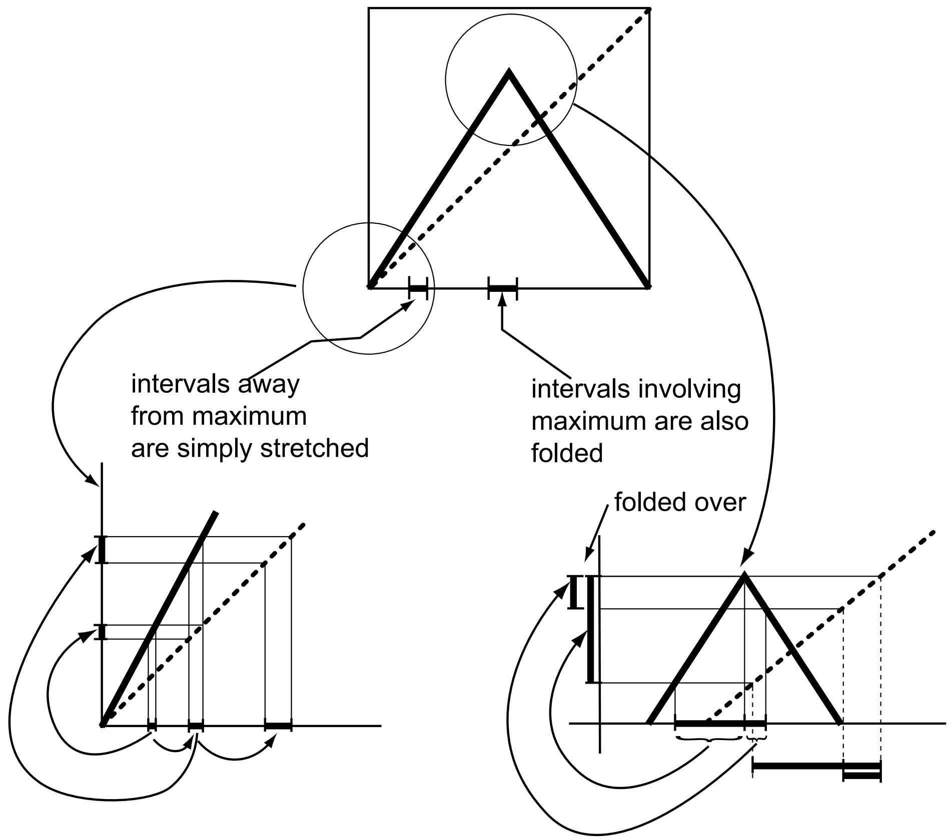


But: all points x confined to $[0,1] \Rightarrow$ separation limited by 1

Resolution of paradox:

"Stretching and Folding"

Example: tent map



Thus:

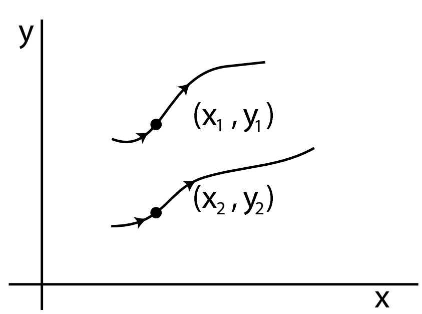
- For bounded attractor positive Lyapunov exponent implies stretching and folding.
- For one-dimensional maps folding implies that the map is not invertible

5.4 Two-dimensional Maps

For flows

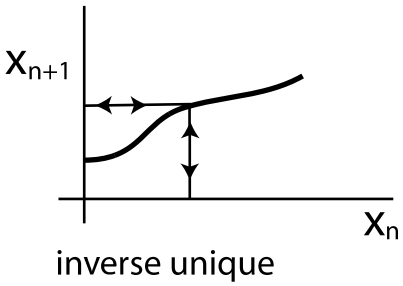
$$\underline{\dot{x}} = \underline{I}(\underline{x})$$

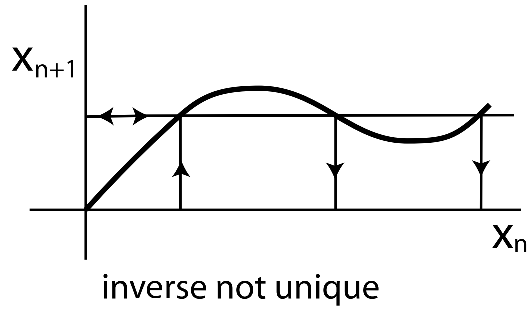
Each state has
– unique future
– unique past



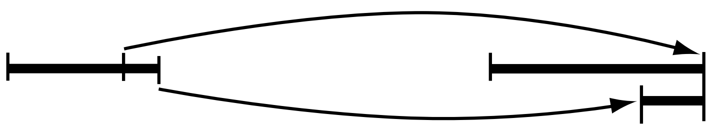
i.e., orbits do not intersect (except at fixed points)
⇒ in maps that are derived from such flows each state must have
– unique image: $\underline{x}_{n+1} = \underline{f}(\underline{x}_n)$
– unique pre-image: $\underline{x}_n = \underline{f}^{-1}(\underline{x}_{n+1})$

⇒ in 1 dimension $f(x)$ must be monotonic





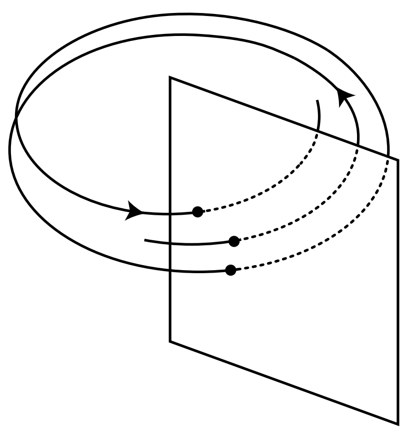
Chaotic dynamics require stretching and *folding*
⇒ chaotic 1-dimensional maps non-invertible



for pre-image need to know from which of the two layers to start from
in 2 dimensions this may be possible.

2-d maps can exhibit chaos:

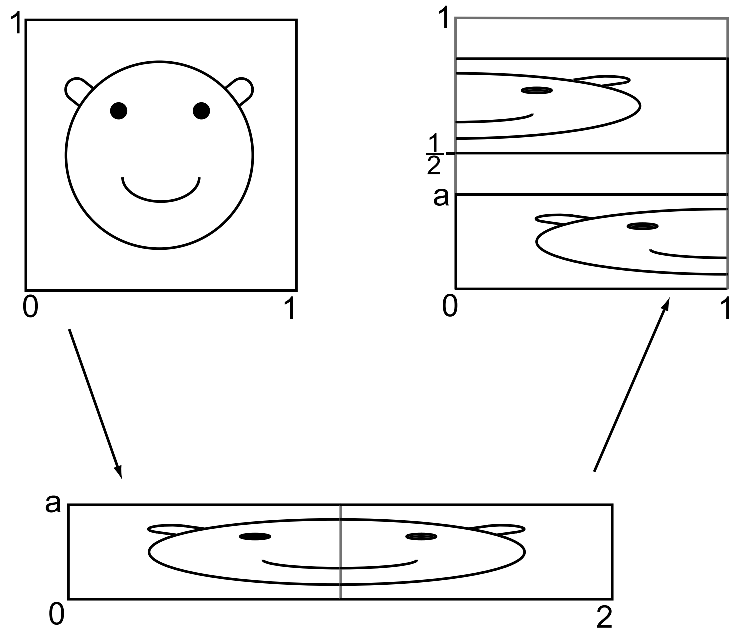
Poincar'e section of 3-d flow (e.g., Lorenz system)



Flow can be run backward: ⇒ map invertible.

2-d maps more representative of chaotic flow than 1-d maps

Example: Dissipative Baker's Map



$$(x_{n+1}, y_{n+1}) = \begin{cases} (2x_n, ay_n) & \text{for} \quad 0 \leq x_n < \frac{1}{2} \\ (2x_{n-1}, ay_n + \frac{1}{2}) & \text{for} \quad \frac{1}{2} \leq x_n \leq 1 \end{cases}$$

Require $0 < a \leq \frac{1}{2}$

Notes:

- map exhibits stretching in x-direction

  Consider $(x_n + \delta_n, y_n)$ and (x_n, y_n)

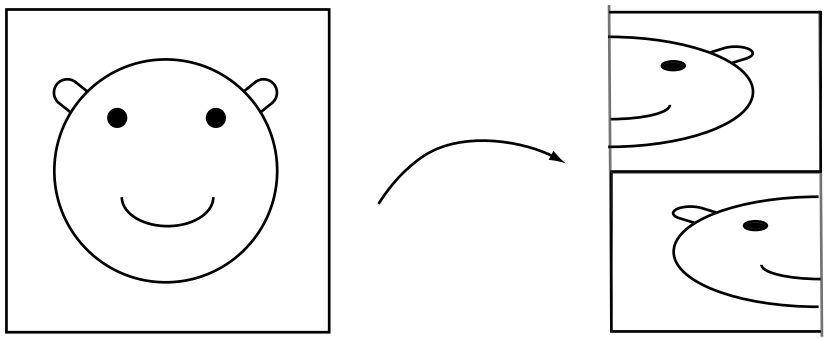
  $$\delta_{n+1} = 2\delta_n \qquad \text{except if } x_n < \frac{1}{2} \text{ and } x_n + \delta_n > \frac{1}{2}$$

  for small δ_n this happens very rarely

  $$\Rightarrow \lambda_1 = \ln 2$$

- map discontinuous: folding replaced by cutting.
- in the y-direction contraction
- For $a = \frac{1}{2}$ area is preserved by map: system is not dissipative.

Example: Conservative Baker's Map



$$(x_{n+1}, y_{n+1}) = \begin{cases} (2x_{n_1}, \frac{1}{2}y_n) & 0 \le x_n < \frac{1}{2} \\ (2x_{n-1}, \frac{1}{2}y_n + \frac{1}{2}) & \frac{1}{2} \le x_n \le 1 \end{cases}$$

Simple description of dynamics in 'binary' notation:

$$x_n = a_1\frac{1}{2} + a_2\frac{1}{4} + a_3\frac{1}{8} + \cdots \qquad y_n = b_1\frac{1}{2} + b_2\frac{1}{4} + b_3\frac{1}{8} + \cdots$$

written as

$$(x_n, y_n) = \ldots b_3b_2b_1.a_1a_2a_3\ldots$$

Calculate (x_{n+1}, y_{n+1}):

For $0 \le x_n < \frac{1}{2}$

$$\begin{aligned} x_{n+1} &= 2x_n = \underbrace{a_1}_{0} + a_2\frac{1}{2} + a_3\frac{1}{4} + \ldots = .\,a_2a_3a_4\ldots \\ y_{n+1} &= \frac{1}{2}y_n = b_1\frac{1}{4} + b_2\frac{1}{8} + \ldots = \ldots b_3b_2b_1 0\,. \end{aligned}$$

$\frac{1}{2} \le x_n \le 1$

$$\begin{aligned} x_{n+1} &= a_1 + a_2\frac{1}{2}a_3\frac{1}{4} + \ldots - 1 = .\,a_2a_3a_4\ldots \\ y_{n+1} &= b_1\frac{1}{4} + b_2\frac{1}{8} + \ldots + \frac{1}{2} = \ldots b_2b_1 1\,. \end{aligned}$$

Thus:

$$(x_{n+1}, y_{n+1}) = \ldots b_3b_2b_1a_1.a_2a_3a_4\ldots$$

• The dynamics are given by a simple **shift** in the binary representation of the *initial conditions.*

Conclusions:

- Depending on initial conditions the map has
  - periodic orbits of arbitrary period ("rational" initial conditions), countably many
  - aperiodic orbits, ("irrational" initial conditions) uncountably many
- each iteration amplifies error in x-direction by factor of 2 ($\lambda_1 = \ln 2$)

  Specific example:
  monitor only whether $x > \frac{1}{2}$ or $x < \frac{1}{2}$, i.e., monitor only first digit after binary point
  if initial condition is known with resolution 2^{-m}, i.e. a_m is the last known digit,
  $\Rightarrow$ after m iterations a_{m+1} determines "left" or "right":
  outcome completely unknown since only $a_1 \dots a_m$ are known:
  deterministic system behaves like completely random coin toss.

Note:

- Long-term behavior strongly affected by dissipation
  Even for weak dissipation ($a \sim \frac{1}{2}$) only initial behavior similar to that of conservative system. Thus chaotic behavior of dissipative system has to be studied separately.

5.5 Diagnostics

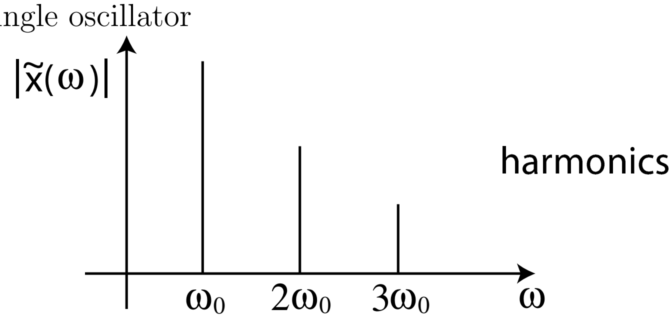
5.5.1 Power Spectrum

For periodic signals: frequency

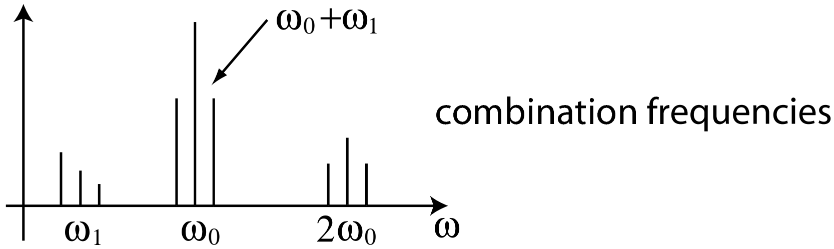
extension: **spectrum**

$$\tilde{x}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} x(t)\, e^{-i\omega t} dt$$

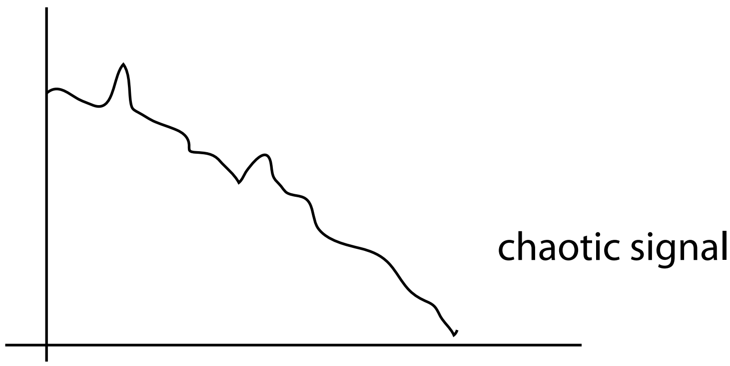
periodic signal: single oscillator



2 coupled oscillators



chaotic signal



broadband with possible peaks

Realistic time series:

- finite duration $T \Rightarrow$ lowest frequency $\omega_{min} = \frac{2\pi}{T}$
- finite sampling rate $\Delta t \Rightarrow$ highest resolved frequency $\omega_{max} = \frac{2\pi}{\Delta T}$

$$\tilde{x}_k = \sum_{j=0}^{N-1} x(t_j)\, e^{-i\omega_k t_j}$$

with

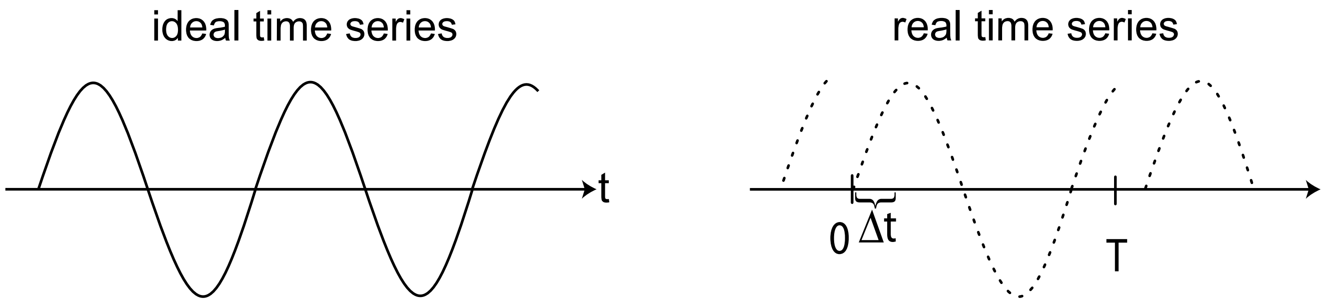
$$t_j = j\Delta t \qquad \omega_k = \frac{2\pi}{T}k \qquad T = N\Delta t$$

Only discrete frequencies in the spectrum

Demos: Cross 1

Why is there a broad "peak" even for the periodic signal?

Fourier series assumes signal **periodic** with period T
but
Time series in general **not periodic** with period T



dotted line indicates values at discrete times

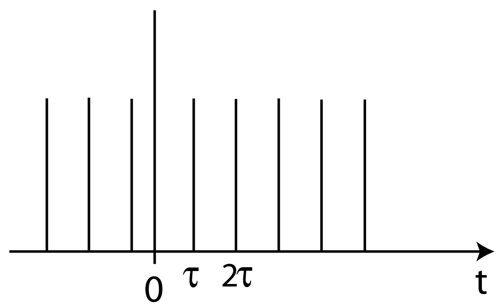
Real time series is **not smooth** at $t = T$

Express realistic time series in detail:

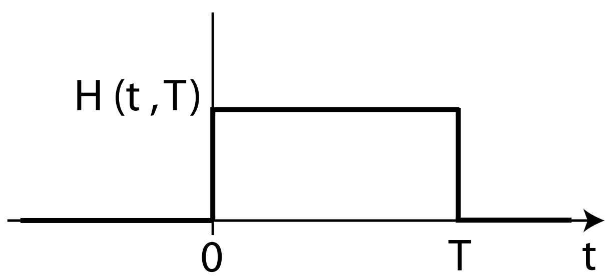
$$\hat{x}(t) = \{(x(t)\, H(t,T)) \otimes S(t,T)\}\ S(t,\Delta T)$$

with

$$S(t,\tau) = \sum_{n=-\infty}^{\infty} \delta(t - n\tau)$$



Window function

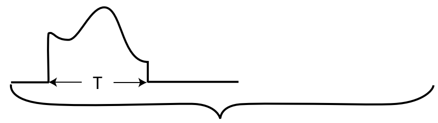


Convolution:

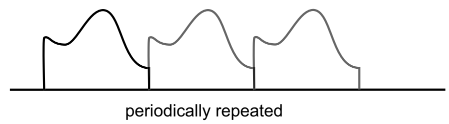
$$\begin{aligned} f(t) \otimes S(t,T) &= \int_{-\infty}^{\infty} f(t')S(t-t',T)\,dt' = \\ &= \int f(t') \sum_{n=-\infty}^{\infty} \delta(t-t'-nT)dt' \\ &= \sum_{n=-\infty}^{\infty} f(t-nT) \end{aligned}$$

Then:

$$\hat{x}(t) \;=\; [(x(t)H(t,T))$$



$$\otimes S(t,T)]$$



$$S(t,\Delta t) \qquad \text{sampled at discrete times}$$

Now we can take usual Fourier transform of $\hat{x}(t)$ to get idea of the transform of realistic data

Fourier transformation and convolution:

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{-i\omega t} f(t) \otimes g(t)\, dt &= \int_{-\infty}^{\infty} e^{-i\omega t} \int_{-\infty}^{\infty} f(t')g(t-t')\, dt'\, dt \\
&= \int \underbrace{\int e^{-i\omega(t-t')} g(t-t')}_{\tilde{g}(\omega)} \underbrace{e^{-i\omega t'} f(t')}_{\tilde{f}(\omega)} \, dt\, dt'
\end{aligned}$$

Thus:

- Fourier transform of a convolution is a product
- Fourier transform of a product is a convolution

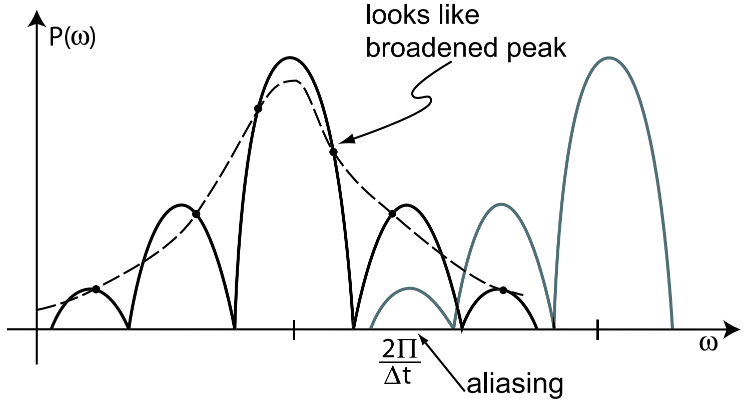
$$\hat{x}(t) \to \left[\left(\tilde{x}(\omega) \otimes \tilde{H}(\omega,T)\right) \tilde{S}(\omega,T)\right] \otimes \tilde{S}(\omega,\Delta t)$$

with

$$\begin{aligned}
\tilde{H}(\omega,T) = \int_0^T e^{-i\omega t} dt = \frac{i}{\omega}\left[e^{-i\omega T} - 1\right] \\
\Rightarrow \qquad |\tilde{H}(\omega,T)| \sim \left|\frac{\sin \frac{1}{2}\omega T}{\frac{1}{2}\omega}\right|
\end{aligned}$$

$$\tilde{S}(\omega,\Delta t) = \int e^{-i\omega t} \sum_n \delta(t - n\Delta t)\, dt = \underbrace{\sum_n e^{-i\omega n \Delta t}}_{\text{sequence of spikes}}$$

$$\underbrace{\left[\underbrace{\left(\tilde{x}(\omega) \otimes \tilde{H}(\omega,T)\right)}_{\text{broadened peak}} \tilde{S}(\omega,T)\right]}_{\text{sampled at discrete frequencies } \omega_k = 2\pi k/T} \qquad \otimes \qquad \underbrace{\tilde{S}(\omega,\Delta t)}_{\text{spectrum periodically repeated}}$$



One cannot see the oscillations since only discrete frequencies are available: minima are at $|\omega T| = 2\pi k$ except for $k = 0$ and frequencies are separated by $2\pi k/T$.

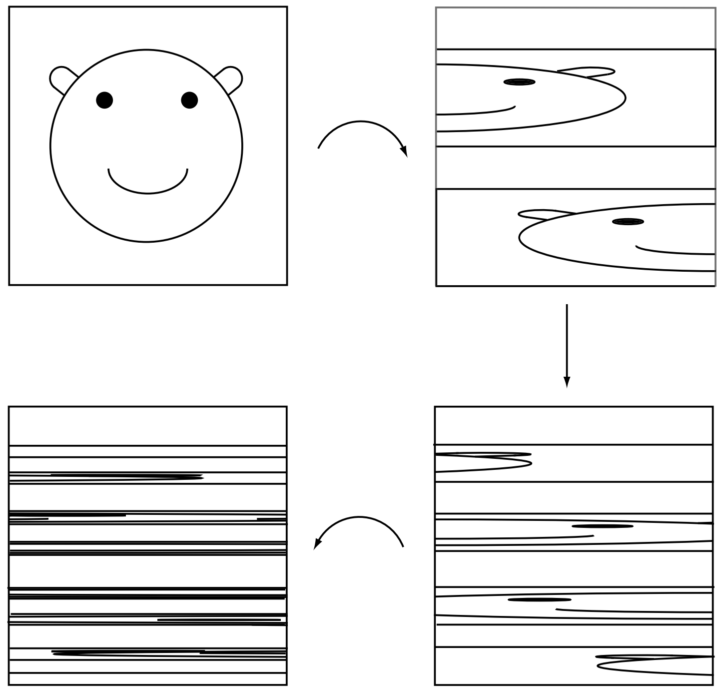
Thus:

- need sufficiently long time series:
  T large $\Rightarrow \omega_{min}$ small
- need sufficiently fine sampling:
  ω_{max} large to avoid aliasing
- 'windowing' to minimize broadening

5.5.2 Strange Attractors. Fractal Dimensions

Chaotic attractors have complex geometry: characterize it quantiatively

Example: Dissipative baker's map
Question: to which set of points do random initial conditions convert?



Any initial condition is mapped into the 2 stripes, which are then mapped into 4 separate thinner stripes....

In the y-direction the attractor becomes extremely intricate: infinitely many *lines*:

- *lines* $\rightarrow$ 1 dimension
- finite number of iterations still *stripes* $\rightarrow$ 2 dimensions
- $n \rightarrow \infty$ infinitely many lines $\Rightarrow 1 < d < 2$

Compare to Cantor set

What is the dimension of the set for $n \rightarrow \infty$?

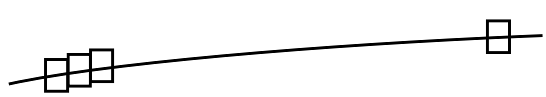
Box Dimension:
Count the minimal number N of boxes of size ϵ that are needed to cover the attractor. Then

$$d_b = \lim_{\epsilon \rightarrow 0} \frac{\ln N}{\ln \frac{1}{\epsilon}}.$$

This would correspond to $N \approx (1/\epsilon)^{d_b}$.

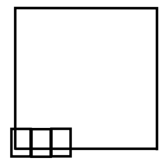
Examples:

i) Line



$$N = \frac{L}{\epsilon} \qquad \Rightarrow \qquad d_b = \lim_{\epsilon \to 0} \frac{\ln L - \ln \epsilon}{-\ln \epsilon} = 1$$

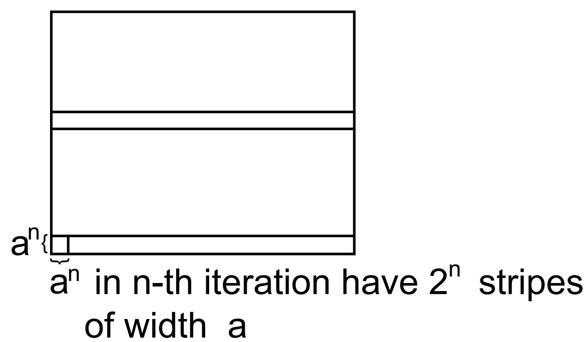
ii) Surface



$$N = \frac{L^2}{\epsilon^2} \qquad \Rightarrow \qquad d_b = \lim_{\epsilon \to 0} \frac{\ln L - \ln \epsilon}{-\ln \epsilon} = 2$$

iii) Attractor of baker's map

pick boxes of size $(a^n)^2$:



2^n stripes:

$$N = 2^n \, \frac{1}{a^n} \qquad \to \qquad d_b = \lim_{\epsilon \to 0} \frac{\ln N}{\ln \frac{1}{\epsilon}} = \lim_{n \to \infty} \frac{\ln \left(\frac{a}{2}\right)^{-n}}{\ln a^{-n}} = \lim_{n \to \infty} \frac{\ln a - \ln 2}{\ln a}$$

$$\to d_b = 1 + \frac{\ln \frac{1}{2}}{\ln a}$$

Thus:

- for $a \to \frac{1}{2} \quad d \to 2$
- for $a \to 0 \quad d \to 1$
- for general a: $1 < d < 2$

Note:

- Box dimension does not depend on dynamics on attractor, only its geometry $\Rightarrow$ define also other dimensions

Correlation Dimension:

For a fixed point $\mathbf{x}$ on the attractor determine the number $N_{\mathbf{x}}(\epsilon)$ of other points on the attractor that lie within a ball of radius ϵ. Then

$$N_{\mathbf{x}}(\epsilon) \sim \epsilon^{d_c}$$

determines the pointwise dimension. Average over $\mathbf{x}$ on the attractor gives

$$C(\epsilon) = < N_{\mathbf{x}}(\epsilon) >_{\mathbf{x}} \sim \epsilon^{d_c}$$

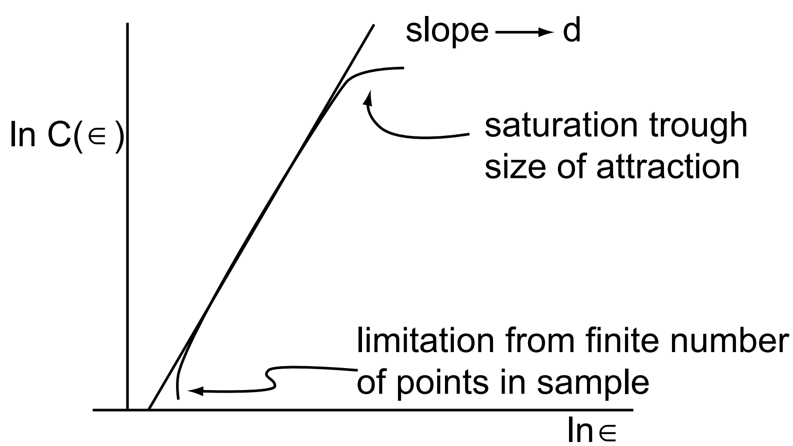
Note:

- Dynamics do enter correlation dimension: where are the points dense, where not, i.e. where in phase space is the system more often?
- one can show $d_c \leq d_b$
  but usually $d_c \sim d_b$

Note:

- there are further dimensions:
  whole spectrum of dimensions generated by weighing the probability of finding points in a small ball with different powers

Practically:

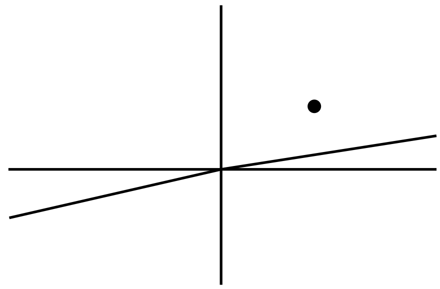


$\Rightarrow$ need sufficently many points to see power laws.

Lyapunov Dimension:

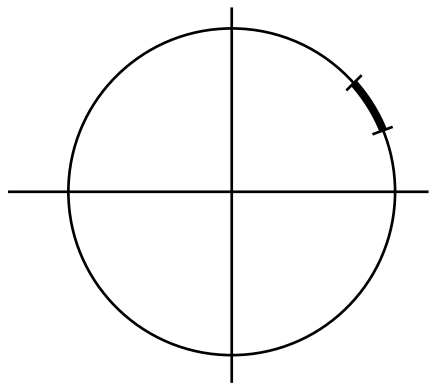
Include dynamics explicitly in the definition of the dimension

Consider dimension of a box that neither grows nor shrinks under the dynamics point attractor



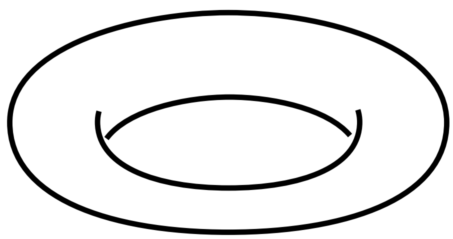
any box with $d \geq 1$ shrinks to a point: $d_L = 0$

line attractor



line segments along the attractor are transported along orbit without volume change (on average), but area covering the width of the attractor shrinks to a line: $d_L = 1$

torus:



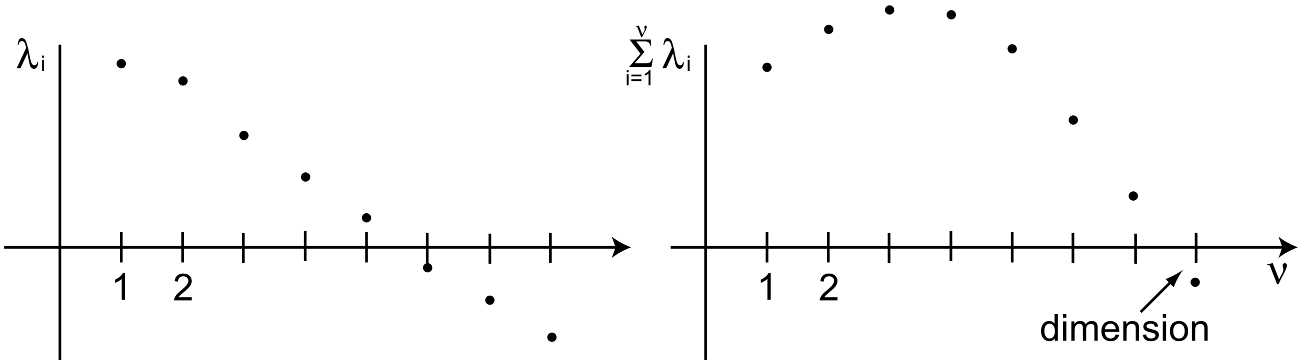
area transported along orbit, but three-dimensional volume would shrink: $d_L = 2$

Growth of ν-dimensional volume in phase space is given by expansions in the ν directions

$$V(t) = L_1 e^{\lambda_1 t} \, L_2 e^{\lambda_2 t} \, L_3 e^{\lambda_3 t} \, \ldots L_\nu e^{\lambda_\nu t}$$

for $V = const.$ need

$$\sum_{i=1}^{\nu} \lambda_i = 0$$



Linear interpolation of

$$f(n) = \sum_{i=1}^{n} \lambda_i$$

$$f(d_L) = 0 \approx f(\nu) + [f(\nu+1) - f(\nu)]\ (d_L - \nu) \qquad \Rightarrow \qquad d_L = \nu + \frac{f(\nu)}{f(\nu+1) - f(\nu)}$$

Thus, for ν such satisfying $\sum_{i=1}^{\nu} \lambda_i > 0$ but $\sum_{i=1}^{\nu+1} \lambda_i < 0$ the Lyapunov dimension is given by

$$d_L = \nu + \frac{1}{|\lambda_{\nu+1}|} \sum_{i=1}^{\nu} \lambda_i$$

Note:

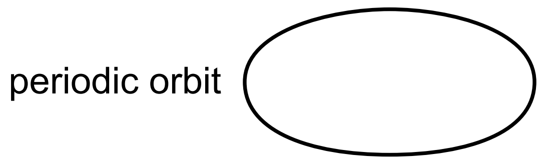
- d_L gives a measure of how many degrees of freedom are "active"

6 Summary

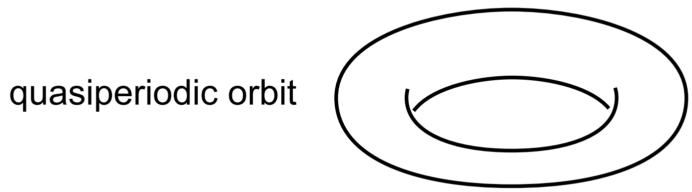
Dissipative Dynamical Systems:

- long-time behavior given by attractors:
  fixed points

  periodic orbits



  quasiperiodic orbits



  strange attractors

- *qualitative* changes in behavior
  - instabilities
  - bifurcations ⇒ new solutions, sequence of bifurcations, period-doubling cascade

- reduction of dynamics:
  separation of time scales ⇒ adiabatic elimination of fast degrees of freedom
  - near bifurcations: center manifold reduction
  - continuous symmetries: slow long-wave dynamics
  - conservation laws: slow long-wave dynamics (e.g. Navier-Stokes equations)

- symmetries can play an important role:
  establish form of equation for reduced dynamics ⇒ scaling

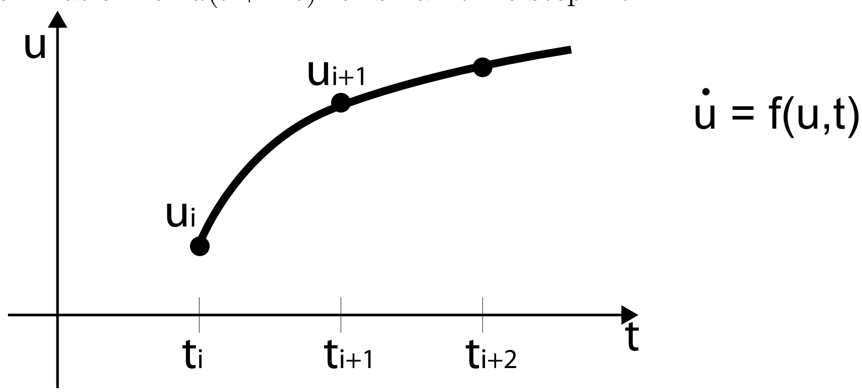
7 Insertion: Numerical Methods for ODE

Discuss a few important methods and core issues for numerical solution of

$$\dot{x} = f(x,t)$$

Consider *finite-difference methods* for time stepping:
continuous analytical solution is replaced discrete sequence of values

Seek approximation for $u(t + \Delta t)$ for small time step Δt.



Notation:

- Use u for numerical solution and x for exact solution
- $t_j = j \cdot \Delta t$, $u_j = u(t_j)$

7.1 Forward Euler

There are two ways to look at this approximation

i) Taylor Expansion

$$u_{j+1} = u_j + \Delta t \left.\frac{du_j}{dt}\right|_{t=t_j} + \underbrace{\frac{1}{2}\Delta t^2 \left.\frac{d^2u}{dt^2}\right|_{t=t^*}}_{\text{Error}}$$

Notes:

- the time t^* is not known: this term constitutes the error term

Using $\dot{x} = f(x,t)$:

$$u_{j+1} = u_j + \Delta t\, f(u_j, t_j) + \mathcal{O}(\Delta t^2)$$

Notes:

- Local error $\mathcal{O}(\Delta t^2)$
- Global error: integrate from 0 to $T = N\Delta t$

$$\Rightarrow E^{(global)} \approx \sum_{j=1}^{N} E_j^{(local)} \sim N\, E_j^{(local)} \sim \frac{T}{\Delta t}\mathcal{O}(\Delta t^2) = \mathcal{O}(\Delta t)$$

  First-order scheme.

$\Rightarrow$ expect scheme to approximate exact solution better and better as $\Delta t \to 0$.

But: if unstable scheme will **not converge** $\to$ later.

Assessment of accuracy:

in practical situation error is not explicitly available (no exact solution)

compare $u_N^{(\Delta t)}$ with $u_N^{(\Delta t/2)}$

$$\begin{aligned} u_N^{(\Delta t)} &= u_N^{(ex)} + a\Delta t \\ u_N^{(\Delta t/2)} &= u_N^{(ex)} + a\frac{\Delta t}{2} \\ \Rightarrow \quad & u_N^{(\Delta t)} - u_N^{(\Delta t/2)} = a\frac{\Delta t}{2} \end{aligned}$$

Thus: difference is of the order of the error

ii) Integral Representation

Solution of differential equation can be written as

$$u_{j+1} = u_j + \int_{t_j}^{t_{j+1}} f(u,t)dt$$

need to approximate integral

Left-end-point rule:

$$\int_{t_j}^{t_{j+1}} f(u,t)dt = f(u_j,t_j)\,\Delta t$$

again

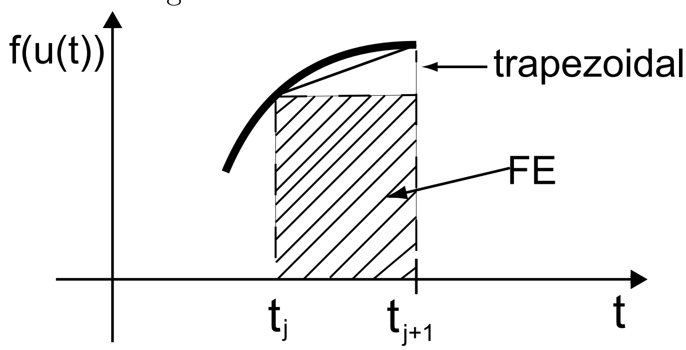
$$u_{j+1} = u_j + \Delta t\, f(u_j,t_j) + \mathcal{O}(\Delta t^2)$$

Note:

- more accurate (higher-order schemes) by

– higher-order Taylor expansion
– higher-order approximation of integral:
  Adams-Bashforth and predictor-corrector schemes ⇒ homework.

E.g. trapezoidal rule for integral



7.2 Crank-Nicholson

Approximate time derivative at mid-point

$$\frac{du}{dt} = \frac{u_{j+1} - u_j}{\Delta t} \qquad \text{at } t_j + \frac{1}{2}\Delta t$$

Need to approximate right-hand-side of diff.eq. also at $t_j + \Delta t/2$

$$\frac{u_{j+1} - u_j}{\Delta t} = \frac{1}{2}\left\{f(u_{j+1}) + f(u_j)\right\}$$

Need to solve for u_{j+1}:

⇒ **implicit** scheme difficult for *nonlinear* equation

Approximate u_{j+1}

$$f(u_{j+1}) = f(u_j + \underbrace{\Delta u}_{u_{j+1}-u_j}) = f(u_j) + \frac{df}{du}\Delta t$$

Insert in differential equation

$$\frac{\Delta u}{\Delta t} - \frac{1}{2}\frac{df}{du}|_{u_j}\,\Delta u = f(u_j)$$

yields the difference scheme

$$(u_{j+1} - u_j)\left[\frac{1}{\Delta t} - \frac{1}{2}\frac{df}{du}\right] = f(u_j) + \mathcal{O}(\Delta t^3)$$

Notes:

- Crank-Nicholson is 2^{nd}-order scheme
- Crank-Nicholson is very stable (⇒ below), very reliable

7.3 Runge-Kutta

2^{nd}-order:

$$
\begin{aligned}
K_1 &= \Delta t\, f(t_j, u_j) \\
K_2 &= \Delta t\, f(t_j + \frac{1}{2}\Delta t, u_j + \frac{1}{2}K_1) \\
u_{j+1} &= u_j + K_2
\end{aligned}
$$

Note: $u_j + K_1/2$ is a better approximation fo u during $[t_j, t_{j+1}]$. Use it in f.

4^{th}-order:

$$
\begin{aligned}
k_1 &= \Delta t\, f(t_j, u_j) \\
k_2 &= \Delta t\, f(t_j + \frac{1}{2}\Delta t, u_j + \frac{1}{2}k_1) \\
k_3 &= \Delta t\, f(t_j + \frac{1}{2}\Delta t, u_j + \frac{1}{2}k_2) \\
k_4 &= \Delta t\, f(t_j + \Delta t, u_j + k_3)
\end{aligned}
$$

$$u_{j+1} = u_j + \frac{1}{6}\{k_1 + 2k_2 + 2k_3 + k_4\} + \mathcal{O}(\Delta t^5)$$

Note:

- RK4 is very efficient scheme, and it is quite robust (stable).

7.4 Stability

In each time step errors are made

- truncation error ($\mathcal{O}(\Delta t^p)$)
- round-off error

Question: do these errors grow/accumulate catastrophically?
If yes: scheme **unstable** and therefore **useless**.

Depending on the type of equations at hand more or less stringent stability requirements may be useful.

For simplicity: discuss only linear equations.

Definition: A difference scheme is stable for $\Delta t \to 0$ if there are C and α such that

$$\| \mathbf{y}(t) \| \leq C e^{\alpha(t-t_0)} \| \mathbf{y}(t_0) \|$$

with C and α independent of initial condition $\mathbf{y}(t_0)$ and Δt.

Notes:

- for stability require growth to be bounded by an exponential with *fixed* growth rate
- if exact solutions are known not to grow at all it may be useful to require that numerical solution does not grow either.

Fundamental Theorem:

If a scheme is **stable** and **consistent** then it **converges**

$$u_j \to x(t_j) \;\text{ for }\; \Delta t \to 0$$

and if local error is $\mathcal{O}(\Delta t^{p+1})$ then global error is $\mathcal{O}(\Delta t^p)$.

Thus:

- consistent: error in each time step can be made small ($p > 0$)
- interpretation: stability guarantees that local error does not grow too much (growth rate is bounded)
  $\Rightarrow$ total error goes to 0 as error in each step goes to 0

Sketch of Proof: need to track growth of error.

introduce time evolution operator $S(t_2, t_1)$

$$\begin{aligned} \text{numerical } u_{j+1} &= S(t_{j+1}, t_j) u_j \\ \text{exact } x_{j+1} &= S(t_{j+1}, t_j) x_j + \underbrace{E_T(t_j)}_{\text{truncation error}} \end{aligned}$$

For **linear** differential equation: error evolves as

$$\begin{aligned} e_{j+1} &\equiv x_{j+1} - y_{j+1} = S(t_{j+1}, t_j) e_j + E_T(t_j) \\ e_{j+2} &= S(t_{j+2}, t_{j+1}) e_{j+1} + E_T(t_{j+1}) = \\ &= S(t_{j+2}, t_{j+1}) \left[S(t_{j+1}, t_j) e_j + E_T(t_j) \right] + E_T(t_{j+1}) \\ &= S(t_{j+2}, t_j) e_j + S(t_{j+2}, t_{j+1}) E_T(t_j) + E_T(t_{j+1}) \\ \Rightarrow e_n &= \underbrace{S(n\Delta t, t_0) e_0}_{\substack{\text{propagation} \\ \text{of initial error}}} + \sum_{\ell=1}^{n} \underbrace{S(n\Delta t, \ell\Delta t)\, E_T\left((\ell-1)\Delta t\right)}_{\substack{\text{propagation of} \\ \text{truncation error at} \\ \text{time } t_{\ell-1} = (\ell-1)\Delta t}} \end{aligned}$$

Scheme consistent: $E_T = \mathcal{O}(\Delta t^{p+1})$

Stability of scheme

$$\| S(n\Delta t, \ell\Delta t)\, v(\ell\Delta t) \| \leq C e^{\alpha(n-\ell)\Delta t} \| v(\ell\Delta t) \|$$

equation linear: same bound for error

$$\begin{aligned}
e_n &\leq C e^{\alpha n\Delta t} e_0 + \sum_{\ell=1}^{n} C \underbrace{e^{\alpha(n-e)\Delta t}}_{\leq e^{\alpha n\Delta t}} \underbrace{E_T((\ell-1)\Delta t)}_{K(t)\Delta t^{p+1} = \mathcal{O}(\Delta t^{p+1})} \\
&\leq C e^{\alpha n\Delta t}\{e_0 + n\cdot\Delta t K(t)\Delta t^p\} \\
e_n &\leq C e^{\alpha t_{\max}} \underbrace{\{e_0 + t_{\max} K(t)\}}_{\substack{\text{bounded for} \\ \text{fixed interval } [0, t_{max}]}} \Delta t^p
\end{aligned}$$

7.4.1 Neumann analysis

Consider

$$\dot{x} = \lambda x \qquad \text{with solution} \qquad x = e^{\lambda t} x_0$$

Allow λ to be complex for oscillations. For linear equation Fourier ansatz for numerical solution

$$\mathbf{u}_j = z^j \mathbf{u}_0$$

z^j corresponds to $e^{\lambda t_j}$, λ and z complex.

Stability of forward Euler scheme:

$$u_{j+1} = u_j + \Delta t\, \lambda u_j = (1 + \Delta t\, \lambda) u_j$$

With Fourier ansatz

$$u_j = z^j u_0 \qquad \Rightarrow \qquad z = 1 + \Delta t\, \lambda$$

check growth

$$|z| = |1 + \Delta t \lambda| = \begin{cases} 1 + \Delta t\, \lambda & \text{for} \quad \lambda \in \mathcal{R} \text{ and } \Delta t\, \lambda > -1 \\ 1 + \Delta t^2 \lambda_i^2 & \text{for} \quad \lambda = i\lambda_i \in i\mathcal{R} \end{cases}$$

use

$$\begin{aligned}
1 + \xi &\leq e^{\xi} \qquad \text{for } \xi \in \mathcal{R} \\
\Rightarrow |z| &\leq \begin{cases} e^{\Delta t\, \lambda} \\ e^{\Delta t^2 \lambda_i^2} \end{cases} \Rightarrow |z|^n \leq \begin{cases} e^{n\Delta t \lambda} = e^{\lambda\, t_{\max}} & \lambda\Delta t > -1 \\ e^{n\Delta t^2 \lambda_i^2} = e^{\lambda_i^2 \Delta t\, t_{\max}} & \lambda = i\lambda_i \in i\mathcal{R} \end{cases}
\end{aligned}$$

Thus:

- for $\lambda\Delta t > -1$ and for $\lambda \in i\mathcal{R}$ bounded by exponential
  $\Rightarrow$ stable according to above definition

- oscillatory case: numerical growth (although exact solution does not grow), growth rate $\rightarrow 0$ for $\Delta t \rightarrow 0$

- $\lambda\Delta t < -1$: scheme oscillates and oscillations **grow** (for $\lambda\Delta t < -2$) although exact solution decays monotonically: **unacceptable**.

Note:

- in oscillatory case one may not accept any growth

  $\Rightarrow$ forward Euler method considered unstable for $\lambda \in i\mathcal{R}$

- Neumann-stable: $|z| \leq 1$.
  Forward Euler scheme only Neumann-stable for $\lambda\Delta t > -1$ and $\lambda \in \mathcal{R}$