

Problem Set 4 Solutions

1. $\lim_{x \rightarrow 0} x^2 k_2(x)$

$$= \lim_{x \rightarrow 0} x^2 (2x^{-2}) = \boxed{2}$$

2. $\lim_{x \rightarrow \infty} x \left([J_p(x)]^2 + [F_p(x)]^2 \right)$

$$= \lim_{x \rightarrow \infty} x \left(\left[\sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi}{4} - \frac{p\pi}{2}\right) \right]^2 + \left[\sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\pi}{4} - \frac{p\pi}{2}\right) \right]^2 \right)$$

$$= \boxed{\frac{2}{\pi}}$$

3. Show $\frac{d}{dx} [x^p I_p(ax)] = ax^p I_{p-1}(ax)$

$$I_p(ax) = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)^{2n+p} a^{2n+p} x^{2n+p}}{n! (n+p)!}$$

$$x^p I_p(ax) = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)^{2n+p} a^{2n+p} x^{2n+2p}}{n! (n+p)!}$$

$$\begin{aligned} \frac{d}{dx} [x^p I_p(ax)] &= \sum \frac{\left(\frac{1}{2}\right)^{2n+p} \cdot 2(n+p) a^{2n+p-1} x^{2n+2p-1}}{n! (n+p)!} \\ &= \sum \frac{\left(\frac{1}{2}\right)^{2n+p} \left(\frac{1}{2}\right)^{-1} a \cdot a^{2n+p-1} \cdot x^p \cdot x^{2n+p-1}}{n! (n+p-1)!} \\ &= ax^p \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)^{2n+p-1} a^{2n+p-1} x^{2n+p-1}}{n! (n+p-1)!} \end{aligned}$$

$$= ax^p I_{p-1}(ax)$$

4. a. Using the given form of the solution,

$$\frac{\partial^2 u}{\partial t^2} = -\omega^2 \sin(\omega t) \cos(n\theta) y(r)$$

$$\frac{d}{dr} \left[r \frac{dy}{dr} \right] = \sin(\omega t) \cos(n\theta) (ry'' + y')$$

$$\frac{\partial^2 u}{\partial r^2} = -n^2 \sin(\omega t) \cos(n\theta) y(r)$$

Plug these in to the equation & cancel $\sin(\omega t) \cos(n\theta)$ from every term:

$$-\omega^2 y = \frac{c^2}{r} (ry'' + y') - \frac{c^2 n^2}{r^2} y$$

$$= c^2 (y'' + \frac{1}{r} y') - \frac{c^2 n^2}{r^2} y$$

$$-\omega^2 r^2 y = c^2 r^2 y'' + c^2 r y' - c^2 n^2 y$$

$$c^2 r^2 y'' + c^2 r y' + (\omega^2 r^2 - c^2 n^2) y = 0$$

$$\boxed{r^2 y'' + r y' + \left[\left(\frac{\omega}{c} \right)^2 r^2 - n^2 \right] y = 0}$$

b. Note that the above equation is of the form of the generalized Bessel's equation, with the following parameter values:

$$a = 1$$

$$b = 0$$

$$c = -n^2$$

$$d = \omega^2/c^2$$

$r = \text{arbitrary} \leftrightarrow$ (the parameter from the notes, not the independant variable from part a)

$$s = 1$$

Plug these values in to the forms given in the notes to find

$$y(r) = \begin{cases} c_1 J_n\left(\frac{\omega}{c} r\right) + c_2 J_{-n}\left(\frac{\omega}{c} r\right), & n \neq 0, 1, 2, \dots \\ c_1 J_n\left(\frac{\omega}{c} r\right) + c_2 Y_n\left(\frac{\omega}{c} r\right), & n = 0, 1, 2, \dots \end{cases}$$

For this solution to be physically meaningful, the displacement $u(r, \theta, t)$ needs to be finite everywhere (since drums are not infinitely stretchable in general). Inspect the solution:

$$u = \sin(\omega t) \cos(\omega r) y(r)$$

\sin and \cos are clearly bounded. Looking at $y(r)$, we can see that it is also finite everywhere except when $r=0$. As $r \rightarrow 0$, J_{-n} and Y_n both go to ∞ , leading to the requirement that $c_2 = 0$. (This is exactly the condition $u(0, \theta, t) < \infty$ given in the homework.) The general solution for y is

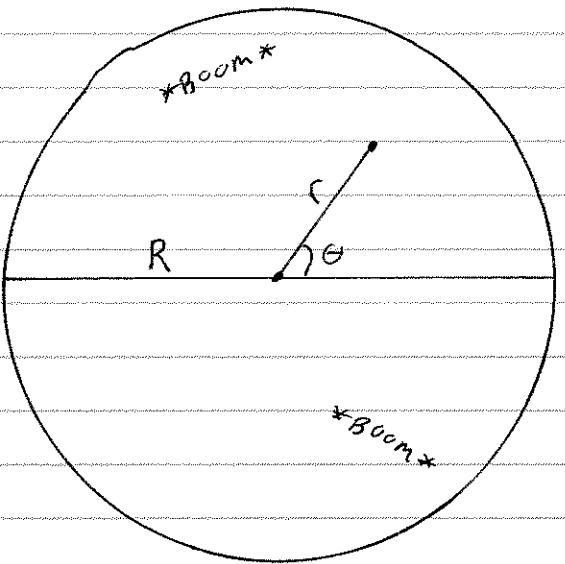
$$y(r) = c_1 J_n\left(\frac{\omega}{c} r\right)$$

Note that the parameter $\frac{\omega}{c}$ gives a characteristic length scale for the problem: ω is a frequency with units of inverse time; c is the speed of sound in the medium with units of length over time. $\frac{\omega}{c}$ therefore has units of inverse length, and $\frac{\omega}{c} \cdot r$ can be thought of as a dimensionless, normalized length scale.

$$c. u(r, \theta, t) = c_1 \sin(\omega t) \cos(n\theta) J_n\left(\frac{\omega}{c} r\right)$$

Recall the physical interpretation of this problem:

A circular drumhead with radius R is bouncing up and down. $u(r, \theta, t)$ gives the height of the drum at the point (r, θ) at time t .



Clearly, for a physically meaningful solution, we should require

$$u(R, \theta, t) = 0$$

i.e., the drumhead must be attached to the frame.

This means that

$$\sin(\omega t) \cos(n\theta) J_n\left(\frac{\omega}{c} \cdot R\right) = 0$$

for all values of $t + \theta$. sin + cos terms
will not be ~~not~~ 0 in general, so the condition
says that

$$J_n\left(\frac{\omega}{c} \cdot R\right) = 0$$

i.e., $\frac{\omega}{c} \cdot R$ must be a zero of the function J_n . Look at some plots of a few J_n & note that they have an infinite number of zeros. Unfortunately, there is no easy way to write down an expression for the zeros of J_n , so we employ the method of Gradshteyn and Ryzhik!

Let $B_{n,i}$ represent the i -th zero of the Bessel function of order n , i.e.

$$J_n(B_{n,i}) = 0 \quad \text{for all } i = 1, 2, 3, \dots$$

Now the boundary condition can be written as

$$\frac{w}{c} \cdot R = B_{n_i}$$

The B_{n_i} are real numbers; they are set + never change. R and c are given properties of the problem. w , however, is a property of the solution which ~~is~~ is the temporal frequency of drumhead oscillations. This condition says that the only functions u that can solve this problem ~~can~~ have a frequency given by

$$w = \frac{c \cdot B_{n_i}}{R}$$

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As a side note, recall that there are no conditions on n ; we can choose it to be any non-negative number.

Also notice that there are an infinite number of w 's that satisfy the condition above. In general, for a given wave mode n , the solution is an infinite sum across all complying frequencies:

Given n ,

$$u(r, \theta, t) = \sum_{i=1}^{\infty} A_i \sin\left(\frac{c \cdot B_{n_i} \cdot t}{R}\right) \cdot \cos(n\theta) \cdot J_n\left(\frac{w}{c} \cdot r\right)$$

$$5. \quad x^4 y'' + d^2 y = 0$$

$$x^2 y'' + d^2 x^{-2} y = 0$$

Match to generalized eqn:

$$a = 0$$

$$b = 0$$

$$r = 0$$

$$c = 0$$

$$s = -1$$

$$d = d^2$$

$$\Rightarrow y(x) = \left[c_1 J_{1/2} \left(-\frac{\alpha}{x} \right) + c_2 J_{-1/2} \left(-\frac{\alpha}{x} \right) \right] x^{1/2}$$

$$= \left[c_1 \sqrt{\frac{-2x}{\pi d}} \sin \left(-\frac{\alpha}{x} \right) + c_2 \sqrt{\frac{-2x}{\pi d}} \cos \left(-\frac{\alpha}{x} \right) \right] x^{1/2}$$

$$= \left[\hat{c}_1 \sin \left(-\frac{\alpha}{x} \right) + \hat{c}_2 \cos \left(-\frac{\alpha}{x} \right) \right] x$$

$$= \left[\hat{c}_1 \exp \left(i \frac{\alpha}{x} \right) + \hat{c}_2 \exp \left(-i \frac{\alpha}{x} \right) \right] x$$

$$6. xy'' - y' + 4x^3y = 0$$

$$x^2y'' - xy' + 4x^4y = 0$$

Matching against the general equation:

$$a = -1$$

$$b = 0 \quad s = 2$$

$$c = 0$$

$$d = 4$$

$$\Rightarrow p = \frac{1}{2}$$

$$\begin{aligned} y(x) &= x [c_1 J_{1/2}(x^2) + c_2 J_{-1/2}(x^2)] \\ &= \hat{c}_1 \sin(x^2) + \hat{c}_2 \cos(x^2) \end{aligned}$$

$$y(0) = 0 \Rightarrow \hat{c}_2 = 0$$

$$y(\sqrt{\pi}) = 0 \Rightarrow \hat{c}_1 \text{ arbitrary}$$

$$y(x) = \hat{c}_1 \sin(x^2)$$