

Problem 1 (25 points).

Use variation of parameters to solve the initial-value problem

$$y'' - 2y' + y = xe^x \quad y(0) = 0, \quad y'(0) = 1.$$

$$Y_H = c_1 \underbrace{e^x}_{Y_1} + c_2 \underbrace{xe^x}_{Y_2} \quad \text{Find } Y_H \text{ correctly}$$

2 + 3

$$Y_P = Y_2 \int \frac{f_{Y_1}}{a_0 w} - Y_1 \int \frac{f_{Y_2}}{a_0 w}$$

$$\begin{aligned} w &= \begin{vmatrix} e^x & xe^x \\ e^x & xe^x + e^x \end{vmatrix} \\ &= xe^{2x} + e^{2x} - xe^{2x} = e^{2x} \end{aligned}$$

7

5

$$Y_P = xe^x \int \frac{xe^x \cdot e^x}{e^{2x}} dx - e^x \int \frac{xe^x \cdot xe^x}{e^{2x}} dx$$

$$= xe^x \cdot \frac{1}{2}x^2 - \frac{1}{3}e^x \cdot x^3 = \frac{1}{6}x^3e^x$$

Find Y_P

3

$$y(x) = c_1 e^x + c_2 xe^x + \frac{1}{6}x^3 e^x$$

BC 1

2

BCS:

$$y(0) = c_1 = 0$$

$$y'(x) = \cancel{c_1 e^x} + c_2 xe^x + c_2 e^x + \frac{1}{6}x^3 e^x + \frac{1}{2}x^2 e^x$$

$$y'(0) = c_2 = 1$$

BC 2

2

$$\Rightarrow y(x) = xe^x + \frac{1}{6}x^3 e^x$$

Final soln

1

Problem 2 (25 points).

Give the general solution for the differential equation

$$xy'' - 2(x+1)y' + 4y = 0,$$

making use of the fact that $y = e^{2x}$ is a solution to that differential equation.

5 pts for forms

$$y_1 = e^{2x}$$

$$r(x) = \exp\left(\int -\frac{2x+2}{x} dx\right) = \exp\left(\int -2dx - 2\ln x\right)$$

$$= x^{-2} e^{-2x}$$

5 pts

8 pts (form)

$$y_2 = y_1 \int \frac{dx}{p y_1^2} = e^{2x} \int \frac{dx}{x^{-2} e^{-2x} e^{4x}}$$

$$= e^{2x} \int \frac{x^2}{e^{2x}} dx$$

5 pts

15 pts to
get here
if no forms

$$u = x^2 \quad v = -\frac{1}{2} e^{-2x}$$

$$du = 2x dx \quad dv = e^{-2x} dx$$

$$= \left[-\frac{1}{2} x^2 e^{-2x} + \frac{1}{2} \int 2x e^{-2x} dx \right] e^{2x} + 4$$

$$u = 2x \quad v = -\frac{1}{2} e^{-2x}$$

$$du = 2 dx \quad dv = e^{-2x} dx$$

$$= \left\{ -\frac{1}{2} x^2 e^{-2x} + \frac{1}{2} \left[-\frac{1}{2} 2x e^{-2x} + \frac{1}{2} \int e^{-2x} dx \right] e^{2x} \right\} + 4$$

$$= \left(-\frac{1}{2} x^2 e^{-2x} - \frac{1}{4} \cdot 2x e^{-2x} + \frac{1}{2} \left(-\frac{1}{2} e^{-2x} \right) \right) e^{2x}$$

$$= \left(-\frac{1}{2} x^2 e^{-2x} - \frac{1}{2} x e^{-2x} - \frac{1}{4} e^{-2x} \right) e^{2x}$$

$$= \boxed{-\left(\frac{1}{4} + \frac{1}{2}x + \frac{1}{2}x^2\right) e^{-2x}}$$

$$\boxed{\begin{aligned} y(x) &= c_1 e^{2x} + c_2 \left(\frac{1}{4} + \frac{1}{2}x + \frac{1}{2}x^2\right) e^{-2x} \\ &= c_1 e^{2x} + c_2 \left(\frac{1}{2} + x + x^2\right) e^{-2x} \end{aligned}}$$

+ 2

Problem 3 (25 points).

Without computing the particular solution determine for which values of a the boundary value problem

$$y'' - y = ae^{-x} + 1 \text{ for } 0 < x < 1 \quad \text{with} \quad y'(0) = y(0) \quad y'(1) = y(1)$$

has infinitely many solutions? For which values does it have no solution at all?

$$\begin{aligned} y_h: \quad y_h'' - y_h = 0 &\rightarrow y_h = c_1 e^x + c_2 e^{-x} + 3 \\ \text{b.c.: } y_H &= \hat{c}_1 e^x + \hat{c}_2 e^{-x} : \quad y_H'(0) = \hat{c}_1 - \hat{c}_2 = \hat{c}_1 + \hat{c}_2 \\ &\quad \# \hat{c}_2 = 0 + 3 \\ &\quad y_H'(1) = \hat{c}_1 e \stackrel{!}{=} \hat{c}_1 e \Rightarrow \hat{c}_1 \text{ arbitrary.} \\ y_H &= e^x + 3 \end{aligned}$$

\Rightarrow Fehlchen

$$\int_0^1 e^x (ae^{-x} + 1) dx = \int_0^1 a + e^x dx = a + e - 1 \stackrel{!}{=} 0$$

\Rightarrow need $a = 1 - e$ to have infinitely many solutions

Without using Fehlchen -8

$a \neq 1 - e$: no solution. / +3

$$R_0 = 1 \quad R_1 = 0 \quad P_0 = i \quad P_1 = -1$$

$$Q_0 = \frac{1}{4} \quad Q_1 = -\frac{1}{2} \quad +2$$

indicial

$$\sigma(\sigma-1) + \frac{1}{4} = 0$$

$$\sigma^2 - \sigma + \frac{1}{4} = 0$$

$$\sigma = \frac{1 \pm \sqrt{1-1}}{2}$$

$$\sigma_+ = \sigma_- = \frac{1}{2} \quad +2$$

Recursion:

g

$$\left[(n+\sigma)(n+\sigma-1) + \frac{1}{4} \right] A_n = - \left[- \underbrace{(n+m+\sigma)}_{n-\frac{1}{2}} - \frac{1}{2} \right] A_{n-1}$$

$$\left[n^2 + 2n\sigma + \sigma^2 - n - \sigma + \frac{1}{4} \right] A_n = \left[n - \left(\frac{1}{2} + \frac{1}{2} + \frac{1}{2} \right) \right] A_{n-1}$$

$$\left[n^2 + n - n \right] A_n = n A_{n-1} \quad \text{using algebra until: } -2$$

$$A_n = \frac{1}{n} A_{n-1} + 5$$

$$\Rightarrow A_1 = A_0 \quad A_2 = \frac{1}{2} A_1 = \frac{1}{2} A_0 \quad A_3 = \frac{1}{3} A_2 = \frac{1}{6} A_0$$

$$(A_n = \frac{1}{n!} A_0) + 5$$

$$y = A_0 \sum_{n=1}^{\infty} \frac{x^n}{n!} + 5$$

$\sigma_+ = \sigma_- \Rightarrow$ only 1 power series

+4
reason

+4 solution

2 pts for both and possibility of
the form of multi-