This brief review appeared in *Pattern Formation in Continuous and Coupled Systems – a Survey Volume,* Vol. 115, Institute for Mathematics and its Applications, p.215, M. Golubitsky, D. Luss, and S. Strogatz, eds., 1999.

LOCALIZED STRUCTURES IN PATTERN-FORMING SYSTEMS*

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Abstract. A number of mechanisms that lead to the confinement of patterns to a small part of a translationally symmetric pattern-forming system are reviewed: nonadiabatic locking of fronts, global coupling and conservation laws, dispersion, and coupling to additional slow modes *via* gradients. Various connections with experimental results are made.

1. Introduction. Over the past years investigations of pattern formation have been quite successful, in particular investigations of spatially and temporally periodic patterns have reached a mature state. In recent years there have been quite a few experimental observations that go beyond this framework, in which spatially localized patterns are found, i.e. structures in which certain patterns extend only over a small part of the spatially homogeneous system. In most cases, far away from the localized pattern the structures are asymptotic to the trivial, unstructured state or to a different, periodic state.

A classic example of such localized structures are propagating pulses of excitation in nerve conduction systems (e.g. [60, 33]). Over the years quite a large number of qualitatively different localized patterns has been identified in a variety of dissipative systems. In convection of binary mixtures one-dimensional wave pulses and domains of waves have been found [34, 4, 36, 28]. In pure-fluid convection in narrow channels confined domains of large convection rolls have been found embedded in a pattern of rolls of smaller wavelength [23]. Qualitatively similar states arise in Taylor vortex flow [3] and in parametrically excited surface waves in ferrofluids [31]. In the Taylor system the vortices with small wavenumber turn out to show additional fine-scale turbulence. In Taylor vortex flow of a viscoelastic fluid striking, as yet unexplained two-vortex states have been observed [21]. Solitary propagating waves have been seen in chemical systems [49], in gas discharge systems [7], and also in parametrically driven surface waves [30]. Recently two observations have found particular interest. On the surface of vertically vibrated granular material circular, solitary waves ('oscillons') arise, which due to the temporal symmetry of the system occur in two symmetrically related forms that can bind and form chains and other larger arrangements [61]. In electroconvection of nematic liquid crystals stable long and narrow domains of convection waves

*This work was partially supported by grants from DOE (DE-FG02-92ER14303) and NASA (NAG3-2113).

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nucleate spontaneously from the very weak noise in the system [15].

The wide range of qualitatively different localized structures cannot be understood with a single mechanism, and in fact quite a few different mechanisms appear to be relevant. A comprehensive review and classification of these mechanisms would be valuable. This goal is, however, beyond the scope of the present paper. Instead, it is intended to provide a brief discussion of some of the important aspects of the mechanisms. The topic of sec.2 is the stabilization through the interaction of fast and slow spatial scales. Localization can also occur due to a conservation law or global coupling (sec.3). The effect of dispersion in wave systems is briefly discussed in sec.4. Localization through the coupling to an additional field *via* gradients is reviewed in some detail in sec.5.

2. Locking of fronts. A quite general situation in which localized states are expected to exist arises in bistable systems in which fronts can connect the two coexisting states. The combination of two opposite fronts leads to a localized structure. The simplest description of such a situation is given by a single-component reaction-diffusion equation. In the context of pattern-forming systems the translation symmetry leads to a Ginzburg-Landau equation for the complex amplitude A of the pattern

$$\partial_t A = \partial_x^2 A + \lambda A + c|A|^2 A - |A|^4 A. \tag{2.1}$$

This equation is valid for weak bistability, i.e. $c > 0$ small. Eq.(2.1) admits front solutions $A_f^{\pm}(x - x_0, t)$ located at x_0 and connecting the basic state $A = 0$ and a patterned state $A = A_0 \exp(iqx)$. For long times the front is expected to approach a front without any phase winding, $A = \mathcal{A} exp(i\psi)$ with $\psi = const.$, i.e. the wavenumber of the localized pattern is given by the critical wavenumber [62]. It is therefore reasonable to focus on the case of A real, i.e. a nonlinear diffusion equation. For general parameters the fronts will propagate and one of the two states will invade the other. For one value of the control parameter, λ_0, however, the fronts are stationary. A localized state can then be obtained by a combination of these two front solutions separated by a distance L. Due to the interaction between the fronts a stationary state will be obtained for suitably adjusted values of λ. For large values of L the dynamics of the fronts can be described asymptotically by an equation for L alone,

$$\frac{dL}{dt} = 2v(\lambda) - ce^{-L/\xi}, \tag{2.2}$$

where the velocity v of a single front vanishes at λ_0 and ξ characterizes the width of the fronts. The interaction coefficient c is positive and the fronts attract each other. Since the interaction decays with distance the attraction renders the localized state unstable. Thus, stable localized states

consisting of two fronts can only be obtained if a repulsive component arises through additional contributions. This is possible in various ways.

For a steady pattern a quite general mechanism can stabilize localized structures [39]. In the asymptotic expression (2.2) the velocity of widely separated fronts vanishes only for a single value of the control parameter λ. In addition, the interaction between the fronts is monotonically attractive, reflecting the monotonic nature of the fronts. In the full equations, which capture also the underlying pattern, the front velocity can vanish, however, over a *finite* range of parameter values [39] due to an interaction between the position x_0 of the front and the underlying pattern. This interaction is lost within the envelope equation (2.1) through the introduction of multiple spatial scales.

By modifying the projection that leads to the solvability condition (2.2) to be a projection over all of space rather than just one wavelength (as it is done to obtain (2.1)) a modified evolution equation for the front position x_0 can be obtained [5],

$$\frac{dx_0}{dt} = v(\lambda) + f(x_0), \tag{2.3}$$

where $f(x_0)$ is periodic with the periodicity of the underlying pattern. Due to the second term a finite locking range $\Delta\lambda$ over which the front is stationary is obtained. It is exponentially small in the steepness of the front, i.e. $f(x_0) \sim \lambda^{\nu} exp(-\alpha/\sqrt{\lambda})$, as is typical for nonadiabatic terms [39]. It is not clear whether the prefactor of the exponential given in [5] contains all the terms of the relevant order.

Due to the oscillatory character of the patterned fronts their interaction will also be modified. No detailed analytical calculation of this appears to have been done so far. It has, however, been studied in quite some detail numerically [52, 53, 13]. This work is motivated by the recent observation of localized excitations (oscillons) of parametrically excited waves in granular material [61]. These excitations cover only a single wavelength of the pattern and due to the subharmonic response consist in alternate phases of the driving of a single peak or a single crater. They arise in a parameter regime in which the transition to spatially periodic waves is subcritical and predominantly to square rather than stripe patterns.

Since the subharmonic instability of the surface waves arises from a Floquet multiplier crossing the unit circle at -1, the small-amplitude and slow-time behavior can be described by the same type of equation as that obtained from a real eigenvalue crossing 0, with the additional requirement that the resulting equations are equivariant under flipping the sign of the amplitude of the pattern. The latter expresses the two equivalent states that are phase-shifted with respect to each other by one period of the driving.

To capture the non-adiabatic locking in a description of this system the fast spatial scales have to be kept in the description. Therefore one

is led to the use of order-parameter equations (e.g. [35]) of the Swift-Hohenberg type. As expected, stable localized states are found in one dimension [52] as well as in two dimensions [53, 13]. In two dimensions the stable range appears to be noticeably larger than in one dimension [13]. Of course, the description of such a short localized state using the interaction of widely separated fronts can at most give qualitative insight. However, the numerical results are consistent with the exponential scaling of the locking range expected from (2.3) [13]. In two dimension the locking mechanism requires that the pattern be periodic in both directions as it is the case with square and hexagonal patterns. In the experiments square patterns are observed in the relevant regime [61]. The case of localized structures arising from hexagonal patterns has been discussed earlier [2].

As in the experiments various bound states of oscillons of alternating polarity (peak *vs.* crater) are found [13, 53]. This multitude of solutions is due to the non-monotonic interaction between the oscillons. Such stationary multi-hump solutions have been discussed in great detail in the context of homoclinic orbits in reversible systems [10, 54]. In the simple order-parameter model, in addition, weakly bound states of equal polarity are found as well, which are, however, quite sensitive to noise. A more detailed analysis raises the question whether this localization mechanism is indeed sufficient to describe the experiments or whether additional mechanisms are relevant [13, 58]. In a number of other phenomenological models oscillons have been found as well, but the localization mechanisms have not been identified clearly [58, 63, 50, 16].

The same mechanism has also been invoked to explain the localized waves in electroconvection of nematic liquid crystals ('worms') [59]. There the transition to the extended waves is, however, most likely supercritical [57] and therefore no fronts connecting the basic state and the extended waves exist.

3. Global coupling and conserved quantities. Single localized states can also be stabilized through a global coupling or the presence of a conservation law. Consider, for example, the simple extension of (2.1),

$$\partial_t A = \partial_x^2 A + \lambda A + c|A|^2 A - |A|^4 A - \kappa A \int_{-\infty}^{\infty} |A|^2 \, dx. \tag{3.1}$$

Now a single domain of $A = \mathcal{A}$, embedded in a domain with $A = 0$, cannot grow to an arbitrary size L due to the ever increasing damping of A with L. Thus, stable domains can be obtained. This mechanism has been studied in detail in the context of current filaments in semiconductors [1, 55] and gas discharge systems.

A similar mechanism arises if the bifurcating amplitude is a conserved quantity. A simple case is given by the Ginzburg-Landau equation for

solutions. Their localization is due to a balance between the amplitude dependence of the oscillation frequency and the linear dispersion. Due to the symmetries of the system the solitons form a continuous four-parameter family of solutions, characterized by the position, phase, amplitude $\mathcal{A}$ and velocity v of the soliton,

$$A = \mathcal{A}\,\mathrm{sech}(\mathcal{A}(x - vt))e^{\frac{i}{2}(\mathcal{A}^2 - v^2)t + ivx}. \tag{4.2}$$

In the presence of weak dissipation or parametric forcing [17], which introduces terms of the form γA^* (cf. 5.8) with A^* denoting the complex conjugate of A, the symmetries involving the amplitude, the velocity, and the phase (in the case of forcing) are broken and the parameters of the solitons become slow functions of time [64, 37, 56, 46], e.g.,

$$\frac{d\mathcal{A}}{dt} = 2(\lambda - d_r v^2)\mathcal{A} + \frac{2}{3}(2c_r - d_r)\mathcal{A}^3 + \frac{16}{15}p_r\mathcal{A}^5, \tag{4.3}$$

where c_r denotes the real part of c, etc. Stable solitary waves correspond to stable fixed points of (4.3) and the corresponding equation for v. Thus, to obtain stable localized waves arising from a Hopf bifurcation within this framework the bifurcation to the extended traveling waves must be subcritical ($p_r < 0$). In fact, the localized waves exist only for parameter values for which also extended waves exist (although possibly unstably). The perturbed solitons of the nonlinear Schrödinger equation were invoked [37, 56] to model the localized wave trains that have been observed in binary-mixture convection [34, 4, 36, 28]. Indeed, for large dispersion the envelope of the experimentally observed wave train looks quite similar to a typical soliton of the nonlinear Schrödinger equation.

In other regimes, in which dispersion is weaker, the wave pulses are better characterized as a pair of stably bound fronts connecting the conductive state with the nonlinear wave state. The interaction between the fronts is affected by dispersion and the amplitude dependence of the frequency. In the limit of weak dispersion again an evolution equation for the length L of the localized state can be derived [32, 22],

$$\frac{dL}{dt} = 2v(\lambda) - c_1 e^{-\frac{L}{\xi}} + \frac{c_2}{L}. \tag{4.4}$$

The potential for a repulsive interaction ($c_2 > 0$) arises from the last term in (4.4). It is due to the differential phase winding, which arises from the amplitude dependence of the frequency and which leads to a gradient in the wavenumber [32].

Recently, cases of strong dispersion have been investigated in the Ginzburg-Landau equation in which stable saturated localized waves arise even in regimes in which spatially periodic waves blow up in finite time [40, 25]. Even though the cubic dissipative term is non-saturating and no fifth-order term is present to prevent blow-up, it turns out that the amplitude dependence of the frequency can be large enough to generate strong

gradients in the wave number which in turn lead to strong dissipation *via* the diffusion term.

5. Gradient-coupling to an additional field. By introducing a second field one can obtain very robust localized structures. A classic case is that of two coupled reaction-diffusion equations. This type of system has been studied in great detail and it will suffice to mention a few keywords [60, 33]. In the paradigmatic case one variable acts as an activator and satisfies a nonlinear equation with an S-shaped nullcline of the reaction term while the equation for the inhibitor is linear. In the lumped system (no spatial dependence) one can then distinguish three classes of dynamics depending on the intersection of the nullclines of the two reaction terms: oscillatory, excitable, and bistable. The type of spatial structure that is obtained depends strongly on the ratio of the diffusion coefficients. In the excitable regime and if both coefficients are of similar size one obtains traveling pulses that have been widely used to model in particular nerve conduction [60]. If the diffusion of the inhibitor is fast compared to that of the activator it spreads well ahead (and behind) of the activator and localized *stationary* structures are obtained [26].

Less well investigated than the reaction-diffusion systems are systems in which the interaction between the different modes is through the gradients of one of the fields. Such an interaction arises naturally in secondary bifurcations off a periodic pattern. There the amplitude of the bifurcating mode is coupled to the phase of the underlying pattern, which is a slow mode due to the translation symmetry of the system [12]. The bifurcating amplitude depends, however, not on the phase itself but only on its gradient, the wavenumber. Two cases of localized structures described by equations of this type have been investigated [9, 48, 51]. The gradients can also arise from the advective nature of the interaction [58, 44, 47].

If a one-dimensional steady pattern undergoes a secondary Hopf bifurcation the complex amplitude A of the oscillations and the phase ϕ of the underlying pattern satisfy an equation of the form [12]

$$\partial_t A = \lambda A + d\partial_x^2 A + c|A|^2 A + f\partial_x \phi\, A, \tag{5.1}$$

$$\partial_t \phi = \delta\partial_x^2 \phi + h\partial_x |A|^2 + iw(A^*\partial_x A - A\partial_x A^*). \tag{5.2}$$

The coefficients in (5.1) are complex while those in (5.2) are real. It turns out that (5.1,5.2) have an exact localized solution of the form [51]

$$A = A_0 \operatorname{sech}(kx) e^{i\omega t + i\psi}, \qquad \frac{d\phi}{dx} = Q_0 + Q_1 \operatorname{sech}^2(kx), \tag{5.3}$$

with $d\psi/dx = B_0 \tanh(kx)$ and the six coefficients satisfying certain algebraic relationships. Numerical simulations show that this solution can in fact be stable. An important feature of this localized solution is that

through the coupling proportional to f the local growth rate of the oscillatory mode is modified by the wavenumber of the underlying pattern and is strongly increased inside the localized structure while it is reduced outside. This localized solution appears to capture the essence of the localized oscillations found in square electroconvection patterns of a nematic liquid crystal [27].

A somewhat similar situation is found for patterns undergoing a parity-breaking bifurcation, i.e. an instability that breaks the reflection symmetry of the pattern and induces a drift of the pattern. Such instabilities can also lead to localized structures in the form of domains of traveling waves that drift through the otherwise stationary pattern. They have been observed in directional solidification [19], viscous fingering [14], and in Taylor vortex flow [65].

The parity-breaking instability of a steady pattern can be described by an equation for the real amplitude A of the asymmetric mode that breaks the reflection symmetry and again for the phase ϕ of the underlying pattern [11, 12, 18],

$$\partial_t A = (\lambda + \lambda_1 \partial_x \phi) A - A^3 + d\partial_x^2 A + b\partial_x^2 \phi + h.o.t. \tag{5.4}$$

$$\partial_t \phi = A. \tag{5.5}$$

Note that these equations exhibit an inhomogeneous scaling in the small parameter measuring the distance from threshold, the amplitude and the space and time scales. The existence of localized drift waves in (5.4,5.5) can be seen quite easily [48, 9]. Assuming a steady solution ($q(x) \equiv \partial_x \phi(x)$, $A(x)$) in a frame moving with velocity v the wavenumber of the underlying pattern is given by $q = -A/v + q_\infty$ and the amplitude A satisfies

$$d\partial_x^2 A + (v - \frac{b}{v})\,\partial_x A = -\frac{d}{dA}\left\{\frac{1}{2}(\lambda + \lambda_1 q_\infty)A^2 - \frac{\lambda_1}{3v}A^3 - \frac{1}{4}A^4\right\}. \tag{5.6}$$

A homoclinic orbit in space connecting $A = 0$ with itself exists if the 'friction' $v - b/v$ vanishes, yielding for the velocity $v = \pm\sqrt{b}$. Numerical simulations show that this state can be stable. It is noteworthy that it can only exist if it drifts: for $v = 0$ the wavenumber q would diverge.

It should be pointed out that both secondary bifurcations discussed here are supercritical for the spatially periodic states. Nevertheless, stable localized structures are possible. They correspond to homoclinic rather than heteroclinic structures in space.

Gradient coupling can also arise in systems undergoing a primary bifurcation to a patterned state. In a phenomenological model for surface waves in vibrated granular material a coupling of the surface oscillation amplitude A to the local averaged thickness of the layer has been intro-

duced leading to [58]

$$\partial_t A = \gamma A^* - (1 - i\omega)A + (1 + ib)\Delta A - |A|^2 A - \rho A, \tag{5.7}$$

$$\partial_t \rho = \alpha \nabla \cdot (\rho \nabla |A|^2) + \beta \Delta \rho. \tag{5.8}$$

Here the gradient coupling models the expulsion of material from strongly oscillating regions. The term involving A^* arises from the parametric forcing of the system at twice the resonant frequency of the oscillatory mode A. In direct numerical simulations and by using a shooting method two-dimensional localized structures similar to the experimentally observed oscillons were obtained. They appear in a regime in which square patterns arise in a subcritical bifurcation. As in the two cases discussed above, it appears to be important that the local growth rate is enhanced inside the oscillon and reduced outside. As in the experiments [61] and other models for oscillons [63, 50, 13] bound states of localized states with opposite polarity are found.

In traveling-wave systems gradient coupling arises quite naturally if the waves advect a quantity that is dynamically relevant, i.e. evolves on a time scale comparable to that of the bifurcating amplitude. This has been studied in quite some detail motivated by experiments on traveling waves in binary-mixture convection [34, 4, 36, 28] and in electroconvection of nematic liquid crystals [15].

In binary-mixture convection, motivated by the anomalously slow drift of the pulses, the advection of a concentration mode by the traveling wave was considered [45]. This mode can be important since mass diffusion is very slow in liquids. It was found that such an advection not only affects the pulse velocity [46] but can also be sufficient to localize a traveling wave structure. The equations that were used to study this mechanism describe the evolution of the complex wave amplitude A and of a real concentration mode C,

$$\partial_t A + s\partial_x A = d\partial_x^2 A + (\lambda + fC)A + cA|A|^2 + p|A|^4 A, \tag{5.9}$$

$$\partial_t C = \delta \partial_x^2 C - \alpha C + h\partial_x |A|^2. \tag{5.10}$$

In general, the coefficients in (5.9) are complex while those in (5.10) are real. The derivation of these equations from the Navier-Stokes equation shows that in principle quite a few additional coupling terms arise [46].

To concentrate on the localization by the concentration mode C rather than by dispersion all coefficients in (5.9) are assumed to be real. Since the system is bistable ($c > 0$, $p < 0$) fronts exist that connect the conductive state $A = 0$ with the traveling-wave state $A = A_0$. Their interaction is strongly affected by the advected field. Focussing on the effect of the advection, evolution equations for the velocities $v_{l,t}$ of the leading and of the trailing front can be derived in the case of weak diffusion of A and C,

small group velocity and weak coupling [24],

$$v_l = s - \frac{\gamma}{|v_l|} + sgn(v_l)\rho, \tag{5.11}$$

$$v_t = s - \frac{\gamma}{|v_t|} + 2\gamma\frac{e^{-\alpha L/|v_l|}}{|v_l|} - sgn(v_t)\rho, \tag{5.12}$$

where ρ measures the control parameter λ, γ is proportional to the coupling h, and L is the (time-dependent) length of the pulse. Since C decays over distances much larger than A the attractive interaction between the fronts that was obtained in (2.2) is negligible in this regime. Eqs.(5.11, 5.12) show that the sign of the interaction mediated by C depends on the direction of propagation of the pulse. It is repulsive only if the whole pulse drifts opposite to the linear group velocity (for $\gamma s > 0$). This can be understood with simple arguments considering the effect of C on the local growth rate of A and its consequence for the velocity of the respective fronts. An essential ingredient is that $|C|$ is smaller at the trailing front than at the leading front [24].

It turns out that the advected field can even lead to localized structures if the initial bifurcation is supercritical ($c < 0$). Somewhat similar to the case of the parity-breaking bifurcation, for $d = 0$ and $\alpha = 0$ the coupled equations (5.9, 5.10) can be reduced to a single equation for C that has the form of a particle in a (cubic) potential [38],

$$\begin{aligned}\frac{1}{2}(s-v)\frac{\delta}{h}\partial_x^2 C + \left(\frac{\delta}{h} + \frac{v}{2h}(s-v) - \left(\frac{2v\delta}{h^2} + \frac{\delta}{h}\right)C - \frac{\delta^2}{h^2}\partial_x C\right)\partial_x C \\ = -\frac{d}{dC}\left\{\frac{v}{2h}C^2 - \frac{v}{3h}\left(\frac{v}{h}+1\right)C^3\right\}.\end{aligned} \tag{5.13}$$

Again the velocity v of the pulse is an eigenvalue and is determined by the condition that the 'work' done by the 'friction' vanishes over the homoclinic orbit that connects $C = 0$ with itself. Since the friction is nonlinear in this system the velocity has to be determined numerically. Direct numerical simulation of (5.9,5.10) (with $d > 0$) shows that these traveling-wave pulses can be stable [44]. As in the parity-breaking case (5.6) it appears to be crucial that the pulse drift ($v \neq 0$).

The advection of a slow mode by traveling waves appears also to be relevant to understand recent observations of localized waves ('worms') in electroconvection of nematic liquid crystals [15]. Due to the anisotropy of this system the initial Hopf bifurcation leads to the competition of waves traveling in four symmetrically related directions that are oblique to the preferred direction characterized by the director of the liquid crystal. In analogy to zig-zag patterns these waves may be termed left- and right-zigs and -zags, respectively. The worms are made up of right-traveling zigs and zags and drift slowly to the left (or vice versa). A surprising aspect of the system is that the initial bifurcation to the periodic waves is supercritical

[57], but the worms are nucleated well below that Hopf bifurcation [6]. The usual coupled Ginzburg-Landau equations for the two participating wave amplitudes alone are therefore insufficient to describe the worms. An extension that includes the advection of a slow mode similar to the

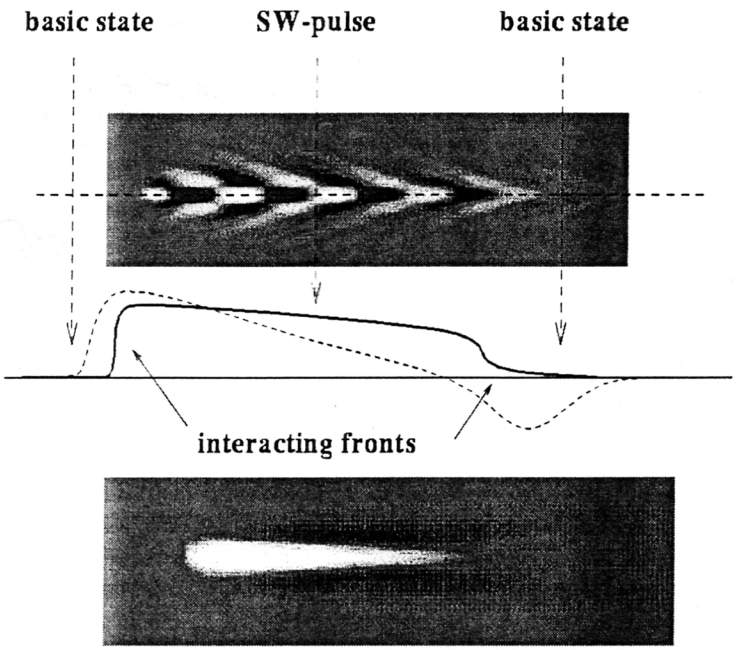

FIG. 1. *Zig- and zag-wave (top) and C-field (bottom) in numerical simulations of a worm. The sketch in the center indicates schematically the amplitude of the standing-wave pulse (solid line) and of the C-field (averaged over y, dashed line).*

concentration mode in binary-mixture convection has been considered [47],

$$\partial_t A = -\mathbf{u}_A \cdot \nabla A + \mu A + b_x \partial_x^2 A + b_y \partial_y^2 A + 2a\partial_{xy}^2 A + fCA + c|A|^2 A + g|B|^2 A, \tag{5.14}$$

$$\partial_t B = -\mathbf{u}_B \cdot \nabla B + \mu B + b_x \partial_x^2 B + b_y \partial_y^2 B - 2a\partial_{xy}^2 B + fCB + c|B|^2 B + g|A|^2 B, \tag{5.15}$$

$$\partial_t C = \delta \Delta C - \alpha C + \mathbf{h}_A \cdot \nabla |A|^2 + \mathbf{h}_B \cdot \nabla |B|^2. \tag{5.16}$$

Here A and B are the amplitudes for the right-traveling zig- and zag-waves, respectively. Within these equations the localization of the worms can be understood to arise from the combination of two different mechanisms. In a one-dimensional reduction in the y-direction transverse to the worm (5.14–5.16) reduce to the equations for two counterpropagating waves each advecting C in opposite directions. Numerical simulations show that again

a localized structure can exist stably already below the Hopf bifurcation although that bifurcation is supercritical ($c < 0$). Similar to the traveling-wave pulse this standing-wave pulse is homoclinic in space, but in contrast to the traveling-wave pulses no analytic description like (5.13) is available as yet.

Given the coexistence of the standing-wave pulses (not of the extended waves) with the basic, non-convective state there exist also fronts that connect the basic state at $x = \pm\infty$ with the standing-wave pulse (see fig.1). In analogy to the interaction between fronts discussed in the case of binary-mixture convection (5.11,5.12) one may expect that these fronts form a stable worm if the worm drifts opposite to the x-component of the linear group velocity of the two wave components. Indeed, the experimentally observed worms show this behavior [15].

6. Conclusion. Quite a few experimentally observed localized structures in dissipative systems can be understood qualitatively with the mechanisms discussed in this paper. With respect to quantitative comparisons the results are somewhat limited, yet. Given the variety of different structures and mechanisms it would be of great interest to condense them into paradigmatic cases, which will depend on the symmetries of the underlying pattern (steady, traveling, oscillatory,...) and the symmetries of the coupling to additional slow modes if present. Another relevant distinction will be whether the localized structures are homoclinic or heteroclinic in space. While for some of the presented mechanisms analytical insight has been gained in limiting cases, this is not the case for all of them. For instance, a systematic treatment of localization through nonadiabatic effects, which are quite general for steady structures and for standing waves, would be valuable. In contrast to the exponentially small interaction between localized structures the formation of localized structures by such an interaction between *fronts of patterns* has not been treated much.

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