Multivariable Calculus Math 215

Spring 2003

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Midterm 2003, May 7

NAME:

Mean 77

Problem	Points	Max
1		10
2		20
3		20
4		25
5		25
total		100

Show all your work. For full credit it is not sufficient to give the correct answer; the write-up needs to show how the answer is obtained.

Guide to letter grades

A	$\geq$	92	14 students
A−	$\geq$	89	3
B+	$\geq$	85	2
B	$\geq$	80	0
B −	$\geq$	70	7
C+	$\geq$	62	1
C	$\geq$	57	4
C−	$\geq$	54	3
D	$\geq$	40	3
F	$\leq$	39	1

1. (10pts.) Evaluate

$$\int_0^1 \int_0^x \int_y^x xyz \, dz \, dy \, dx.$$

$$\int = \int_0^1 dx \int_0^x dy \; \frac{1}{2} z^2 \Big|_{z=y}^{x} = \int_0^1 dx \int_0^x dy \; \frac{1}{2}(x^2 - y^2) =$$

$$= \frac{1}{2}\int_0^1 dx \; \frac{1}{2} y x^2 - \frac{1}{6} y^3 \Big|_0^x = \frac{1}{2}\int_0^1 dx \; \frac{1}{2}x^3 - \frac{1}{6}x^3 =$$

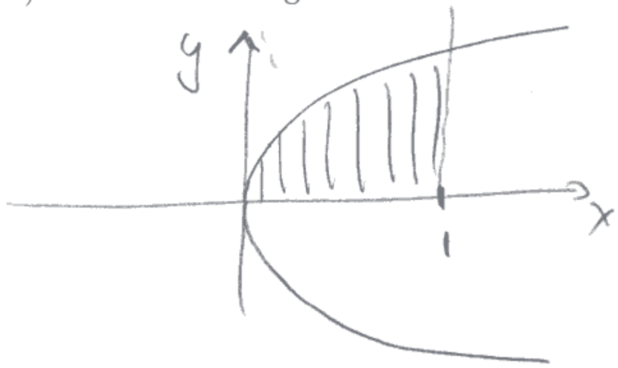
$$= \frac{1}{2}\,\frac{1}{3}\,\frac{1}{4}\,x^4 \Big|_0^1 = \frac{1}{24}.$$

2. (20 pts.) Consider the integral

$$\int_0^1 \int_{y^2}^1 \sin\left(\frac{\pi y}{\sqrt{x}}\right)\, dx\, dy.$$

a) Sketch its domain of integration.
b) Evaluate the integral.

a)



$x = y^2$

$y = \pm\sqrt{x}$

b) $\int_0^1 dy \int_{y^2}^1 dx \, \sin \frac{\pi y}{\sqrt{x}} = \int_0^1 dx \int_0^{\sqrt{x}} dy \, \sin \frac{\pi y}{\sqrt{x}} =$

change order of integration

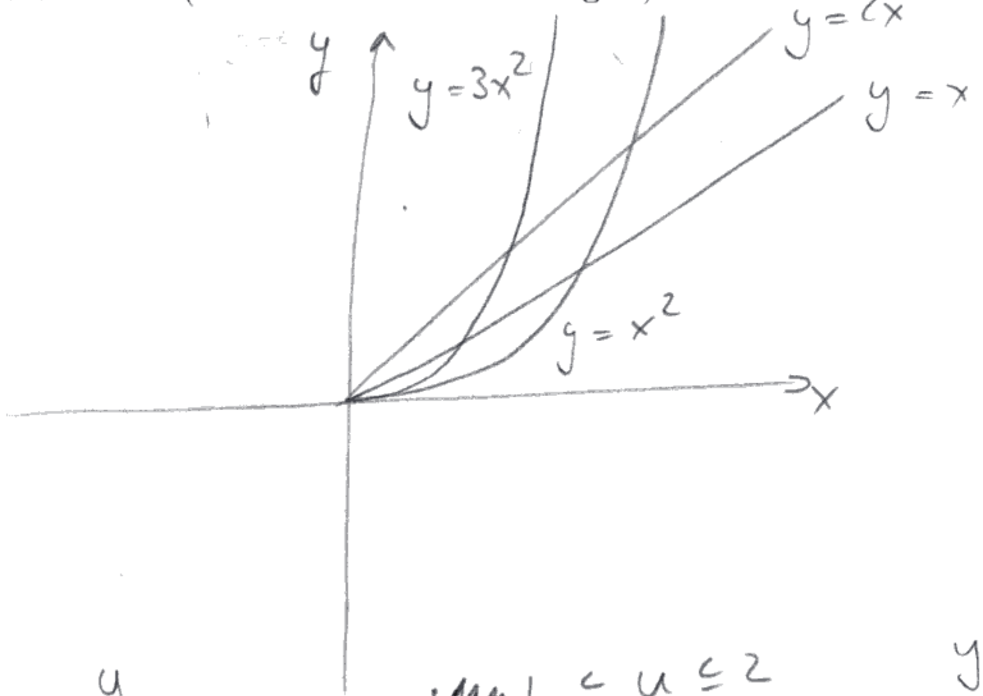
$= \int_0^1 dx \left(- \frac{\sqrt{x}}{\pi} \cos \frac{\pi y}{\sqrt{x}} \right) \Big|_{y=0}^{\sqrt{x}} = -\frac{1}{\pi} \int_0^1 dx \, {+\sqrt{x}} \, \underbrace{\left[\cos \pi - \cos 0 \right]}_{-2} =$

$= +\frac{2}{\pi} \, \frac{2}{3} x^{\frac{3}{2}} \Big|_0^1 = \frac{4}{3\pi}.$

3. (20 pts.) A domain is bounded by the curves

$$y = x, \quad y = 2x, \quad y = x^2, \quad y = 3x^2.$$

Use a coordinate transformation $u = u(x, y)$, $v = v(x, y)$ to transform the domain to a rectangle in the (u, v)-plane and write down an integral in u and v for the area of the domain. (**Do not evaluate the integral**).



$u = \frac{y}{x} \Rightarrow 1 \le u \le 2$

$v = \frac{y}{x^2} \Rightarrow 1 \le v \le 3$

$y = ux = vx^2$

$\Rightarrow x = \frac{u}{v}$

$y = \frac{u^2}{v}$

$$A = \iint_D dx\,dy = \iint_D \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

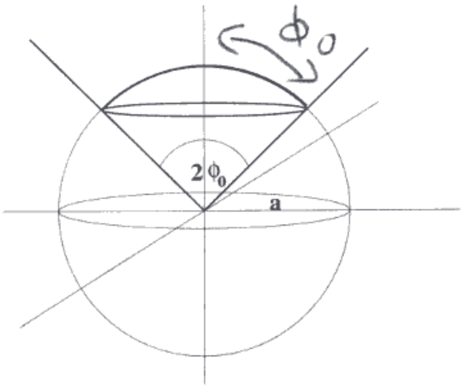
$$\frac{\partial(x,y)}{\partial(u,v)} = -\frac{1}{v}\frac{u^2}{v^2} - \left[-\frac{u}{v^2}\cdot\frac{2u}{v}\right] = \frac{u^2}{v^3}$$

$$A = \int_1^2 du \int_1^3 dv \, \frac{u^2}{v^3}.$$

4. (25 pts.) Calculate the mass of a solid body that is bounded above by a sphere with radius a and center at the origin ($(x, y, z) = 0$) and below by a cone with opening angle $2\phi_0$, as sketched below. The density ρ of the body is given by

$$\rho(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}}.$$

Do not use cartesian coordinates for the integration.



$$M = \iiint \rho(r)\, dV$$

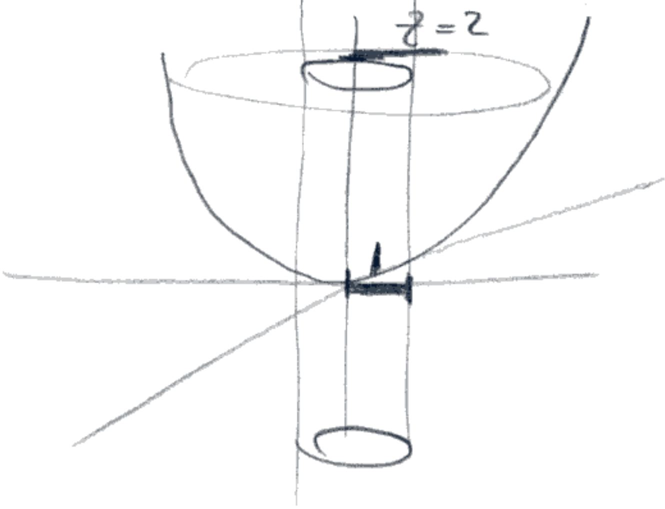
Use spherical coordinates: $\rho(\underline{r}) = \frac{1}{R}$

$$M = \int_0^{2\pi}\int_0^{\phi_0}\int_0^{a} \frac{1}{R} R^2 dR \sin\phi \, d\phi \, d\theta =$$

$$= 2\pi \int_0^a R \, dR \int_0^{\phi_0} \sin\phi \, d\phi = 2\pi \, \frac{1}{2} a^2 \cdot (-\cos\phi)\Big|_0^{\phi_0} =$$

$$= \pi a^2 \left[-\cos\phi_0 + 1\right] = \pi a^2 (1 - \cos\phi_0)$$

5. (25 pts.) The paraboloid $z = (x^2 + y^2)/2$ is intersected by a cylinder $x^2 + y^2 = 1$ and a plane $z = 2$. Calculate the surface area of the part of the paraboloid outside the cylinder and below the plane.
Do not use cartesian coordinates for the integration.



Paraboloid:

$\underline{r} = (x, y, \varphi(x,y)) =$

$= (x, y, \frac{1}{2}(x^2+y^2))$

$$S = \iint_D \sqrt{1 + \left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2}\, dx\, dy$$

$$= \iint_D \sqrt{1 + x^2 + y^2}\, dx\, dy =$$

polar coordinates

$$= \iint \sqrt{1 + r^2}\, r\, dr\, d\theta$$

inner radius given by cylinder: $r = 1$

outer radius given by plane: $z = 2 \Leftrightarrow 2 = \frac{1}{2}(x^2+y^2)$

$\Rightarrow r = 2$

$$S = \int_0^{2\pi} \int_1^2 \sqrt{1+r^2}\, r\, dr\, d\theta = \frac{2\pi \, 2}{2 \cdot 3} (1+r^2)^{3/2} \Big|_1^2 =$$

$$= \frac{2\pi}{3} \left[1 + 4 - 1 - 1\right] = 2\pi.$$