

Problem Set 1 Solutions

1.11.7

$$xy' + x + 3y = 0$$

$$y' + \frac{3}{x}y = -1$$

$$\begin{aligned} p(x) &= \exp\left(\int \frac{3}{x} dx\right) \\ &= \exp(3 \ln x) = (e^{\ln x})^3 = x^3 \\ x^3 y' + 3x^2 y &= -x^3 \\ \frac{d}{dx}(x^3 y) &= -x^3 \\ x^3 y &= -\frac{1}{4}x^4 + A \end{aligned}$$

$$y(x) = -\frac{1}{4}x + Ax^{-3}$$

Note:

All the problems from the notes are worth 2 points.

Problem #2 is worth 4 points, for a total of 22 points.

2. (a)

For $0 \leq t \leq T$: (assuming no spike)

$$\begin{cases} \tau V' + V = I_0 \\ V(0) = 0 \end{cases}$$

$$\begin{aligned} V' + \frac{1}{\tau} V &= I_0 / \tau \\ \exp\left(\frac{t}{\tau}\right) V' + \frac{1}{\tau} \exp\left(\frac{t}{\tau}\right) V &= \frac{I_0}{\tau} \exp\left(\frac{t}{\tau}\right) \\ \frac{d}{dt} \left(e^{\frac{t}{\tau}} V \right) &= \frac{I_0}{\tau} e^{\frac{t}{\tau}} \end{aligned}$$

$$\begin{aligned} e^{\frac{t}{\tau}} V &= I_0 e^{\frac{t}{\tau}} + A \\ V(t) &= I_0 + A e^{-\frac{t}{\tau}} \end{aligned}$$

Apply initial condition:

$$V(0) = I_0 + A = 0 \Rightarrow A = -I_0$$

$$V(t) = I_0 (1 - e^{-\frac{t}{\tau}}) \quad \text{for } 0 \leq t \leq T$$

For $T < t$:

$$\begin{cases} \tau V' + V = 0 \\ V(T) = I_0 (1 - e^{-\frac{T}{\tau}}) \end{cases}$$

$$V' + \frac{1}{\tau} V = 0$$

$$V(t) = A e^{-\frac{t}{\tau}}$$

Apply initial condition:

$$\begin{aligned} V(T) &= A e^{-\frac{T}{\tau}} = I_0 (1 - e^{-\frac{T}{\tau}}) \\ \Rightarrow A &= I_0 (e^{\frac{T}{\tau}} - 1) \end{aligned}$$

$$V(t) = I_0 (e^{\frac{T}{\tau}} - 1) e^{-\frac{t}{\tau}} \quad \text{for } T < t$$

The spiking condition:

In order for a spike to be generated, I_o must be large enough to cause $V(t) > V_{threshold}$ for $0 \leq t \leq T$.

Note that setting $I_o > V_{thrs}$ does not guarantee a spike.

Since $V(t)$ increases monotonically during current injection, we can say this:

If $V(T)$ is greater than the threshold voltage, then a spike was fired at some point between 0 and T .

$$V(T) > V_{thrs}$$
$$I_o (1 - e^{-T/\tau}) > V_{thrs}$$

Therefore, any input current I_o such that

$$I_o > \frac{V_{thrs}}{1 - e^{-T/\tau}}$$

will generate a spike.

$$2. (b) \quad \tau V' + V = I_0 t e^{-\alpha t}$$

$$V' + \frac{1}{\tau} V = I_0 \frac{t}{\tau} e^{-\alpha t}$$

$$\frac{d}{dt} \left[e^{\frac{t}{\tau}} V \right] = I_0 \frac{t}{\tau} \exp\left(\frac{t}{\tau} - \alpha t\right)$$

$$e^{\frac{t}{\tau}} V = \frac{I_0}{\tau} \int t \exp\left(\frac{t}{\tau} - \alpha t\right) dt$$

$$= \frac{I_0}{\tau} \int t \exp\left[t\left(\frac{1}{\tau} - \alpha\right)\right] dt$$

$$u = t \quad v = \frac{1}{A} e^{At}$$

$$du = dt \quad dv = e^{At} dt$$

$$= \frac{I_0}{\tau} \left[\frac{t}{A} e^{At} - \frac{1}{A^2} e^{At} \right] + C_1$$

$$V(t) = \frac{I_0}{\tau} e^{-\frac{t}{\tau}} \left[\frac{t}{A} e^{At} - \frac{1}{A^2} e^{At} \right] + C_1 e^{-\frac{t}{\tau}}$$

Apply initial condition:

$$V(0) = \frac{I_0}{\tau} \left[\frac{-1}{A^2} \right] + C_1 = 0$$

$$\Rightarrow C_1 = \frac{I_0}{\tau A^2}$$

Plug C_1 in & rearrange:

$$V(t) = \frac{I_0}{\tau} e^{-\alpha t} \left[\frac{t}{\frac{1}{\tau} - \alpha} - \frac{1}{(\frac{1}{\tau} - \alpha)^2} \right] + \frac{I_0}{\tau} \left(\frac{1}{\tau} - \alpha \right)^{-2} e^{-\frac{t}{\tau}}$$

(Intermediate variables like A should not appear in the final answer)

$$1.11.11 \quad y'' + 6y' + 5y = 0$$

Homogeneous, constant - coefficient,

$$m^2 + 6m + 5 = 0$$

$$(m+1)(m+5) = 0$$

$$\boxed{y(x) = c_1 e^{-x} + c_2 e^{-5x}}$$

$$1.11.12 \quad y'' + 6y' + 9y = 0$$

$$m^2 + 6m + 9 = 0$$

$$(m+3)^2 = 0$$

$$\boxed{y(x) = c_1 e^{-3x} + c_2 x e^{-3x}}$$

$$1.11.13 \quad y'' + 6y' + 13y = 0$$

$$m^2 + 6m + 13 = 0$$

$$m = \frac{-6 \pm \sqrt{6^2 - 4 \cdot 13}}{2}$$

$$= -3 \pm 2i$$

$$y(x) = c_1 e^{(-3+2i)x} + c_2 e^{(-3-2i)x}$$

or

$$y(x) = e^{-3x} (\hat{c}_1 \cos 2x + \hat{c}_2 \sin 2x)$$

$$11.19 \quad x^2 y'' + 7xy' + 13y = 0$$

This equation is equidimensional. It is not necessary to go through all the calculations associated with the change of variables presented in the notes; instead, based on that discussion, make the educated guess

$$y = x^r$$

$$\Rightarrow x^2 \cdot r \cdot (r-1)x^{r-2} + 7x \cdot rx^{r-1} + 13x^r = 0$$
$$r(r-1)x^r + 7rx^r + 13x^r = 0$$

$$r^2 + 6r + 13 = 0$$

$$r = -3 \pm 2i$$

$$y(x) = A x^{-3+2i} + B x^{-3-2i}$$

$$\text{or: } = x^{-3}(Ax^{2i} + Bx^{-2i})$$
$$= x^{-3}(A e^{i2\ln x} + B e^{-i2\ln x})$$

$$y(x) = x^{-3}(\hat{A} \cos[2\ln x] + \hat{B} \sin[2\ln x])$$

$$1.11.20 \quad xy'' + 3y' = x^2$$

Let $z = y'$ to reduce to first order

$$xz' + 3z = x^2$$

$$z' + \frac{3}{x}z = x$$

$$\rho(x) = \exp\left(\int \frac{3}{x} dx\right) = x^3$$

$$\frac{d}{dx} \left[x^3 z \right] = x^4$$

$$x^3 z = \frac{1}{5} x^5 + A$$

$$z(x) = \frac{1}{5} x^2 + A x^{-3}$$

$$\boxed{y(x) = \frac{1}{5} x^3 + \hat{A} x^{-2} + B}$$

$$1.11.22 \quad y'' - 3y' + 2y = xe^x + xe^{-x}$$

Homogeneous equation:

$$y_H'' - 3y_H' + 2y_H = 0$$

$$y_H(x) = c_1 e^x + c_2 e^{2x}$$

Split the RHS into two parts & find particular solutions using undetermined coefficients:

$$y_{p1}'' - 3y_{p1}' + 2y_{p1} = xe^x$$

$$\text{Guess } y_{p1} = Axe^x + Bx^2e^x$$

$$\text{Then } y_{p1}' = Axe^x + Ae^x + Bx^2e^x + 2Bxe^x$$

$$y_{p1}'' = Axe^x + Ae^x + Ae^x + Bx^2e^x + 2Bxe^x + 2Bxe^x + 2Be^x$$

\Rightarrow

$$(2A + 2B - 3A)e^x + (2B + 2B + A - 6B - 3A + 2A)x e^x$$

$$+ (B - 3B + 2B)x^2 e^x = xe^x$$

0

Balancing terms gives two equations for the coefficients

$$2A + 2B - 3A = 0$$

$$2B + 2B + A - 6B - 3A + 2A = 1$$

$$A = 2B$$

$$B = -\frac{1}{2}$$

$$A = -1$$

$$\Rightarrow y_{p_1}(x) = -xe^x - \frac{1}{2}x^2e^x$$

$$y_{p_2}'' - 3y_{p_2}' + 2y_{p_2} = xe^{-x}$$

Guess $y_{p_2} = Ce^{-x} + Dxe^{-x}$

+ proceed similarly to find

$$y_{p_2}(x) = \frac{5}{36}e^{-x} + \frac{1}{6}xe^{-x}$$

The final solution is $y = y_H + y_{p_1} + y_{p_2}$,

$$y(x) = C_1 e^x + C_2 e^{2x} + \left(\frac{5}{36} + \frac{1}{6}x\right)e^{-x} - \left(x + \frac{1}{2}x^2\right)e^x$$

$$1.11.27 \quad y'' + 4y' + 4y = \frac{e^{-2x}}{x^2}$$

Homogeneous equation:

$$y_H'' + 4y_H' + 4y_H = 0$$

$$(m+2)^2 = 0$$

$$y_H(x) = c_1 \underline{e^{-2x}} + c_2 \underline{x e^{-2x}}$$

$$\gamma_1 \qquad \qquad \gamma_2$$

Using variation of parameters,

$$W(y_1, y_2) = \begin{vmatrix} e^{-2x} & x e^{-2x} \\ -2e^{-2x} & (1-2x)e^{-2x} \end{vmatrix} = e^{-4x}$$

$$y_p = -\gamma_1 \int \frac{f \cdot y_2}{W \cdot g_0} dx + \gamma_2(x) \int \frac{f \cdot y_1}{W \cdot g_0} dx$$

$$= -e^{-2x} \int \frac{e^{-2x} \cdot x e^{-2x}}{x^2 e^{-4x}} dx + x e^{-2x} \int \frac{e^{-2x} \cdot e^{-2x}}{x^2 e^{-4x}} dx$$

$$= -e^{-2x} \ln x - e^{-2x}$$

$$y(x) = y_H + y_p$$

$$= c_1 e^{-2x} + c_2 x e^{-2x} - e^{-2x} \ln x - e^{-2x}$$

$$y(x) = \hat{c}_1 e^{-2x} + c_2 x e^{-2x} - e^{-2x} \ln x$$

$$1.11.29 \quad (x-1)y'' - xy' + y = 0 \quad y_1 = e^x$$

$$\begin{aligned} p(x) &= \exp\left(\int \frac{-x}{x-1} dx\right) && \text{Let } u = x-1 \quad du = dx \\ &= \exp\left(-\int \frac{u+1}{u} du\right) = \exp(1-x-\ln(x-1)) \\ &= \frac{e^{1-x}}{x-1} \end{aligned}$$

$$y_2(x) = y_1 \int \frac{dx}{p \cdot y_1^2} = e^x \int \frac{x-1}{e^{2x} e^{1-x}} dx$$

$$\begin{aligned} &= e^x \int (x-1) e^{-x-1} dx \\ &\quad u = x-1 \quad v = -e^{-x-1} \\ &\quad du = dx \quad dv = e^{-x-1} dx \end{aligned}$$

$$\begin{aligned} &= \left[-(x-1)e^{-x-1} + \int e^{-x-1} dx \right] e^x = (1-x)e^{-x-1} - e^{-x-1} \\ &= (-xe^{-x-1}) e^x \end{aligned}$$

$$= -\frac{x}{e}$$

$$\boxed{y(x) = c_1 e^x + c_2 x}$$