Subcritical Dissipation in Three-Dimensional Superflows

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Three-dimensional (3D) superflows past a circular cylinder are studied by numerically integrating the nonlinear Schrödinger equation. 3D initial data are built from the two-dimensional (2D) stationary vortex nucleation solutions. Quasistationary half-ring vortices, pinned at the sides of the cylinder, are generated after a short time. On a longer time scale, either 3D vortex stretching induces dissipation and drag, or the vortex is absorbed by the cylinder. The corresponding 3D critical velocity is found to be well below the 2D one. The implications for experiments in Bose-Einstein condensed gas and low-temperature helium are discussed.

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Dilute Bose-Einstein condensates have been recently produced experimentally [1]. The dynamics of these compressible nonlinear quantum fluids is accurately described by the nonlinear Schrödinger equation (NLSE), also called the Gross-Pitaevskii equation [2], allowing direct quantitative comparison between theory and experiment [3]. The NLSE can also be considered to describe the dynamics of superfluid 4He, at temperatures low enough for the normal fluid to be negligible. In the homogeneous two-dimensional (2D) NLSE flow past a disk, Frisch et al. [4] found the existence of a transition to dissipation due to the periodic emission of pairs of counterrotating vortices. This transition was shown to occur at a critical Mach number $M_{2D}^c$ always greater than 0.35 [5]. Vortex formation and pressure drag above $M_{2D}^c$ were extensively studied [6]. In a recent experiment, Raman et al. have studied dissipation in a Bose-Einstein condensed gas by moving a blue detuned laser beam through the condensate at different velocities [7]. In their inhomogeneous condensate, they observed a critical Mach number for the onset of dissipation $M_{2D}^c/1.6$.

The main problem addressed in this Letter is the characterization of an intrinsically three-dimensional (3D) vortex stretching mechanism in the superflow around a cylinder. Such a mechanism is made plausible by recent 3D NLSE simulations and experiments in 4He [8] demonstrating the existence of vortex stretching and inertial range Kolmogorov scaling in superfluid turbulence. This 3D mechanism generates drag and can be responsible for the subcritical (below $M_{2D}^c$) dissipation observed by Raman et al. [7].

We study the effect of a moving cylinder of diameter $D$ in a 3D superfluid at rest described by the action

$$\mathcal{A} = \int dt \left\{ \sqrt{2} \, c \xi \int d^3 x \left[ \frac{i}{2} \left( \psi \frac{\partial \psi}{\partial t} - \bar{\psi} \frac{\partial \bar{\psi}}{\partial t} \right) - \mathcal{F} \right] \right\},$$

where $\psi$ is a complex field and $\bar{\psi}$ is its conjugate. The coherence length $\xi$ and the speed of sound $c$ (for a mean fluid density $\rho_0 = 1$) are the physical parameters characterizing the superfluid. The energy $\mathcal{F}$ reads

$$\mathcal{F} = \mathcal{E} - \bar{P} \cdot \vec{U},$$

where

$$\mathcal{E} = c^2 \int d^3 x \left( -1 + V(\vec{x}) \right) |\psi|^2 + \frac{1}{2} |\psi|^4$$

$$+ \xi^2 |\nabla \psi|^2$$

and

$$\bar{P} = \sqrt{2} \, c \xi \int d^3 x \frac{i}{2} (\psi \nabla \bar{\psi} - \bar{\psi} \nabla \psi)$$

is the superfluid momentum. The repulsive potential $V(r) = (V_o/2) \{ \tanh[4(r - D/2)/\ell] - 1 \}$ represents the cylinder, whose velocity $\vec{U}$ is imposed by the $\bar{P} \cdot \vec{U}$ term. In the calculations presented below, $V_o = 10$ and $\ell = \xi$. With these values, the field inside the cylinder is negligible ($|\psi| \sim 0$) and the boundary layer is well resolved with a mesh adapted to the coherence length.

The NLSE is the Euler-Lagrange equation corresponding to (1),

$$\frac{\partial \psi}{\partial t} = i \frac{c}{\sqrt{2} \xi} \left[ \Omega(\vec{x}) \psi - |\psi|^2 \psi + \xi^2 \nabla^2 \psi \right] + \vec{U} \cdot \nabla \psi,$$

where $\Omega(\vec{x}) = 1 - V(|\vec{x}|)$.

The NLSE (4) can be mapped into two hydrodynamic equations by applying Madelung’s transformation: $\psi = \sqrt{\rho} \exp(i \varphi/\sqrt{2} \xi)$, where $\rho$ and $\vec{v} = \nabla \varphi - \bar{U}$ are the fluid density and velocity relative to the cylinder. We thus obtain the continuity equation $\partial \rho/\partial t + \nabla \cdot (\rho \vec{v}) = 0$ and the Bernoulli equation $\partial \varphi/\partial t + \vec{v} \cdot (\rho \vec{v})/2 - \bar{U}^2/2 + c^2 \rho - \bar{U}(\Omega(\vec{x})) - c^2 \xi^2 \nabla^2 \sqrt{\rho}/\sqrt{\rho} = 0$. The last term of this equation is a dispersive supplementary “quantum pressure” term that is relevant only at length scales smaller than $\xi$. Note that the NLSE (4) admits vortical solutions of characteristic core size $\sim \xi$. These are topological defects.
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apparent in the figure.

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\( \) critical Mach number

\( \) cal integrations of the NLSE (4). Beyond the saddle-node

\( \) and two vortices, respectively, near the sides of the cylin-

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ders. The corresponding dynamics was studied by numerical

integrations of the NLSE (4). Beyond the saddle-node

critical Mach number \( M_{2D} \), no stationary solution exists

and (dissipative) periodic vortex nucleation occurs. No

subcritical dissipation was observed in this 2D system. The

value of \( M_{2D} \) was found to depend on \( \xi / D \), with a lower

bound at \( M_{2D} \sim 0.35 \), reached in the limit \( \xi / D \to 0 \) [5].

We used the 2D laminar stationary solution \( \psi_{0V}(x,y) \)
(corresponding to branch \( 0V \)) and the one-vortex unstable
stationary solution \( \psi_{1V}(x,y) \) (branch \( 1V \)) to construct the
3D initial condition \( \psi_{3D}(x,y,z) = f_1(z)\psi_{1V}(x,y) +
[1 - f_1(z)]\psi_{0V}(x,y) \). The function \( f_1(z) \), defined by

\[ f_1(z) = \frac{\tanh[(z - z_1)/\Delta_z] - \tanh[(z - z_2)/\Delta_z]}{2}, \]

takes the value 1 for \( z_1 \leq z \leq z_2 \) and 0 elsewhere (\( \Delta_z \)

is an adaptation length). The surface \( |\psi_{3D}| = 0.5 \) is shown in
Fig. 2a, for \( \xi / D = 0.025 \), \( |\hat{U}|/c = 0.26 \), and \( \Delta_z =
2\sqrt{2} \xi \) in the \( [L_x \times L_y \times L_z] \) periodicity box (\( L_z / D =
2.4\sqrt{2} \pi \), \( L_y / D = 1.2\sqrt{2} \pi \), and \( L_x / D = 0.4\sqrt{2} \pi \).

The cylinder surface and the initial condition vortex line,
with both ends pinned to the left side of the cylinder, are
apparent in the figure.

The resulting dynamical evolution is obtained by integrating
the NLSE, with a 3D version of the code used in our
previous studies [5]. It can be schematically described
in terms of short-time and long-time dynamics.

Under short-time dynamics, the initial pinned vortex line
rapidly contracts, evolving through a decreasing number of
half-ring-like loops, down to a single quasistationary half
ring (see Figs. 2b–2d). If the initial vortex line is long
enough to contract to quasistationary half rings, the evolu-
tion always takes place near the plane perpendicular to the
flow. If the vortex line is too small, it moves upstream and
collapses against the cylinder.

The diameter \( d \) of a stationary vortex ring in an infinite
Eulerian flow with no obstacle is given by [9]

\[ |\hat{U}|/c = (\sqrt{2} \xi /d) [\ln(4d/\xi) - K], \]

where \( |\hat{U}| \) is the flow velocity at infinity and the vortex
core model constant \( K \sim 1 \) is obtained by fitting the nu-
merical results in [10]. This equation can be used to check
that the half-ring state (Fig. 2d) is quasistationary. Indeed,
the local flow velocity \( v \) in a low-Mach number Eulerian
flow around a cylindrical obstacle varies from \( v = |\hat{U}| \) at
infinity to \( v = 2|\hat{U}| \) at both sides of its surface. For the
values used in Fig. 2, local velocities therefore range from
\( v = 0.25 \) to \( v = 2 \times 0.25 \). Equation (5) implies that

FIG. 1. Plot of the energy of the stationary solutions \( |\mathcal{F}(M) -
\mathcal{F}(0)| \) versus Mach number \( (M = |\hat{U}|/c) \) for \( \xi / D = 1/10, 1/20, 1/40 \).

Stable branch: solid line

(0V). Unstable symmetric branch: long-dashed line

(1V). The saddle-node

bifurcation is marked by \( M_1^s \), with \( M_1^s(1/10) = 0.429, 
M_2^s(1/20) = 0.4 \), and \( M_3^s(1/40) = 0.383 \).

FIG. 2. Short-time dynamics of a vortex pinned to the cylinder
for run 6 (see Table I below). The surface \( |\psi| = 0.5 \) is shown at
times (a) \( t = 0 \), (b) \( t = 40\sqrt{2} \xi /c \), (c) \( t = 60\sqrt{2} \xi /c \), and
(d) \( t = 80\sqrt{2} \xi /c \). Note that the half ring formed on (d)
has the diameter needed to be approximately stationary [see
text below Eq. (5)].
The diameter \( d \) in Fig. 3 (inset), behavior. The diameter of the half ring shown below is indicated when vortex stretching takes place (NS: no stretching), except for run 6 [see text below Eq. (7)].

On a longer time scale, the quasistationary half ring can evolve in two opposite ways: it starts moving either upstream or downstream. When the half ring is driven downstream, the vortex loop is continuously stretched while the pinning points move towards the back of the cylinder. When the half ring moves upstream, it eventually collapses against the cylinder, generating a laminar superflow. In order to distinguish between the two situations we have carried out 3D runs summarized in Table I. Most runs were performed at Mach numbers \( M \) slightly different from that of the 2D stationary solutions \( M_{2D} \). The value of \( \Delta M = (M - M_{2D})/M \) is indicated in the table. These 3D computations are rather expensive, e.g., to integrate the NLSE up to the situation in Fig. 3b (run 2) necessitates a resolution of \( 256 \times 128 \times 256 \) and 25 hours of CPU on a Cray 90 machine.

Figure 3 shows the long-time dynamics for a stretching case: run 2 of Table I. The inset in Fig. 3b pictures the corresponding quasistationary half ring for size comparison. Note that, as the vortex loop grows, its rear part remains oblique to the flow (see Fig. 3a).

The runs of Table I are displayed schematically in Fig. 4. The runs with vortex stretching are labeled by circles and those without by \( \times \). All runs were performed at Mach numbers \( M \) below \( M_{2D} \) indicated in Fig. 4 as a solid line. The experimental [7] critical Mach number and value of \( \xi/D \) are marked by an asterisk.

For \( 1/30 < \xi/D < 1/20 \), there is a frontier between the dissipative and nondissipative cases that can be drawn approximately as the dashed line in Fig. 4, which corresponds to the expression \( R_s = 5.5 \) with

\[
R_s = |\ddot{U}|D/c\xi = MD/\xi.
\]
This superfluid “Reynolds” number is defined in the same way as the standard (viscous) Reynolds number $Re = |\vec{U}|/\nu$ (with $\nu$ the kinematic viscosity). In the superfluid turbulent ($R_s \gg 1$) regime, $R_s$ was shown to be equivalent to the standard (viscous) Reynolds number $Re$ [8]. Note that, for a Bose condensate of particles of mass $m$, the quantum of velocity circulation around a vortex, $\Gamma = 2\sqrt{2} c \xi$, has the Onsager-Feynman value $\Gamma = h/m$ ($h$ is Planck’s constant) and the same physical dimensions $L^2 T^{-1}$ as $\nu$.

In order to study quantitatively the transition to dissipation, we define the nondimensional drag

$$C_s = 2(d|\vec{P}|/dt)/\rho_0 |\vec{U}|^2 S,$$

where the surface $S$ facing the flow is the diameter $D$ of the cylinder times the height of the vortex loop obtained at the end of the run. Typically, the superfluid momentum (3) as a function of time (data not shown) first oscillates during the short-time dynamics. In the case of vortex stretching it then grows linearly. In this case, $d|\vec{P}|/dt$ is estimated by a linear fit to $|\vec{P}|$. The case of run 6 is special. Although vortex stretching took place, it was impossible to reliably determine $C_s$ because the height of the vortex loop kept on growing during the run. This may be due to the lower value of $L_s/D$ in this run. The values of $C_s$ are displayed in Table I. The obtained order-one values of $C_s$ demonstrate that 3D vortex stretching is an efficient dissipative mechanism.

In our numerical system, the initial vortex loop was imposed extrinsically. In an experimental setting, fluctuations strong enough to nucleate the initial vortex loop are needed to commence vortex stretching. The vortex nucleation by thermal or quantum fluctuations has been studied by different groups [11,12]. Ihas et al. proposed a nucleation barrier formed by an unstable stationary vortex loop that is pinned to the walls bounding the superfluid. Interestingly, this solution is equivalent to the quasistationary half-ring solution towards which our numerical system is driven naturally before starting vortex stretching.

In summary, our numerical results demonstrate the possibility of a subcritical (below $M_{2D}$) and efficient ($C_s \sim 1$) drag mechanism, based on 3D vortex stretching. Our computations were performed for values of $\xi/D$ comparable to those in Bose-Einstein condensed gas experiments [7]. In the context of superfluid $^4$He flow, the experimental critical velocity is known to depend strongly on the system’s characteristic size $D$. It is often found to be well below the Landau value (based on the velocity of roton excitation) except for experiments where ions are dragged in liquid helium. Feynman’s alternative critical velocity criterion $R_s \sim \log(D/\xi)$ is based on the energy needed to form vortex lines. It produces better estimates for various experimental settings but does not describe the vortex nucleation mechanism [9]. It would be very interesting to determine experimentally the dependence of the critical Mach number on the parameter $\xi/D$ and the nature (2D or 3D) of the excitations.

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