Problem Set 2 Solutions

3.4.1
$$\begin{cases} y''' - 2y'' + y' = 1 + xe^x & x > 0 \\ y(0) = 0 \\ y'(0) = 0 \\ y''(0) = 1 \end{cases}$$

Let $z = y' \Rightarrow z'' - 2z' + z = 1 + xe^x$

$z_H(x) = c_1 e^x + c_2 xe^x$

Guess $A + Bx^2e^x + Cx^3e^x$ & find

$z_p = 1 + \frac{1}{6}x^3e^x$

$z(x) = c_1 e^x + c_2 xe^x + \frac{1}{6}x^3e^x + 1$

Integrate & collect terms:

$y(x) = c_1 e^x + c_2 xe^x + c_3 + x - \frac{1}{2}x^2e^x + \frac{1}{6}x^3e^x$

BCs: $y(0) = c_1 + c_3 = 0$

$y'(0) = c_1 + c_2 + 1 = 0$

$y''(0) = -1 + c_1 + 2c_2 = 1$

$\Rightarrow c_1 = -4$

$c_2 = 3$

$c_3 = 4$

$$\boxed{y(x) = 4 - 4e^x + x + 3xe^x - \frac{1}{2}x^2e^x + \frac{1}{6}x^3e^x}$$

3.4.2 $\begin{cases} e^x y''' + \sin x \cdot y'' + \cos x \cdot y' + x^6 y = 0 & x > 0 \\ y(0) = 0 \\ y'(0) = 0 \\ y''(0) = 0 \end{cases}$

Note that $y(x) = 0$ is a solution, which must be unique because this is an IVP.

3.4.3 $\begin{cases} xy'' - (x+1)y' + y = x^2 e^{2x} & x > 1 \\ y(1) = 0 \\ y'(e) = e \end{cases}$

Homogenous solutions:

$y_1(x) = x+1$ (given)

$$y_2(x) = y_1 \int \frac{dx}{p \cdot y_1^2} = (x+1)\int \frac{xe^x}{(x+1)^2}\,dx$$

integrate by parts:

$$= \frac{e^x}{(1+x)} \cdot (x+1) = e^x$$

$y_H(x) = c_1(x+1) + c_2 e^x$

Guess $(A+Bx)e^{2x}$ or use variation of parameters:

$y_p(x) = -\frac{1}{2}e^{2x} + \frac{1}{2}xe^{2x}$

$y(x) = c_1(x+1) + c_2 e^x + \frac{1}{2}e^{2x}(x-1)$

BCs: $y(1) = 2c_1 + ec_2 = 0$

$y'(1) = c_1 + ec_2 + \frac{1}{2}e^2 = e$

$\Rightarrow c_1 = \frac{1}{2}e^2 - e \qquad c_2 = 2 - e$

$$\boxed{y(x) = (\tfrac{1}{2}e^2 - e)(x+1) + (2-e)e^x + \tfrac{1}{2}e^{2x}(x-1)}$$

3.4.5 $\begin{cases}(x-1)y'' - xy' + y = (x-1)^2 & 0 \le x \le \frac{1}{2} \\ y(0) = 0 \\ y(\frac{1}{2}) = 0\end{cases}$

Find homogenous soln:

$y_1 = x$ given

$$y_2 = x\int \frac{dx}{\rho(x)x^2} \qquad \rho(x) = \exp\left(\int \frac{-x}{x-1}\,dx\right) = \frac{e^{-x}}{x-1}$$

$$= x\int \frac{e^x(x-1)}{x^2}\,dx$$

$$= x\int \frac{e^x}{x}dx - x\underbrace{\int \frac{e^x}{x^2}\,dx}$$

$u = e^x \qquad v = -x^{-1}$

$du = e^x\,dx \qquad dv = x^{-2}\,dx$

$$= \cancel{x\int \frac{e^x}{x}dx} - x\left[\frac{-e^x}{x} + \cancel{\int \frac{e^x}{x}dx}\right]$$

$$= e^x$$

$$y_h(x) = c_1 x + c_2 e^x$$

Apply the FAT to determine if the BVP has a soln:

Use BCs with y_h to find complete solution of homogenous problem

$$y_h(0) = c_2 = 0$$
$$y_h(\tfrac{1}{2}) = \tfrac{1}{2}c_1 = 0$$

$$\Rightarrow c_1 = c_2 = 0$$

The homogenous problem has only a trivial solution; therefore the BVP has a unique soln

Guess $A + Bx + Cx^2$ or use variation to find y_p, + arrive at

$$y(x) = c_1 x + c_2 e^x - 1 - x - x^2$$

Apply BCs:

$$y(0) = c_2 - 1 = 0 \Rightarrow c_2 = 1$$

$$y(\tfrac{1}{2}) = \tfrac{1}{2} c_1 + \sqrt{e} - 1 - \tfrac{1}{2} - \tfrac{1}{4} = 0$$

$$\Rightarrow c_1 = \tfrac{7}{2} - 2\sqrt{e}$$

$$\boxed{\begin{aligned} y(x) &= \left(\tfrac{7}{2} - 2\sqrt{e}\right) x + e^x - 1 - x - x^2 \\ &= \left(\tfrac{5}{2} - 2\sqrt{e}\right) x + e^x - 1 - x^2 \end{aligned}}$$

3.4.8 $\begin{cases} 4y'' + y = x & -\pi \le x \le \pi \\ y(-\pi) = 0 \\ y(\pi) = 0 \end{cases}$

Apply the FAT:

$y_H(x) = c_1 \cos\frac{1}{2}x + c_2 \sin\frac{1}{2}x$

$y_H(-\pi) = -c_2 = 0 \Rightarrow c_2 = 0$

$y_H(\pi) = c_2 = 0 \Rightarrow c_2 = 0$

But c_1 is free, & thus there are an infinite number of solutions to the homogenous problem.

In order for a solution to the BVP to exist, the solvability condition must be satisfied, so check it:

$$\int_{-\pi}^{\pi} x \cdot \cos\frac{1}{2}x \, dx \stackrel{?}{=} 0$$

These functions (x and $\cos\frac{1}{2}x$) are orthogonal, so the solvability condition does hold.

The particular solution is $y_p = x$

$y(x) = c_1 \cos\frac{1}{2}x + c_2 \sin\frac{1}{2}x + x$

Apply BCs + find

$$\boxed{y(x) = c_1 \cos\frac{1}{2}x - \pi \sin\frac{1}{2}x + x}$$

Note the solution is not unique, as expected.

3.4.10 Integrate twice + apply ICs

$$y(x) = 1 - \cos x$$

3.4.15
$$\begin{cases} 9y'' + y = x e^{-x^2} & -3\pi \le x \le 3\pi \\ y(-3\pi) = 0 \\ y(3\pi) = 0 \end{cases}$$

Apply the FAT:

$$\begin{cases} 9y'' + y = 0 \\ y(-3\pi) = y(3\pi) = 0 \end{cases} \Rightarrow y_H(x) = c_1 \cos \tfrac{1}{3}x + c_2 \sin \tfrac{1}{3}x$$

$$y_H(-3\pi) = 0 \Rightarrow c_1 = 0$$
$$y_H(3\pi) = 0 \Rightarrow c_1 = 0$$

So the homogenous problem has non-trivial solutions, + we need to check the solvability condition:

$$\int_{-3\pi}^{3\pi} x e^{-x^2} \sin \tfrac{1}{3}x \, dx$$

Holy crap! What to do?
Note that the integrand can be split up into a product of functions like this:

$$f(x) = e^{-x^2}$$
$$g(x) = x \sin \tfrac{1}{3}x$$

Note that f is positive for any value of x.
g is positive or zero for all x between -3π + 3π.
The integral of a positive function (i.e., $f \cdot g$), can't be zero. Therefore, the solvability condition doesn't hold, + the BVP has no solution.

The End.