Honors Calculus 252-2

Winter 2002

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Problem Set 3

to be handed in Friday January 25, 2002, in class

1. (a) Consider a triangle with sides given by vectors $\vec{A}$, $\vec{B}$ and $\vec{B}-\vec{A}$. Show that the three medians intersect at the point given by $(\vec{A}+\vec{B})/3$. This point is called the *centroid.*

   (b) The *center of mass* of a region $\mathcal{R}$ with density $\rho(x,y)$ has coordinates $(\bar{x},\bar{y})$ where

$$\bar{x}=\frac{1}{M}\int\int_{\mathcal{R}} x\rho(x,y)\;dx\,dy \qquad \bar{y}=\frac{1}{M}\int\int_{\mathcal{R}} y\rho(x,y)\;dx\,dy \qquad M=\int\int_{\mathcal{R}} \rho(x,y)\;dx\,dy$$

   Compute the center of mass of the triangle with vertices $(0,0),(h,0),(a,b)$ (h and b are positive) and unit density, $\rho(x,y)=1$. Show that the center of mass and the centroid are the same (Hint: Use horizontal slices).

2. Compute by definition the Jacobian

$$\frac{\partial(R,\phi,\theta)}{\partial(x,y,z)}$$

   where (R,ϕ,θ) are spherical coordinates. Show that the answer is the reciprocal of

$$\frac{\partial(x,y,z)}{\partial(R,\phi,\theta)}=R^2\sin\phi.$$

3. The gravitational attraction, $\vec{f}$, of a volume $\mathcal{V}$ with density $\rho(x,y,z)$ on a mass M at the origin is given by (G is the gravitational constant)

$$\vec{f}=GM\int\int\int_{\mathcal{V}}\frac{\rho(x,y,z)\hat{R}}{R^2}dxdydz$$

   where

$$R^2=x^2+y^2+z^2 \qquad \hat{R}=\frac{\vec{R}}{R}$$

   (a) Using spherical coordinates compute the gravitational attraction of an inverted cone with vertex at the origin and base in the plane $z=b$ with radius a centered on the z-axis. Assume a uniform density $\rho(x,y,z)=\rho_0$. Use the symmetry of the cone to conclude that two of the three cartesion components of the force vanish. Write down integral expressions for all three components and identify the integration variable the integration over which makes those two components vanish. Compute the third component of the force.

   (b) Repeat the calculation in cylindrical coordinates. Make sure your answers agree.

4. In cylindrical coordinates (r,θ,z), a torus has equation

$$(r-a)^2+z^2=b^2\ .$$

   (a) Write and evaluate an integral for the volume of the torus in cylindrical coordinates.

(b) Change the coordinate system from cylindrical coordinates (r, θ, z) to toroidal coordinates (ρ, θ, ϕ) where

$$r = a + \rho\cos(\phi) \qquad \theta = \theta \qquad z = \rho\sin(\phi)\ .$$

Write and evaluate an integral for the volume of the torus in toroidal coordinates.

5. 7.3.24, 7.3.25

6. Compute the integral

$$\int\int_S z dS$$

Where S is the surface $z = xy$ in the region $0 < x < 1$, $0 < y < 1$.

7. Compute the area of the portion of the surface $z = x^2 - y^2$ above the unit circle $x^2 + y^2 \leq 1$.