SINGULAR PERTURBATIONS IN NOISY DYNAMICAL SYSTEMS

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Four previous awardees were my teachers and inspirations

Peter Lax, 1968    Kurt Friedrichs, 1979

Joe Keller, 1983   Jurgen Moser, 1984
Two spoke on Asymptotics and Applications
I follow their footsteps, speak on same topic

Singular Perturbations in in
Noisy Dynamical Systems

• Asymptotics studies the local behavior of functions
  – Functions may be known \textit{a-priori}
  – For some functions we may only have hints
e.g., satisfy DEs + BCs or ICs
  – Perturbation Theory
    * Regular perturbations
    * Singular perturbations

• Asy series often divergent
• Abel: invention of the devil
• Diff. bet. convergent and asymptotic
• Asymptotic often superior
• Abel comment not relevant
Regular Perturbation

- Small changes in model lead to small changes in behavior
- Results little noted nor long remembered

Singular Perturbation (SP)

- Small changes in model lead to large changes in behavior
- Results deeper and more interesting
- Can occur if perturbation random
We employ singular perturbation methods in noisy dynamical systems

- The method of matched asymptotic expansions (MAE)

The Exit Problem + Applications
The Exit Problem

• Deterministic dynamical system perturbed by white noise
  – ”Derivative” of Brownian motion
  – Brownian motion is nowhere differentiable

• Example: particle in a potential well
  – Deterministic problem has a stable equilibrium
  – Particle suffers random collisions with smaller, lighter particles of medium
    * Particles exit the well
    * Rare event (not low probability, low frequency)

![Potential well](image_url)
Questions

1. How long to exit?
   Mean Free Passage Time (MFPT)

2. From where on the boundary (rim)
   does exit occur?

• Each quantity satisfies a deterministic BVP
  (Kolmogorov backward equation)

• When noise is small, the resulting BVP is a
  singular perturbation problem

• Solve the BVP by
  singular perturbation methods (MAE)
L. Prandtl - Boundary Layer Theory

- 1904 Prandtl ICM Talk In Heidelberg
  - revolutionized fluid mechanics
- Pre-Prandtl few solutions of Navier Stokes were known
- Low viscosity flow over solid
  - Ignore viscosity away from the boundary
  - Consider Euler equations, not Navier Stokes
  - Viscosity important only in thin layer near the boundary (boundary layer) where the solution varies rapidly

Upon hearing talk, Felix Klein arranged position in Gottingen, mecca of Math., Sci.

Prandtl undoubtedly great F.M.
flawed human being, apologist for nazi regime

- Boundary layer theory was later generalized and systematized: Friedrichs, Wasow, MAE; later by others
Idea of Matched Asymptotic Expansions

- Outer expansion
  \[ \sum a_j(x)\epsilon^j \]

- Stretching Transformation
  \[ \xi = \frac{x - x_0}{\epsilon^\alpha} \]
  \( x_0 \) layer location, \( \alpha \) layer width

- Boundary layer Expansion
  \[ \sum b_j(\xi)\epsilon^j \]
  rapidly varying

- Matching inner and outer expansion: smooth connection
MAE
• Successful for many problems & applications
• Not always - “Failure of MAE”
  – Problem exhibits boundary layer resonance
  – “Spurious Solutions”
• MAE not successful on the exit problem
• Caused some to claim ”failure of MAE”

Here we present a physical and four mathematical arguments which modify or augment MAE so it is successful for the Exit Problem

We restrict 1D linear DEs and limit technical detail though extensions to higher dims, limit cycle escape, different noise, nonlinear problems
Brownian Motion

• 1827 Robert Brown: Pollen grains in water agitated, irregular motion

• 1785 Jan Ingenhausz: Carbon dust in alcohol, less systematic, possible

Stigler Law of Eponomy, which states

”No Discovery Named For its Original Discoverer”
Brownian Motion

- 1905 Albert Einstein: Explanation of Brownian motion; 1906 Smoluchowski independently same result:
  - Motion due to collisions with smaller, lighter particles in which they’re suspended
  - Probabilistic description $O(10^{21})$ collisions/sec can’t observe collisions, nor path
  - Beginning of stochastic modeling
  - Two forces: collisions + viscous drag
  - Process is diffusive:
    \[ p_t = D p_{xx}, \quad D = \frac{kT}{6\pi\eta a} \]
    collisions modeled by diffusion
  - Confirmed existence of atoms then topic of debate
  - 1908 Perrin, later Nobel Prize experimental confirmation
• 1908 Langevin
first stochastic differential equation – (SDE)

\[ m\ddot{x} + 6\pi \eta a \dot{x} + R, \quad D = \frac{kT}{6\pi \eta a} \]

SDE solution only known statistically. 
\( x \) is the particle position, and \( R \) is a random force modeling the collisions.
1D Random Walk Collision Model

- Particle at $x$, jump right $r(x)$, jump left $\ell(x)$, no jump $1 - r(x) - \ell(x)$
- Jump size $\epsilon$, jump time $\delta t$ small
- $p(x, y, t)$ probability to reach $x(t) = y$ given $x(0) = x$

\[
p_{\tau} = L^*p = \frac{\epsilon}{2}[(r + \ell)p]_{yy} - [(r - \ell)p]_y
\]

$\tau = \epsilon t$ (long time scale)
- hardly any motion on shorter scales

- If $r = \ell$, no drift – pure diffusion

\[
p_{\tau} = (rp)_{yy}
\]

- Intimate connection between probability & partial differential equations
N. Van Kampen asked: Why Do Stochastic Processes Enter Physics?

He answers: Many phenomena which evolve in time in an extremely complicated way, well beyond the possibility of calculation or observation, have some average properties that can be observed and obey simple laws. The use of probability is justified by our ignorance of the precise microscopic state. Nevertheless macroscopic variables are observable and can be calculated.

- Process goes from $y$ at time $s$ to $x$ at time $t$
- $p(x, t|y, s)$ satisfies $p_t = L^* p = dp_{xx}$
- $p$ describes the time evolution of a probability density function
• $x, t$ forward variables (to where it’s going)  
  $y, s$ backward variables (from where coming)  
• pure Brownian motion $p_t = L^*p = p_{xx}$ forward Kolmogorov eq.  
  $p_t = Lp$ backward Kolmogorov eq.  
• 1923 Wiener formalized mathematical theory of Brownian motion  
  – Wiener process $w$  
    “derivative” $dw$ (white noise)  
• deterministic dynamical system  
  $\dot{x} = b(x)$  
perturbed by small white noise  

SDE  
\[ dx = b(x)dt + \sqrt{2}\epsilon dw, \]  
SIE - Ito, Stratonovich
• Kolmogorov forward operator
  \[ L^* p = \epsilon p_{xx} - (bp)_x \]

  Kolmogorov backward operator
  \[ L p = \epsilon p_{yy} + bp_y \]

  \( L, L^* \) are adjoints

• We’ll use boundary value problems for \( L p \) to compute MFPT & distribution of exit points in the exit problem

• Deterministic force derived from potential

  \[ V(x) = \frac{x^2}{2} \]

  force = \(-V' = -x, \)

  \( D = (-a, b), \quad a, b > 0 \)

  small random perturbation (white noise)

• MFPT \( \tau \) free Brownian particle

  \[ L\tau = \epsilon \tau'' = -1 \quad \text{in} \ D \]

  \[ \tau = 0 \quad \text{on} \ \partial D \]
• Follows from Ito’s formula

\[ dx_\epsilon = b(x_\epsilon)dt + \sqrt{2}\epsilon dw \]

• \( f(x_\epsilon) = f(x) + \int_0^t Lf ds + \int_0^t Mf dw \)

\( L \) is the backward operator,

\[ Mf = \frac{\partial f}{\partial x}, \text{any } f \]

Last term is a stochastic integral

• MFPT satisfies

\[ Lv = \epsilon v'' - xv' = -1 \text{ in } D \]

\[ v = 0 \text{ on } \partial D \]

• Set \( f = v, t = T, T \) is first passage time to \( \partial D \)

\[ v(x_\epsilon(T)) = v(x) - T + \int_0^T Mvdw \]

Take expectation,

Use \( E(\text{Stochastic Integral}) = 0 \) and BC

\[ v(x) = \tau \]
• Similarly, $u(x)$ satisfies

$$Lu = \epsilon u'' - xu' = 0 \text{ in } D$$

$$u = \phi \text{ on } \partial D$$

• Set $f = u$, $t = T$,

$$u(x_\epsilon(T)) = u(x) + \int_0^T Mudw$$

$$E(\text{Stochastic integral}) = 0 \text{ and BC}$$

$$u(x) = E(x_\epsilon(T))$$

$$u(x) = \int_{\partial D} \phi(y) \rho(x, y) dy$$

• $\rho$ is probability density of exit points
  $=$ Green’s function of Dirichlet problem
• MFPT for free Brownian particle

\[ \tau = \frac{(a + x)(b - x)}{\epsilon} \]

algebraically large in \( \epsilon \)

• Brownian particle in force field

\[ L\tau = \epsilon \tau'' - x\tau' = -1 \quad \text{in } D \]

\[ \tau = 0 \quad \text{on } \partial D \]

− Can show

\[ \tau = O \left( e^{\frac{1}{\epsilon}} \right) \]

exponentially large in \( \epsilon \)

− Takes longer time to

overcome potential barrier
• Probability distribution of exit points

\[ u(x) = \int_{\partial D} \phi(y) \rho(x, y) \, dy \]

• In our two point boundary value problem

\[ u = P_{-a} \alpha + P_b \beta \]

\( P_{-a}, P_b \) probabilities to exit at \(-a, b\)

• Use MAE to find uniform asymptotic solution

  – Reduced problem \((\epsilon = 0)\)
  – Cannot satisfy both boundary conditions, boundary layer(s) necessary

\[ u \sim c_0 + (\alpha - c_0) e^{-a\xi} + (\beta - c_0) e^{-b\eta} \]

\[ \xi = \frac{x + a}{\epsilon}, \quad \eta = \frac{b - x}{\epsilon} \]

• But what is \(c_0\)?
• Uniform expansion consists of outer + BL
  – Outer $O(1)$
  – Boundary layer goes from $O(1)$ to exponentially small

• Appropriate to ask:
  enough functions to represent solution?
  i.e., enough to span the solution space?

• If not, need to add more functions

• All MAE conditions employed, no answer

• No help from h.o.t.

• Though solution is unique
  asymptotic solution not unique

• 1 parameter family of possible
  asymptotic solutions,
  some called “Spurious solutions”

• Some declared ”Failure Of MAE”

• Goal – Rescue (modify or augment)
We Present Intuitive Argument and 4 Mathematical Arguments To Rescue MAE
Intuitive Argument

• Exit path should be shortest to exit point
  – 1 exit point, probability 1
  – N exit points, probability \( \frac{1}{N} \)

• Thus,

\[
a < b, \quad c_0 = \alpha, P_{-a} = 1, P_b = 0
\]

(No left boundary layer)

\[
b < a, \quad c_0 = \beta, P_{-a} = 0, P_b = 1
\]

(No right boundary layer)

\[
a = b, \quad c_0 = \frac{\alpha + \beta}{2}, \quad P_{-a} = P_b = \frac{1}{2}
\]

(2 boundary layers)

• However, intuition is not conclusive

• We next present 4 different mathematical arguments to show these results are correct
(I): Modify (Matkowsky 1975)

- Replace standard MAE boundary layers
  \[(\alpha - c_0) e^{-\frac{-a(x+a)}{\epsilon}}, \quad (\beta - c_0)e^{-\frac{-b(b-x)}{\epsilon}}\]
  by JWKB boundary layer function
  \[A(x)e^{\frac{\phi(x)}{\epsilon}}\]

- \(\phi\) satisfies Eikonal equation
  \[(\phi')^2 + x\phi' = 0\]

- \(A\) satisfies transport equation
  \[xA' + A = 0\]

- Two solutions
  \[\phi' = 0, \quad \text{(outer)}\]
  \[\phi' = -x\]

  so \(\phi = \frac{K^2 - x^2}{2}\)

  - \(\phi\) quadratic - not linear
  - Want \(\phi \geq 0, \phi = 0\) at boundaries
  - Choose \(K = \max(a, b)\)
• $A(x) = \frac{a_0}{x}$, $a_0$ constant

• Note:

$$\phi \rightarrow a(x + a) \text{ as } x \rightarrow -a,$$
$$\phi \rightarrow b(b - x) \text{ as } x \rightarrow b$$

reduces to standard MAE construction

• Note: single boundary layer function describes multiple boundary layers

• Note: apparent singularity gone, no effect outside boundary layers & using Friedrichs mollifier

• Results same as intuitive argument
(II): Augment (Grasman, Matkowsky 1977)

- Introduce variational problem whose Euler Lagrange equation is given DE
- Use MAE family as admissible functions
- Set first variation to zero
- Same result as intuitive & (I)
(III): Augment (Matkowsky, Schuss 1977)

- Replace variational condition by orthogonality condition
- $(p^s, Lu) = 0$, $(f, g) = \int_{-a}^{b} fg \, dx$

- Stationary Kolmogorov forward equation
  
  $$L^* p^s = 0$$

  so $p^s = Ce^{-\frac{x^2}{2\epsilon}}$, $C$ normalization constant

- Variational condition (Ritz)
- Orthogonality condition (Galerkin)
- Same result as intuitive, (I), (II)
Asymptotics beyond all orders, aka exponential asymptotics

Reason: unable to determine $c_0$. Not enough terms in outer expansion to span solution space

Add exponentially small terms to outer expansion (construct by JWKB)

$$c_0 = \frac{a\alpha e^{\frac{-a^2}{2\epsilon}} + b\beta e^{\frac{-b^2}{2\epsilon}}}{a e^{\frac{-a^2}{2\epsilon}} + b e^{\frac{-b^2}{2\epsilon}}}.$$

Consider the 3 cases

1. $a < b \rightarrow c_0 = \alpha$
2. $a > b \rightarrow c_0 = \beta$
3. $a = b \rightarrow c_0 = \frac{\alpha + \beta}{2}$

Same result as intuitive & (I) (II) (III)
• Again apparent singularity gone as before
  – No effect on interior
  – Only important to match boundary layer
• Solution should depend continuously on data
  – Does not: discontinuous at $a = b$
• Reason: only considered $b - a = O(1)$
• Now consider $b - a = \epsilon d$

$$c_0 = \frac{\alpha + e^{-ad}}{1 + e^{-ad}}$$

$$d \to \infty, \quad c_0 \to \alpha, \quad d \to -\infty, \quad c_0 \to \beta,$$

$$d = 0, \quad c_0 = \frac{\alpha + \beta}{2}$$

• Solution depends continuously on data
  & bridges gap between results
• Result indicates exit doesn’t occur
  at isolated value $c_0 = \frac{\alpha + \beta}{2}$,
  but in a thin layer about that value
KRAMERS MODEL OF CHEMICAL REACTION RATES
BROWNIAN PARTICLE IN FIELD OF FORCE

1940 KRAMERS: FIELD = POTENTIAL WELL

REACTION OCCURS WHEN PARTICLE OVERCOMES POTENTIAL BARRIER
E = HT. OF WELL, $\kappa$ = RATE

$\kappa = \frac{1}{2\tau}$
$\tau$ is MFPT

FACTOR 1/2; AFTER REACHING RIM EQUALLY LIKELY TO EXIT-RETURN

ARRHENIUS LAW
$\kappa = Ae^{-\frac{E}{kT}}$
E=V(b)-V(a)=ACTIVATION ENERGY, T=TEMP A=PREEXPOSITIONAL FACTOR

MERELY STATES $\tau$ is $O\left(\epsilon^{-\frac{1}{\epsilon}}\right)$ $\epsilon = \frac{kT}{E}$
\[ \kappa \text{ INCREASES WITH } T \]
MILK SOURS FASTER IN ROOM THAN FRIDGE

SIMILAR APPLICATIONS IN:
ATOMIC MIGRATION IN CRYSTALS
IONIC CONDUCTIVITY IN CRYSTALS
TRANSITIONS BET. EQUIL. STATES
IN JOSEPHSON JUNCTIONS
TO NAME BUT A FEW

\[ \kappa \text{ EMPLOYED IN} \]

\[ \frac{dC}{dt} = -\kappa C \]

C = CONCENTRATION OF
REACTION COMPONENT
DIFFUSION APPROXIMATION IN NEUTRON TRANSPORT THEORY

NOW CONSIDER SP TO DESCRIBE LARGE TIME APPROXIMATION OF MICROSCOPIC MODEL OF NEUTRON TRANSPORT BY MACROSCOPIC MODEL - DIFFUSION EQ NO SURPRISE: COLLISIONS = DIFFUSION

NEUTRON TRANSPORT THEORY STUDIES NEUTRON POPULATIONS NEUTRONS MAY COLLIDE, ANNIHILATED NEW NEUTRONS BORN BY FISSION IMPT IN NUCLEAR REACTOR DESIGN

MICROSCOPIC MODEL IS LINEAR INTEGRODIFFERENTIAL EQ (LBE) FEW SOLUTIONS AVAILABLE DESIRE SIMPLER MODEL AMENABLE TO ANALYSIS
DIFFUSION APPROXIMATION

PREVIOUS ATTEMPTS UNSATISFACTORY
WE USE SP FOR DIFFUSION APPRX.

NUCLEAR AGE BEGAN WITH
1932: CHADWICK DISCOVERED NEUTRONS
1939: HAHN, MEITNER -FISSION
1942: FERMI et. al. - NUCLEAR REACTOR

MANY NEUTRONS, $O(10^7)$ - CONTINUUM
FAR MORE NUCLEI $O(10^{23})$ IN MEDIUM,
ONLY NEUTRON-NUCLEAR INTERACTION
LINEAR INTEGRODIFFERENTIAL EQ (LBE)

COLLISIONS CHANGE DIRECTION
$\mu \rightarrow \mu'$
$\mu = \cos \theta$
FISSION, LARGE ENERGY RELEASED
ENERGY USED GENERATE ELECTRICITY
LINEAR BOLTZMANN EQ (LBE)
\[
\frac{1}{v} \Psi_{\tau} + \mu \Psi_x + \sigma(x) \Psi - \frac{\sigma(x)c(x)}{2} \int_{-1}^{1} \Psi(x, \mu', \tau) d\mu' = 0,
\]

BOUNDARY CONDITIONS

\[
\Psi(x = 0) = f_1(\mu, \tau) \quad \text{for} \quad \mu > 0
\]
\[
\Psi(x = d) = f_2(\mu, \tau) \quad \text{for} \quad \mu < 0,
\]

+ INITIAL CONDITION.

\(\Psi(x, \mu, \tau)\) IS NEUTRON DISTRIBUTION FCTN IN SLAB GEOMETRY,
PROBABLE NUMBER NEUTRONS AT \(x, \tau\), TRAVELING AT CONST SPEED \(v\)
IN DIRECTION \(\mu = \cos \theta\),
\(\theta\) IS ANGLE \(v\) MAKES WITH HORIZONTAL
\(\sigma(x)\) IS SCATTERING CROSS SECTION,
PROB. THAT NEUTRON INCIDENT ON NUCLEUS RESULTS IN SCATTERING EVENT
(DIRECTION CHANGES \(\mu'\) to \(\mu\))
AVG. INVERSELY PROPORTIONAL TO MEAN FREE PATH \(l\),
AVG. DIST. TRAVELED BET. COLLISIONS.
\(c(x)\) AVG. # SECONDARY NEUTRONS BORN
\[c = 1 \text{ (critical) NEUTRON POPULATION JUST SUSTAINED}\]
\[c > 1 \text{ (} c < 1 \text{) SUPERCRITICAL (SUBCRITICAL), NEUTRON POPULATION GROWS (DECAYS) TO CONTROL NEUTRON GROWTH (SAFETY) CONTROL RODS INSERTED (ABSORB NEUTRONS).}\]
\[v \text{ MICROSCOPIC VELOCITY}\]

COMPLICATIONS
1/2 BC AT EACH BDRY PRESCRIBE INCOMING, NOT OUTGOING CONTINUOUS SPECTRUM FEW SOLUTIONS KNOWN
DESIRE SIMPLER MODELS
FOR REACTOR DESIGN
MORE AMENABLE TO ANALYSIS
DIFFUSION ”APPROXIMATION”
WIDELY USED

PREVIOUS ATTEMPTS
$P_1$ DIFF., ASY DIFF.
$P_1$ DIFFUSION
EXPAND IN LEGENDRE POLYNOMIALS $P_n(\mu)$
TRUNCATE AFTER $N$ TERMS; $P_N$ APPROX
IF $N$ SUFF LARGE, CONVERGENCE
TRUNCATE AFTER 2 TERMS
GET DIFFUSION ”APPROX”
Q: WHY ”APPROX” VALID?

ASY DIFFUSION
REPLACE FINITE BY INFINITE DOMAIN
REPLACE VARIABLE BY CONST COEFFS
CONSIDER SOLUTION AT INFINITY
GET DIFFUSION ”APPROX”
Q: WHY ”APPROX” VALID?
DIFFUSION Eqs have different coefficients close if $c \sim 1$ (near critical)

Different BCS postulated, not derived e.g., Marshak, Mark

To make sense as approximation must be able to answer 4 questions:
In what sense approx?
Conditions for validity?
How good is approx?
How provide corrections (improvement)?

We actually derive a diffusion approximation and answer these questions.
RATHER THAN STOCHASTIC APPROACH
WE EMPLOY SCALING
ELEMENTARY CALCULUS, SP (MAE)

NONDIMENSIONALIZATION
\[ y \equiv \frac{x}{d}, \quad t \equiv \frac{\bar{v}\tau}{d}, \]
\( \bar{v} \) REFERENCE MACROSCOPIC VELOCITY,
NONDIM. SCATTERING CROSS SECTION
\[ a(y) = \frac{\sigma}{\bar{\sigma}}, \]
\( \bar{\sigma} \) REF. SCATTERING CROSS SECTION.
INTRODUCES SMALL PARAMETERS
\[ \epsilon \equiv \frac{l}{d} \ll 1, \quad \delta \equiv \frac{\bar{v}}{v} \ll 1, \]
FORMER
MEAN FREE PATH \( l \ll \) TYPICAL MACROSCOPIC LENGTH,
e.g., SIZE OF REACTOR
LATTER
MACRO VELOCITY \( \ll \) MICRO VELOCITY.
\[ \Psi \rightarrow \psi \]

DEFINITIONS IMPLY
\[ t = \epsilon \delta \tau, \text{ LONG TIME SCALE.} \]
ASSUME
\[ \epsilon, \delta \text{ SAME ORDER} \]
SET \( \epsilon = \delta \), TO GET
\[
\epsilon^2 \psi_t + \epsilon \mu \psi_y + a(y)\psi - a(y)c(y, \epsilon) \int_{-1}^{1} \psi d\mu' = 0.
\]
EQUATION CLEARLY SP TYPE.
EXPAND BOTH \( \psi, c \) IN
ASYMPTOTIC SERIES IN \( \epsilon \)
\[
\psi \sim \sum_n \psi^n(y, t, \mu) \epsilon^n, \quad c \sim \sum_n c_n(y) \epsilon^n
\]
FOR THE OUTER EXPANSION TO BE
VALID IN INTERIOR OF DOMAIN
BCs OBTAINED BY MATCHING TO BL EXPANSION

EQUATING COEFF OF EACH POWER OF $\epsilon$ TO ZERO OBTAIN RECURSIVE EQs FOR COEFFs

\[ L\psi^0 \equiv a[\psi^0 - \frac{c_0}{2} \int_{-1}^{1} \psi^0 d\mu'] = 0, \]

LEARN $\psi^0$ IND. OF $\mu$, $\psi^0 = \psi^0(x, t)$, $c_0 = 1$

\[ L\psi^1 = -\psi^0_y + \frac{ac_1}{2} \int_{-1}^{1} \psi^0 d\mu' \]

LEARN $\psi^1$ LINEAR IN $\mu$, $c_1 = 0$

\[ L\psi^2 = -\psi^1_y + \frac{ac_2}{2} \int_{-1}^{1} \psi^0 d\mu' + \frac{ac_1}{2} \int_{-1}^{1} \psi^1 d\mu' - \psi^0_t. \]

LEARN $\psi^2$ QUADRATIC IN $\mu$

RELATIONS BET. COEFFS

COLLECTING RESULTS GET
DIFFUSION EQUATION

\[ \psi_t^0 = \left( \frac{1}{3a} \psi_y^0 \right)_y + ac_2 \psi^0. \]

NOTE ANGULAR DEPENDENCE (\(\mu\))
DERIVED, NOT ASSUMED AS IN
\(P_1\) DIFFUSION APPROXIMATION.
\(c_2 < 0\) (SUBCRITICAL)
SOLUTION DECAYS TO 0,
BOTH R.H.S. TERMS NEGATIVE
REACTION NOT SUSTAINED
\(c_2 > 0\) (SUPERCRITICAL)
REACTION CAN BE SUSTAINED

TO COMPLETE DERIVATION, MUST SOLVE
BL PROBLEM NEAR EACH BOUNDARY
THEN, MATCH BL TO OUTER EXPANSION,
TO GET BCs FOR DIFFUSION EQUATION.
WE DON’T CARRY THIS OUT,
INVOLVES TOO MUCH DETAIL
cf. HABETLER, MATKOWSKY PAPER
FOR BL ANALYSIS AND MATCHING
WE SUCCESSFULLY ANSWERED QUESTIONS
Q: IN WHAT SENSE IS APPROXIMATION APPROXIMATE?
A: AN ASYMPTOTIC APPROXIMATION.

Q: WHEN IS IT VALID?
A: WHEN \( \epsilon \) IS SMALL
i.e., WHEN MEAN FREE PATH \(<<\) THAN TYPICAL MACROSCOPIC LENGTH,
e.g., THE SIZE OF THE DOMAIN

Q: HOW GOOD IS THE APPROXIMATION?
A: ERROR IS \( O(\epsilon) \)

Q: HOW TO IMPROVE APPROXIMATION?
A: INCLUDE HIGHER ORDER TERMS.

NO INITIAL LAYER ANALYSIS NEEDED
MODEL NOT VALID FOR EARLY TIMES (STARTUP)
Dedication This lecture is dedicated to my teacher, role model, colleague and friend, Joe Keller, of blessed memory.

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