ESAM 311-1 Methods in Applied Mathematics

Fall Quarter 2007

Hermann Riecke

Problem Set 3 **Due Friday November 2, 2007**

1. Consider the solutions to the differential equation

$$2x^2y'' - xy' + (\gamma + \delta\, x)y = 0$$

that are valid near $x = 0$ (4.4.2).

(a) For $\delta = 0$ this equation is equi-dimensional. Calculate its general solution for arbitrary $\gamma \le \frac{9}{8}$.

(b) Use the Frobenius theorem to obtain the general solution for $\gamma = 1$. Employ the approach used in class to write the power series for y^+ and for y^- in terms of elementary functions. Consider the limit $\delta \ll 1$ and perform a Taylor expansion of your solutions to $\mathcal{O}(\delta)$, i.e. $y^+ = y_0^+ + \delta\, y_1^+$ and similarly for y^-. How are these solutions related to the solutions obtained in part (a)?

(c) Can you give a value of γ for which only one of the homogeneous solutions has the form of a power series, $y = x^\sigma \sum_{n=0}^{\infty} A_n x^n$? For that case give both solutions for the case $\delta = 0$.

2. Consider an oscillator (e.g. an LC-circuit) with time-dependent frequency, $\omega^2 = \omega^2(t) = -t$. Note that for $t > 0$ the system is not oscillatory but exhibits only exponential decay, whereas for $t < 0$ it is oscillatory. Rewriting the equation in terms of x instead of t such a system is described by the differential equation

$$y'' - xy = 0.$$

This equation is called Airy's equation. It is also important for the description of a quantum-mechanical particle that impinges upon a smooth potential wall.
Use Frobenius theory to obtain the general solution to this equation. The two homogeneous solutions are called Airy functions and cannot be expressed in terms of other elementary functions (4.4.4).

3. Determine the two values of the constant γ for which **all** solutions of

$$xy'' + (x - 1)y' - \gamma y = 0$$

can be written as power series $y = x^\sigma \sum_{n=0}^{\infty} A_n x^n$ (4.4.7).