Problem Set 5 Solutions

1. a.

$$f(x) = \sin\left(\frac{\pi x}{L}\right) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L} x\right)$$

$$A_m \int_0^L \sin^2\left(\frac{m\pi}{L} x\right) dx = \int_0^L \sin\left(\frac{m\pi}{L} x\right) \sin\left(\frac{\pi}{L} x\right) dx$$

Note that the RHS is zero unless $m = 1$.
This series has only one term, and $A_1 = 1$.

$f(x) = \sin\left(\frac{\pi x}{L}\right)$ is the expansion.

b. $$f(x) = \sin\left(\frac{\pi x}{L}\right) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi}{L} x\right)$$

$$A_m \int_0^L \cos^2\left(\frac{m\pi}{L} x\right) dx = \int_0^L \sin\left(\frac{\pi}{L} x\right) \cos\left(\frac{m\pi}{L} x\right) dx$$

If $m = 0$, $A_0 \cdot L = \frac{2}{\pi} L \Rightarrow A_0 = \frac{2}{\pi}$

If $m = 1$, the RHS is $0 \Rightarrow A_1 = 0$

If $m > 1$,

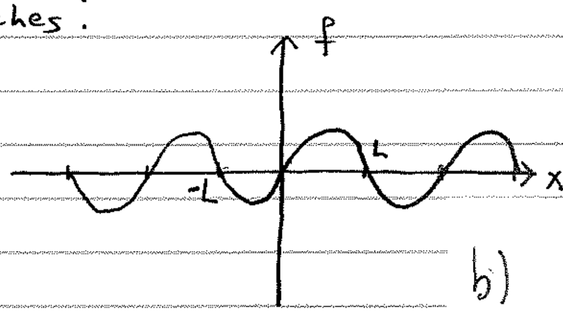
$$A_m \cdot \frac{L\left[2m\pi + \sin(2m\pi)\right]}{4m\pi} = \frac{L \cdot \left[1 + \cos(m\pi)\right]}{\pi(1-m^2)}$$

$$A_m = \frac{2}{\pi} \cdot \frac{1+\cos(m\pi)}{1-m^2} = \frac{2}{\pi} \cdot \frac{1+(-1)^m}{1-m^2}$$

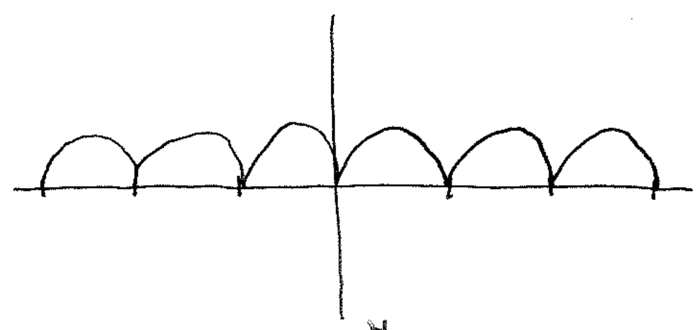
$$\boxed{f(x) = \frac{2}{\pi} + \sum_{n=2}^{\infty} \frac{2\left[1+(-1)^n\right]}{\pi(1-n^2)} \cos\left(\frac{n\pi}{L} x\right)}$$

Sketches:

a)



b)



2. a. $f(x) = 1 - \frac{x}{2} = \sum_{n=1}^{\infty} A_n \sin(n\pi x)$

$$A_m \int_0^1 \sin^2(m\pi x)\,dx = \int_0^1 \left(1-\frac{x}{2}\right)\sin(m\pi x)\,dx$$

$$A_m \cdot \frac{1}{2} = \int_0^1 \sin(m\pi x)\,dx - \frac{1}{2}\int_0^1 x\sin(m\pi x)\,dx$$

$u = x$ $\qquad v = \frac{-1}{m\pi}\cos(m\pi x)$

$du = dx$ $\qquad dv = \sin(m\pi x)\,dx$

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$$A_m = \frac{2 + (-1)^{m+1}}{m\pi}$$

$$f(x) = \sum_{n=1}^{\infty} \frac{2+(-1)^{\cancel{m}+1}}{\cancel{m}\pi} \sin(n\pi x)$$

b. $1 - \frac{x}{2} = \sum_{n=0}^{\infty} A_n \cos(n\pi x)$

$$A_m \int_0^1 \cos^2(m\pi x)\,dx = \int_0^1 \left(1-\frac{x}{2}\right)\cos(m\pi x)\,dx$$

If $m = 0$, $\quad A_m \cdot 1 = \frac{3}{4} \Rightarrow A_0 = \frac{3}{4}$

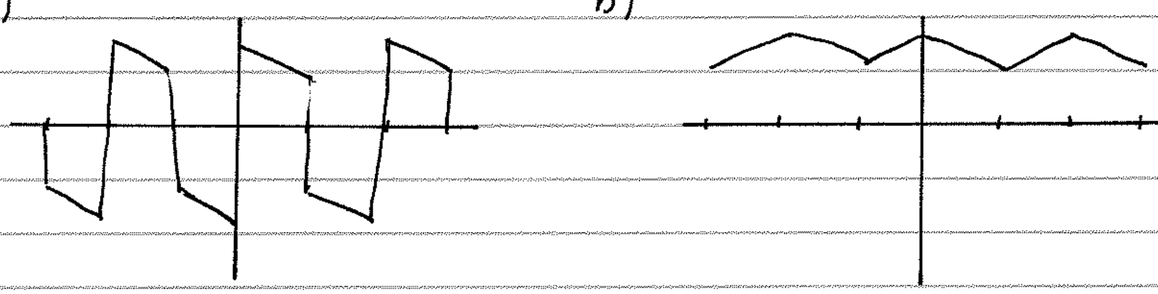
If $m > 0$,

$$A_m \cdot \frac{1}{2} = \frac{1 + (-1)^{m+1}}{2(m\pi)^2}$$

$$A_m = \frac{1+(-1)^{m+1}}{(m\pi)^2}$$

$$f(x) = \sum_{n=\cancel{0}}^{\infty} \frac{1+(-1)^{\cancel{m}+1}}{(\cancel{m}\pi)^2}\cos(n\pi x) + \frac{3}{4}$$

a) $\qquad\qquad$ b)



3. $\tilde{f}(\omega) = \int_{-\infty}^{\infty} e^{-(x-a)^2} e^{-i\omega x} \, dx$

$$= \int_{-\infty}^{\infty} \exp\left[-\left(x^2 - 2ax + i\omega x + a^2\right)\right] dx$$

$$= \int \exp\left[-\left(x^2 + (i\omega - 2a)x + a^2\right)\right] dx$$

want to write as $(x+\bar{x})^2 + \beta$, so complete the square!

$$x^2 + (i\omega - 2a)x + a^2$$
$$x^2 + 2x\bar{x} + \bar{x}^2$$
$$\Rightarrow \quad 2\bar{x} = i\omega - 2a \qquad \bar{x} = \tfrac{1}{2} i\omega - a$$

$$x^2 + (i\omega - 2a)x + \left(\tfrac{1}{2} i\omega - a\right)^2 - \left(\tfrac{1}{2} i\omega - a\right)^2 + a^2$$
$$\left(x + \tfrac{1}{2} i\omega - a\right)^2 + a^2 - \left(\tfrac{1}{2} i\omega - a\right)^2$$
$$\left(x + \tfrac{1}{2} i\omega - a\right)^2 + \tfrac{1}{4}\omega^2 + i\omega a$$

$$= \int_{-\infty}^{\infty} \exp\left[-\left(x + \tfrac{1}{2} i\omega - a\right)^2\right] \exp\left[-\left(\tfrac{1}{4}\omega^2 + i\omega a\right)\right] dx$$

Let $u = x + \frac{1}{2} i\omega - a$

$du = dx$

$$= \exp\left[-\left(\tfrac{1}{4}\omega^2 + i\omega a\right)\right] \int_{-\infty}^{\infty} e^{-u^2} du$$

$$= \sqrt{\pi}\, \exp\left[-\left(\tfrac{1}{4}\omega^2 + i\omega a\right)\right]$$

4. a. The idea is to separate the transient + steady-state parts of the solution, so hypothesize that T can be decomposed as

$$T(x,t) = \hat{T}(x,t) + T_\infty(x)$$

where $\hat{T}$ satisfies the heat equation with homogenous BCs, and T_∞ is the steady-state solution. Plug this in to get

$$\hat{T}_t = D\,\hat{T}_{xx} + D\,T_{\infty\,xx}$$

Since we make the assumption that $\hat{T}$ satisfies the heat equation, $\hat{T}_t - D\,\hat{T}_{xx} = 0$, leaving

$$D\,T_{\infty\,xx} = 0 \quad \Rightarrow \quad T_\infty(x) = c_1 x + c_2$$

So $\hat{T}(x,t) = T(x,t) - c_1 x - c_2$

$\hat{T}$ must satisfy homogeneous BCs by hypothesis:

$$\left.\frac{d\hat{T}}{dx}\right|_{x=0} = \left.\frac{dT}{dx}\right|_{x=0} - c_1 = 0$$

$$= 0 - c_1 = 0 \quad \Rightarrow \quad c_1 = 0$$

$$\hat{T}(L,t) = T(L,t) - c_2 = 0$$

$$T_r - c_2 = 0$$

$$c_2 = T_r$$

$$\boxed{T_\infty(x) = T_r}$$

b. $$\hat{T} = \sum a_n(t)\cos(k_n x) + \sum b_n(t)\sin(\tilde{k}_n x)$$

$$\hat{T}_t = \sum a_n' \cos(k_n x) + \sum b_n' \sin(\tilde{k}_n x)$$

$$\hat{T}_{xx} = -\sum a_n k_n^2 \cos(k_n x) - \sum b_n \tilde{k}_n^2 \sin(\tilde{k}_n x)$$

$$\hat{T}_t = D\hat{T}_{xx}$$

$$\Rightarrow$$

$$\sum a_n' \cos(k_n x) + \sum b_n' \sin(\tilde{k}_n x) = -\sum a_n k_n^2 \cos(k_n x) - \sum b_n \tilde{k}_n^2 \sin(\tilde{k}_n x)$$

Since sines + cosines are orthogonal, we can collect terms + write

$$\boxed{\begin{array}{ll} a_n'(t) + D k_n^2 a_n(t) = 0 & b_n'(t) + D\tilde{k}_n^2 b_n(t) = 0 \\ a_n(t) = A_n \exp(-Dk_n^2 t) & b_n(t) = B_n \exp(-D\tilde{k}_n^2 t) \end{array}}$$

There are three BCs to apply to $\hat{T}$:

(A) $\left.\frac{d\hat{T}}{dx}\right|_{x=0} = 0$

(B) $\hat{T}(L, t) = 0$

(C) $\hat{T}(x, 0) = -T_r$ (b/c $T(x,0) = \hat{T}(x,0) + T_r = 0$)

Starting with (A),

$$\left.\frac{d\hat{T}}{dx}\right|_{x=0} = -\sum a_n k_n \sin(0) + \sum b_n \tilde{k}_n \cos(0) = 0$$

$\Rightarrow b_n \tilde{k}_n = 0 \Rightarrow b_n = 0$ or $\tilde{k}_n = 0$ for all n.

Either way, the second series in $\hat{T}$ drops out, leaving

$$\hat{T} = \sum A_n \exp(-Dk_n^2 t)\cos(k_n x)$$

Moving on to (B),

$$\hat{T}(L,t) = \sum A_n \exp(-Dk_n^2 t)\cos(k_n L) = 0$$

$$\Rightarrow \cos(k_n L) = 0 \Rightarrow k_n = \frac{\pi}{L}\left(n+\frac{1}{2}\right)$$

Leaving

$$\hat{T}(x,t) = \sum A_n \exp\left(-D\left[\frac{\pi}{L}\left(n+\frac{1}{2}\right)\right]^2 t\right)\cos\left(\frac{\pi x}{L}\left(n+\frac{1}{2}\right)\right)$$

Finally, (C):

$$\hat{T}(x,0) = \sum A_n \cos(k_n x) = -T_r$$

i.e, $\hat{T}(x,0)$ is the Fourier cosine expansion of $-T_r$. multiply through by $\cos(k_m x)$ & integrate:

$$A_m = \frac{-T_r \int_0^L \cos\left[\frac{\pi}{L}x\left(m+\frac{1}{2}\right)\right]dx}{\int_0^L \cos\left[\frac{\pi}{L}\left(m+\frac{1}{2}\right)x\right]^2 dx}$$

$$= -T_r \frac{2L\cos(m\pi)}{\pi+2m\pi} \cdot \frac{2(\pi+2m\pi)}{L(\pi+2m\pi-\sin(2m\pi))}$$

$$= \frac{4T_r(-1)^{m+1}}{\pi(2m+1)}$$

So finally,

$$T(x,t) = \sum_{n=0}^{\infty} \frac{4T_r(-1)^{n+1}}{\pi(2n+1)} \exp\left(-D\left[\frac{\pi}{L}\left(n+\frac{1}{2}\right)\right]^2 t\right)\cos\left(\frac{\pi x}{L}\left(n+\frac{1}{2}\right)\right) + T_r$$