

Review Sheet for Final

TR 11

1) $y'' + \lambda y = f(x)$ $y(0) = y'(0)$ $y(1) = y'(1)$

a) $y'' + \lambda y = 0$

$$\begin{cases} \lambda < 0 : \\ \lambda = -\mu^2 : \quad y = c_1 e^{\mu x} + c_2 e^{-\mu x} \end{cases}$$

$$\text{b.c. } x=0 : \quad c_1 + c_2 = \mu c_1 - \mu c_2 \Rightarrow c_2 = \frac{\mu-1}{\mu+1} c_1$$

$$x=1 : \quad c_1 e^\mu + c_2 e^{-\mu} = c_1 \mu e^\mu - c_2 \mu e^{-\mu}$$

$$\begin{aligned} c_1 (e^\mu - \mu e^{-\mu}) &= -c_2 (\mu e^{-\mu} + e^\mu) = \\ &= -\frac{(\mu+1)(\mu-1)}{\mu+1} e^{-\mu} c_1 \end{aligned}$$

$$c_1 (1-\mu) e^\mu = c_1 (1+\mu) e^{-\mu}$$

$$\Rightarrow c_1 = 0 \quad \text{or} \quad \mu = 1 \quad \text{or} \quad \mu = 0$$

\downarrow \downarrow \downarrow
 only trivial ok 2nd w/able
 solution ok ok

$$y_0 = e^x \quad \text{with} \quad \mu_0 = 1 \quad \lambda_0 = -1$$

$\lambda = 0 \Rightarrow y = c_3 x + c_4 \Rightarrow c_3 = 0 = c_4$

$\lambda > 0 \Rightarrow \lambda = \mu^2 : \quad y = c_5 e^{\mu x} + c_6 \sin \mu x$

$$x=0 : \quad c_5 = c_6 \mu$$

$$x=1 : \quad c_5 \cos \mu + c_6 \sin \mu = -c_5 \mu \sin \mu + c_6 \mu \cos \mu$$

$$c_6 \mu \cos \mu + c_6 \sin \mu = -c_6 \mu \sin \mu + c_6 \mu \cos \mu$$

$$\sin \mu = -\mu^2 \cos \mu$$

$$-\sin \mu = 0 \quad \mu = n\pi, \quad n=1, 2, 3, \dots$$

$$y_n = n\pi \cos nx + \sin nx$$

b) $f(x) = 5 \cos 6x + \sin 6x$

Fredholm alternative theorem: need y_H

depending on sign of λ we have already
the solutions y_h in a)

i) $\lambda < 0$: $y_h = c_1 e^{\mu x} + c_2 e^{-\mu x}$

ii) $\lambda = 0$: $y_h = c_3 x + c_4$

iii) $\lambda > 0$: $y_h = c_5 \cos \mu x + c_6 \sin \mu x$

for y_H we need to satisfy b.c.

i) for $x < 0$ this is only possible if $\lambda = 1$

$$y_H = e^x$$

\rightarrow we need to look at $\int_0^1 e^x f(x) dx$

$$\int_0^1 e^x (5 \cos 5x + \sin 5x) dx = e^x \sin 5x \Big|_0^1 = e \sin 5$$

\Rightarrow for $\delta \neq 0$ no solution exists

ii) $\lambda = 0$: $y_H = 0 \Rightarrow$ solution is unique

iii) $\lambda > 0$: from a) we know that $y_H = 0$ unless

λ is an eigenvalue

i.e. for $\lambda \neq (n\pi)^2$ $y_H = 0$

a solution unique

for $\lambda = (n\pi)^2$ we have $y_H = \sqrt{c} n\pi x + \sin n\pi x$

Unfortunately, there is a bug in the assignment (I made an algebra error when I did it first). The condition on c should read $c = m\pi$, m integer.

In that case we have $f(x) = y_m$, i.e. it is an eigenfunction of the problem in a)

Thus: for $\lambda = (n\pi)^2$ we have $y_H = y_n$, i.e. y_H is an eigenfunction
 and $\int_0^1 f(x) y_H dx = \int_0^1 y_m(x) y_n(x) dx$ since it satisfies
 b.c.

The differential equation (1) is already in the form of a Sturm-Liouville eigenvalue problem with $P=1$, $q=0$ and $\tau=1$.

\Rightarrow its eigenfunctions are orthogonal

$$\int y_n y_m \tau dx = 0 \text{ for } n \neq m$$

$$\Rightarrow \int f(x) y_H(x) dx = 0 \text{ if } n \neq m$$

in that case there are infinitely many
solutions

$$\text{if } n = m \quad \int f(x) y_H(x) dx = \int [y_n(x)]^2 dx > 0$$

\Rightarrow there is no solution to this boundary
value problem.

Review sheet for Final

TR, 2.1

2) Legendre's equation

$$(1-x^2)y'' - 2xy' + p(p+1)y = 0$$

with p non-negative

In Frobenius:

$$x^2(1-x^2)y'' - 2x^3y' + p(p+1)x^2y = 0$$

$$R_0 = 1 \quad R_1 = 0 \quad R_2 = -1$$

$$P_0 = 0 \quad P_1 = 0 \quad P_2 = -2$$

$$Q_0 = 0 \quad Q_1 = 0 \quad Q_2 = p(p+1)$$

$$\text{indicial equation: } 6(\sigma-1) + 0 \cdot 6 + 0 = 0$$

$$\Rightarrow \sigma_+ = 1 \quad \sigma_0 = 0 \quad \text{differ by } \sigma_+-\sigma_0 = 1$$

\Rightarrow need to check equation for A_n to see whether y is a power series or contains in addition a term $y_1 \cdot \ln x$

Recursion relation

$$n[n+2\sigma_{\pm} - \sigma_+ - \sigma_-] A_n = - \sum_{m=1}^n [R_m(n-m+\sigma_{\pm})(n-m+\sigma_{\pm}) + P_m(n-m+\sigma_{\pm}) + Q_m] A_{n-m}$$

$$= -[-1(n-2+\sigma_{\pm})(n-2+\sigma_{\pm}-1) + 2(n-2+\sigma_{\pm}) + p(p+1)] A_{n-2}$$

check for $n=1$: r.h.s. vanishes since it involves only R_1, P_1, Q_1

for
 $n \geq 2$

$\Rightarrow y$ has power series form

$$\Delta \sigma_{\pm} = 1: n(n+1) A_n = [(n-1)(n-2) + 2(n-1) - p(p+1)] A_{n-2}$$

$$= [(n-1)n - p(p+1)] A_{n-2}$$

$$A_n = \frac{(n-1)n - p(p+1)}{n(n+1)} A_{n-2}$$

if p is a non-negative integer:

$$(n-1)n - p(p+1) = 0 \quad \text{for } n = p+1$$

$$\Rightarrow A_{p+1} = 0 \Rightarrow A_{p+3} = 0 \text{ etc.}$$

Second root: $n^2 - n - p(p+1) = 0$

$$n = \frac{1 \pm \sqrt{1 + 4p(p+1)}}{2} = \frac{1 \pm \sqrt{(1+2p)^2}}{2}$$

$$= \frac{1 \pm (1+2p)}{2} \quad n_1 = p+1$$

$$n_2 = -p; \text{ not relevant}$$

- for p odd, $p+1$ even: $A_{p+1} = 0$

$$g^+ = x \left[A_0^+ + A_2^+ x^2 + \dots + A_{p-1}^+ x^{p-1} + 0 \right]$$

- for p even the series does not stop $\Rightarrow g^-$ is a true power series

▲ For $g^- = 0$: $n(n-1) A_n = -[-(n-2)(n-3) - 2(n-2) + p(p+1)] A_{n-2}$
 $= +[(n-2)(n-1) + -p(p+1)] A_{n-2}$

for $n-2=p \Rightarrow$ we get $A_{p+2}=0$

- for p even $p+2$ even

$$g^- = 1 \cdot [A_0^- + A_2^- x^2 + \dots + A_p^- x^p + 0]$$

Thus: ... for even and odd p we get one solution that is a polynomial of finite order rather than an infinite series

- the other solution is an infinite series.

Note: we can show that the polynomial solution can be written as

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$$

Rodrigues' formula.

$$P_0 = 1 \quad P_1 = x \quad P_2 = \frac{1}{2} (3x^2 - 1) \quad P_3 = \frac{1}{2} (5x^3 - 3x)$$

Ques: the Legendre polynomials form an orthogonal set:

$$\int_{-1}^1 P_m(x) P_n(x) dx = \frac{2}{2n+1} \delta_{mn} \Rightarrow \begin{cases} 0 & m \neq n \\ \frac{2}{2n+1} & m = n \end{cases}$$

Kronecker delta.

Review for Final

FR 3.1

③ $x^2y'' - y' + 4x^3y = x^3 \quad 0 < x < \infty$
 $y(0) = 0 = y(\infty)$

Does the solution for this boundary-value problem exist?

general solution: $y = y_h + y_p$

homogeneous solution:

Compare with generalized Bessel equation:

$a = -1 \quad b = 0 \quad c = 0 \quad d = 4 \quad s = 2$

since one needs to write the equation in the form

$$x^2y'' - xy' + 4x^4y = x^4$$

$$\Rightarrow \hat{p} = \frac{1}{2}\sqrt{\left(\frac{2}{\epsilon}\right)^2 - 0} = \frac{1}{2} \text{ not an integer}$$

$$d > 0$$

$$\Rightarrow y_h = x \left[c_1 J_{\frac{1}{2}}(x^2) + c_2 J_{-\frac{1}{2}}(x^2) \right]$$

simplify since half-integer index:

$$= x \left[c_1 \sqrt{\frac{2}{\pi x^2}} \sin x^2 + c_2 \sqrt{\frac{2}{\pi x^2}} \cos x^2 \right]$$

$$y_h = \hat{c}_1 \sin x^2 + \hat{c}_2 \cos x^2$$

One is tempted to use the Fredholm alternative theorem, but it requires that the differential equation satisfy $p(x) > 0$ continuous on $[a, b]$ when written in Sturm-Liouville form

$$\frac{d}{dx} \left(p \frac{dy}{dx} \right) + qy = f(y)$$

here we have

$$x^2 \frac{d}{dx} \left(\frac{1}{x} \frac{dy}{dx} \right) + 4x^2 y = x^3$$

$$\text{i.e. } \frac{d}{dx} \left(\frac{1}{x} \frac{dy}{dx} \right) + 4xy = x$$

with $p = \frac{1}{x}$ not ∞ continuous at $x=a=0$

Therefore we need to get an explicit particular solution and then try to satisfy the boundary conditions.

$$\text{by inspection: } y_p = \frac{1}{4}$$

alternatively via variation of parameters

$$W = -2x$$

$$y_p = -\sin x \int \frac{x^3 \cos x^2}{-2x} dx + Cx^2 \int \frac{x^3 \sin x^2}{-2x} dx$$

or via undetermined coefficients

$$y_p = ax^3 + bx^2 + cx + d$$

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$$x(6ax + 2b) = 3ax^2 - 2bx - c + 4x^3 (ax^2 + bx^2 + cx + d) \stackrel{?}{=} x^3$$

Collect powers in x :

$$x^6: \quad a = 0$$

$$x^5: \quad b = 0$$

$$x^4: \quad c = 0$$

$$x^3: \quad 4d = 1 \Rightarrow d = \frac{1}{4}$$

$$x^2: \quad 6a - 3a = 0 \quad \checkmark \text{ since } a \geq 0$$

$$x: \quad 2b - 2b = 0$$

$$x^0: \quad -c = 0 \quad \checkmark \text{ since } c \geq 0 \text{ already.}$$

thus we have

$$y_1 = \hat{c}_1 \sin x^2 + \hat{c}_2 (\cos x^2 + \frac{1}{4})$$

$$y(0) = 0 \Rightarrow \hat{c}_2 + \frac{1}{4} = 0 \Rightarrow \hat{c}_2 = -\frac{1}{4}$$

$$y(\pi) = 0 \Rightarrow \hat{c}_2 \cos \pi + \frac{1}{4} = 0 \Rightarrow \hat{c}_2 = +\frac{1}{4}$$

Cannot satisfy boundary conditions!

If Fredholm was applicable in this case (which it is not)
 we would have done the following steps:

get y_H :

$$\text{b.o.c. : } x = 0 \Rightarrow C_2 = 0$$

$$x = \sqrt{\pi} \Rightarrow C_1 \text{ is arbitrary}$$

$$\Rightarrow y_H = \sin x^2$$

$$\int_a^b f(x) y_H dx = \int_0^{\sqrt{\pi}} x^4 \underbrace{\sin x^2}_{>0} dx \neq 0$$

for $0 < x < \sqrt{\pi}$

if the theorem could be used it would predict
 that no solution exists (which is correct).

4) $f(x) = \begin{cases} 0 & 0 < x < \frac{\pi}{2} \\ x & \frac{\pi}{2} \leq x < \pi \end{cases}$

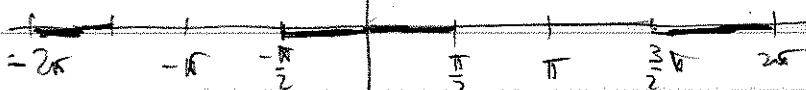
a) Cosine series with period 2π :

$$a_0 = \frac{1}{\pi} \int_0^{\pi} f(x) dx = \frac{1}{\pi} \int_{\frac{\pi}{2}}^{\pi} x dx = \frac{1}{2\pi} x \Big|_{\frac{\pi}{2}}^{\pi} = \frac{1}{2\pi} \left(\pi^2 - \frac{1}{4}\pi^2 \right) = \frac{3}{8}\pi$$

$$\begin{aligned} n \neq 0 \quad a_n &= \frac{2}{\pi} \int_0^{\pi} (\cos nx) \cdot x dx = \frac{2}{\pi} \left[\frac{1}{n} \sin nx \cdot x \right]_0^{\pi} - \\ &- \int_0^{\pi} \frac{1}{n} \sin nx dx = \frac{2}{\pi} \left[\frac{1}{n^2} \cosh nx \right]_0^{\pi} = \frac{2}{\pi} \frac{1}{n^2} [(-1)^n - 1] \end{aligned}$$

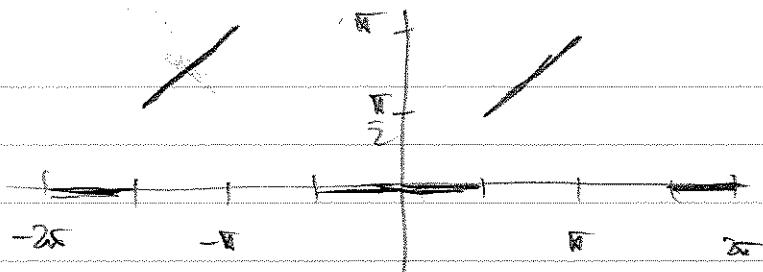
b) Sine series:

$$\begin{aligned} b_n &= \frac{2}{\pi} \int_0^{\pi} \sin nx \cdot x dx = \frac{2}{\pi} \left[-\frac{1}{n} \cos nx \cdot x \right]_0^{\pi} + \frac{1}{n} \int_0^{\pi} \cos nx dx \\ &= -\frac{\pi}{n} \cos n\pi + \frac{1}{n} \int_0^{\pi} \sin nx dx = -\frac{\pi}{n} (-1)^n \end{aligned}$$



Cosine series

Fr 4/2



Sine series

$$f^{(1)}\left(\frac{\pi}{2}\right) = \frac{1}{2} [f\left(\frac{\pi}{2}-0\right) + f\left(\frac{\pi}{2}+0\right)] = \frac{\pi}{4}$$

\uparrow
 $x = \frac{\pi}{2}$
discontinuity

$$f^{(0)}\left(\frac{\pi}{2}\right) = \frac{\pi}{4} \text{ analogously}$$

$$f^{(2)}(\pi) = \frac{1}{2} (f(\pi-0) + f(-\pi+0)) = 0$$

$$f^{(0)}(\pi) = \frac{1}{2} (f(\pi-0) + f(-\pi+0)) = \pi$$

Review Sheet for Final

FR 5.1

$$5) xy'' - 2y' + \frac{5}{4x}y = x \quad 1 < x < 4$$

$$y(1) = -\frac{2}{3}, \quad y(4) = 10$$

inhomogeneous differential equation

\Rightarrow find homogeneous solution

$$xy'' - 2y' + \frac{5}{4x}y = 0$$

is equidimensional, write as

$$x^2y'' - 2xy' + \frac{5}{4}y = 0$$

$$\Rightarrow \text{ansatz } y = x^m$$

(usually no need to do variable transformation)

$$m(m-1) - 2m + \frac{5}{4} = 0$$

$$m^2 - 3m + \frac{5}{4} = 0 \quad m = \frac{3 \pm \sqrt{9-5}}{2} =$$

$$m_1 = \frac{5}{2} \quad m_2 = \frac{1}{2}$$

two different roots \Rightarrow no $\ln x$ needed!

$$y_h = c_1 x^{\frac{5}{2}} + c_2 x^{\frac{1}{2}}$$

inhomogeneous: look for particular solution

$$x^2y'' - 2xy' + \frac{5}{4}y = x^2$$

as undetermined coefficients: $y_p = ax^2 + bx + c$

try first without $bx + c$:

$$x^2 \cdot 2a - 2x \cdot 2ax + \frac{5}{4}ax^2 = x^2$$

$$\Rightarrow 2a - 4a + \frac{5}{4}a = 1$$

$$-\frac{3}{4}a = 1 \quad a = -\frac{4}{3}$$

$$y_g = c_1 x^{\frac{5}{2}} + c_2 x^{\frac{1}{2}} - \frac{4}{3}x^2$$

need to satisfy b.c.

$$x=1: \quad c_1 + c_2 - \frac{4}{3} = -\frac{2}{3}$$

$$c_1 + c_2 = +\frac{2}{3} \Rightarrow c_1 = \frac{2}{3} - c_2$$

$$x=4: \quad c_1 32 + c_2 2 - \frac{4}{3} \cdot 16 = 10$$

$$\frac{2}{3} \cdot 32 - 32c_2 + 2c_2 - \frac{64}{3} = 10$$

$$c_2 = -\frac{1}{3} \quad c_1 = 1$$

$$y = x^{\frac{5}{2}} - \frac{1}{3}x^{\frac{1}{2}} - \frac{4}{3}x^2.$$