ESAM 311-1 Methods in Applied Mathematics

Fall Quarter 2007

Hermann Riecke

Problem Set 5 **Due Wednesday November 28, 2007**

1. Consider the function $f(x) = \sin(\pi x/L)$ on the interval $(0, L)$.

   (a) Expand $f(x)$ in a Fourier sine series of period $2L$. Sketch the function represented by the Fourier sine series on the interval $(-3L, 3L)$.

   (b) Expand $f(x)$ in a Fourier cosine series of period $2L$. Sketch the function represented by the Fourier cosine series on the interval $(-3L, 3L)$.

2. Consider the function $f(x) = 1 - x/2$ on the interval $(0, 1)$.

   (a) Expand $f(x)$ in a Fourier sine series of period 2. Sketch the function represented by the Fourier sine series on the interval $(-3, 3)$.

   (b) Expand $f(x)$ in a Fourier cosine series of period 2. Sketch the function represented by the Fourier cosine series on the interval $(-3, 3)$.

3. Compute the Fourier transform $\tilde{f}(\omega)$ of $f(x) = e^{-(x-a)^2}$.

4. Consider the heat equation

$$\frac{\partial T}{\partial t} = D\frac{\partial^2 T}{\partial x^2} \qquad 0 < x < L$$

   with boundary conditions

$$\left.\frac{\partial T(x,t)}{\partial x}\right|_{x=0} = 0 \qquad T(L,t) = T_r.$$

   (a) Can you find a simple function $T_\infty(x)$ such that $\hat{T}(x,t) = T(x,t) - T_\infty(x)$ satisfies homogeneous boundary conditions at $x = 0$ and at $x = L$? What differential equation does $\hat{T}(x,t)$ satisfy?

   (b) Expand $\hat{T}(x,t)$ in a Fourier series of the form

$$\hat{T}(x,t) = \sum_{n=0}^{\infty} a_n(t)\,\cos k_n x + \sum_{n=0}^{\infty} b_n(t)\,\sin \tilde{k}_n x$$

   and obtain differential equations for $a_n(t)$ and $b_n(t)$. What conditions for b_n and k_n result from the boundary conditions on $\hat{T}(x,t)$? What initial condition does $\hat{T}(x,t)$ have to satisfy if T satisfies the initial condition $T(x,0) = 0$? Determine $T(x,t)$ in terms of T_∞ and a Fourier series.
   Note: you may need the orthogonality condition

$$\int_0^L \cos(n+\frac{1}{2})\frac{\pi}{L}x\,\cos(m+\frac{1}{2})\frac{\pi}{L}x\,dx = 0 \quad \text{for } n \neq m.$$