Problem Set 3 Solutions

1. a. $2x^2 y'' - xy' + \gamma y = 0$

$y = x^r$

$\Rightarrow 2r(r-1) - r + \gamma = 0$

$$r = \frac{3 \pm \sqrt{9 - 8\gamma}}{4}$$

If $\gamma < \frac{9}{8}$, two real roots $r_1 + r_2$

If $\gamma = \frac{9}{8}$, double root $r_1 = r_2 = r$

$\Rightarrow$

$$y(x) = \begin{cases} c_1 x^{r_1} + c_2 x^{r_2}, & \gamma < \frac{9}{8} \\ c_1 x^r + c_2 x^r \ln x, & \gamma = 9/8 \end{cases}$$

1. b. $2x^2y'' - xy' + (1+\delta x)y = 0$

$$x^2y'' - \frac{1}{2}xy' + \left(\frac{1}{2} + \frac{1}{2}\delta x\right)y = 0$$

$R_0 = 1 \qquad P_0 = -\frac{1}{2} \qquad Q_0 = \frac{1}{2}$

$Q_1 = \frac{1}{2}\delta$

$$s(s-1) - \frac{1}{2}s + \frac{1}{2} = 0 \quad \Rightarrow \quad s^2 - \frac{3}{2}s + \frac{1}{2} = 0$$

$s_+ = 1 \qquad s_- = \frac{1}{2}$

$s_+ - s_- = \frac{1}{2}$, not a positive integer, & there are two linearlly independent power series

$$A_n^+\left[(n+1)(n) - \frac{1}{2}(n+1) + \frac{1}{2}\right] = -\sum_{m=1}^{n} A_{n-m}^+ \cdot Q_m$$

$$\Rightarrow A_n^+ = \frac{-\delta A_{n-1}^+}{2(n^2 + \frac{1}{2}n)} = \frac{-\delta A_{n-1}^+}{2n^2 + n}$$

$$A_n^-\left[\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right) - \frac{1}{2}\left(n+\frac{1}{2}\right) + \frac{1}{2}\right] = -\sum_{m=1}^{n} A_{n-m}^- \cdot Q_m$$

$$\Rightarrow A_n^- = \frac{-\delta A_{n-1}^-}{2n^2 - n}$$

Tweak the coefficients:

$$A_n^+ = \frac{-2\delta A_{n-1}^+}{2n(2n+1)} \qquad A_n^- = \frac{-2\delta A_{n-1}^-}{2n(2n-1)}$$

$$= \frac{(-2\delta)(-2\delta)A_{n-2}^+}{(2n+1)(2n)(2n-1)(2n-2)}$$

$\vdots \qquad \vdots$

$$A_n^+ = \frac{(-1)^n \delta^n A_0^+ \cdot 2^n}{(2n+1)!} \qquad A_n^- = \frac{(-1)^n \delta^n A_0^- \cdot 2^n}{(2n)!}$$

Then $y^+ = A_0^+ x \sum_{n=0}^{\infty} \frac{(-1)^n 2^n \delta^n}{(2n+1)!} x^n$

Note that $\sin(t) = \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!} (-1)^n$

$$= t \sum_{n=0}^{\infty} \frac{(-1)^n (t^2)^n}{(2n+1)!}$$

If $t^2 = 2\delta x$, the two series are identical, and so

$$\boxed{y^+ = A_0^+ \sqrt{\frac{2\delta}{x}} \sin(\sqrt{2\delta x})}$$

$$y^- = A_0^- x^{1/2} \sum_{n=0}^{\infty} \frac{(-1)^n (2\delta x)^n}{(2n)!}$$

$$\cos(t) = \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} (-1)^n = \sum \frac{(-1)^n (t^2)^n}{(2n)!}$$

& again $t^2 = 2\delta x$

$$\boxed{y^- = A_0^- x^{1/2} \cos(\sqrt{2\delta x})}$$

c. $R_0 = 1 \qquad P_0 = -\frac{1}{2} \qquad Q_0 = \frac{1}{2}\gamma$

$Q_1 = \frac{1}{2}\delta$

$$s_+ = \frac{\frac{3}{2} + \sqrt{\frac{9}{4} - 2\gamma}}{2} \qquad s_- = \frac{\frac{3}{2} - \sqrt{\frac{9}{4} - 2\gamma}}{2}$$

Want to choose γ s.t. $s_+ - s_- = 0$:

$$s_+ - s_- = 2\sqrt{\frac{9}{4} - 2\gamma} = 0$$

$$\Rightarrow \gamma = \frac{9}{8}$$

$$s_+ = s_- = \frac{3}{4}$$

Then y_1 is a power series and y_2 isn't

$$A_1^+ = -\frac{1}{2}\delta A_0^+$$
$$A_2^+ = \frac{1}{16}\delta^2 A_0^+$$
$$A_3^+ = -\frac{1}{2^5 3^2}\delta^3 A_0^+$$

all terms involve δ except the first, so when $\delta = 0$, y_1 is

$$y_1 = A_0^+ x^{3/4}$$

Then using reduction of order,

$$p(x) = \exp\left(\int \frac{-x}{2x^2}\,dx\right) = x^{-1/2}$$

$$y_2 = A_0^+ x^{3/4} \int \frac{dx}{x^{-1/2} A_0^{+2} x^{3/2}}$$

$$= \frac{1}{A_0^+} x^{3/4} \ln x + B x^{3/4}$$

$$y(x) = A_0^+ x^{3/4} + \frac{1}{A_0^+} x^{3/4} \ln x + B x^{3/4}$$

1. c. Con't

In retrospect, an equally valid solution is to refer to 1.a) + point out that when $\gamma = \frac{9}{8}$, y_1 is a power series + y_2 isn't.

2. $x^2 y'' - x^3 y = 0$

$R_0 = 1 \qquad P_0 = 0 \qquad Q_0 = 0$

$Q_1 = 0$

$Q_2 = 0$

$Q_3 = 1$

$s_+ = 1 \qquad s_- = 0$

$s_+ - s_- = 1$, a positive integer - check to see if y^- is a power series:

$$A_0^- [0 + 0 + 0] = 0 \quad \checkmark$$

So A_0^- is arbitrary

A_1^- can be taken to be 0

$$A_n^+ \big((n+1) n \big) = - \sum_{m=1}^{n} A_{n-m}^+ Q_m$$

$\Rightarrow A_1^+ = A_2^+ = 0$

and $A_n^+ = \dfrac{A_{n-3}^+}{(n+1)n}$ for $n \geq 3$

Similarly, $A_1^- = A_2^- = 0$

$A_n^- = \dfrac{A_{n-3}^-}{n(n-1)}$ for $n \geq 3$

$$y(x) = x \cdot \sum_{n=0}^{\infty} A_n^+ x^n + \sum_{n=0}^{\infty} A_n^- x^n$$

~~the~~ with A_n^+ + A_n^- as given above

More explicitly,

A_0^+ arbitrary

$A_1^+ = A_2^+ = 0$

$A_3^+ = \dfrac{1}{3 \cdot 4} A_0^+$

$$A_4^+ = A_5^+ = 0$$

$$A_6^+ = \frac{1}{3\cdot 4\cdot 6\cdot 7} A_0^+$$

$$A_7^+ = A_8^+ = 0$$

$$A_9^+ = \frac{1}{3\cdot 4\cdot 6\cdot 7\cdot 9\cdot 10} A_0^+$$

etc.

A_0^- arbitrary

$$A_1^- = A_2^- = 0$$

$$A_3^- = \frac{1}{2\cdot 3} A_0^-$$

$$A_4^- = A_5^- = 0$$

$$A_6^- = \frac{1}{2\cdot 3\cdot 5\cdot 6} A_0^-$$

etc.

3. $x^2 y'' + (-x + x^2) y' - \gamma x y = 0$

$R_0 = 1 \qquad P_0 = -1 \qquad Q_0 = 0$

$\qquad\qquad P_1 = 1 \qquad Q_1 = -\gamma$

$s_+ = 2$ and $s_- = 0$, so $s_+ - s_- = 2$, a positive integer.

For both solutions to be power series, must satisfy the condition given in the second case of the Frobenius Theorem, which reduces to

$$A_1^-(1-\gamma) = 0$$

Since $A_1^- = -\gamma A_0^-$, the condition is

$$-A_0^- \gamma (1-\gamma) = 0$$

In general, $A_0^- \neq 0$, so we must have

$$\boxed{\gamma = 0 \text{ or } \gamma = 1}$$