

Problem Set 6 Solutions

1. a. $\begin{cases} y'' - \lambda y = 0 \\ y'(0) = 0 \\ y'(\pi) = 0 \end{cases}$

Case 1: $\lambda > 0$

$$y(x) = c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x}$$

$$y'(x) = c_1 \sqrt{\lambda} e^{\sqrt{\lambda}x} + c_2 \sqrt{\lambda} e^{-\sqrt{\lambda}x}$$

$$y'(0) = 0 \Rightarrow c_1 = c_2$$

$$y'(\pi) = 0$$

$$c_1 \sqrt{\lambda} e^{\sqrt{\lambda}\pi} + c_2 \sqrt{\lambda} e^{-\sqrt{\lambda}\pi} = 0$$

$$e^{\sqrt{\lambda}\pi} = e^{-\sqrt{\lambda}\pi}$$

$$\sqrt{\lambda} = -\sqrt{\lambda} \Rightarrow \lambda = 0 !!!$$

Case 2: $\lambda = 0$

$$y(x) = c_1 x + c_2 \quad y' = c_1$$

$$BC \Rightarrow c_1 = 0$$

$$y(x) = c_2$$

Case 3: $\lambda < 0$

$$y'' + \underbrace{(-\lambda)}_{\text{a positive number}} y = 0$$

$$y(x) = c_1 \sin(\sqrt{-\lambda}x) + c_2 \cos(\sqrt{-\lambda}x)$$

$$y' = c_1 \sqrt{-\lambda} \cos(\sqrt{-\lambda}x) + c_2 \sin(\sqrt{-\lambda}x) \cdot \sqrt{-\lambda}$$

$$y'(0) = c_1 \sqrt{-\lambda} = 0 \quad c_1 = 0$$

$$y'(\pi) = c_2 \sqrt{-\lambda} \sin(\sqrt{-\lambda}\pi) = 0$$

$$\Rightarrow \sqrt{-\lambda} \pi = n\pi$$

$$\lambda_n = -n^2$$

Then the eigenfunctions are

$$y_n(x) = c_n \cos(nx)$$

b. Yes

c. $f(x) = \cos^2 x = \frac{1}{2}(1 + \cos 2x)$

$$\frac{1}{2}(1 + \cos 2x) = \sum_{n=0}^{\infty} c_n \cos nx$$

$$c_n = \frac{\int_0^\pi \frac{1}{2}(1 + \cos 2x) \cos nx dx}{\int_0^\pi \cos^2 nx dx}$$

$$\begin{aligned} & \frac{1}{2} \int_0^\pi (1 + \cos 2x) \cos nx dx \\ & \underbrace{\frac{1}{2} \int_0^\pi \cos nx dx}_{= \begin{cases} 0, & n \neq 0 \\ \frac{\pi}{2}, & n = 0 \end{cases}} + \underbrace{\frac{1}{2} \int_0^\pi \cos 2x \cos nx dx}_{= \begin{cases} 0, & n \neq 2 \\ \frac{\pi}{4}, & n = 2 \end{cases}} \end{aligned}$$

$$\int_0^\pi \cos^2 nx dx = \begin{cases} \frac{\pi}{2}, & n \neq 0 \\ \pi, & n = 0 \end{cases}$$

$$c_0 = \frac{1}{2}$$

$$c_1 = 0$$

$$c_2 = \frac{1}{4}$$

$$c_3 = c_4 = \dots = 0$$

$$f(x) = \frac{1}{2} + \frac{1}{4} \cos 2x = \cos^2 x$$

$$2. \text{ a.) } X'' - 2X' + (1+\lambda)X = 0$$

$$m^2 - 2m + 1 + \lambda = 0 \quad m = 1 \pm \sqrt{-\lambda}$$

$$X(x) = e^x [c_1 e^{(\sqrt{-\lambda})x} + c_2 e^{(-\sqrt{-\lambda})x}] \quad \lambda \neq 0$$

$$X(x) = c_1 e^x + c_2 x e^x \quad \lambda = 0$$

b.)

Case 1: $\lambda = 0$

$$X(0) = c_1 = 0$$

$$X'(L) = c_2 [L e^L + e^L] = 0$$

$$\Rightarrow c_2 = 0 \text{ in general}$$

only trivial solution

Case 2: $\lambda < 0$

Then $\sqrt{-\lambda}$ is real

$$X(0) = c_1 + c_2 = 0 \quad c_2 = -c_1$$

$$X'(L) = c_1 (1 + \sqrt{-\lambda}) e^{(1+\sqrt{-\lambda})L} - c_1 (1 - \sqrt{-\lambda}) e^{(1-\sqrt{-\lambda})L} = 0$$

$$e^{2\sqrt{-\lambda}L} = \frac{1 - \sqrt{-\lambda}}{1 + \sqrt{-\lambda}} \quad \text{Let } a = \sqrt{-\lambda}$$

$$e^{2La} = \frac{1-a}{1+a}$$

This is an unpleasant transcendental equation.

By inspection, $a=0$ is a solution, but this corresponds to $\lambda=0$, which violates our assumption. Are there other solutions?

For positive values of a , the LHS is always greater than one, and the RHS is always less than one, so there are no solutions for $a > 0$, and thus no negative eigenvalues.

Case 3: $\lambda > 0$

Let $\lambda = \mu^2$ and rewrite X :

$$X(x) = e^x [\hat{c}_1 \sin \mu x + \hat{c}_2 \cos \mu x]$$

$$X(0) = c_2 = 0$$

$$X'(L) = c_1 e^L [\mu \cos \mu L + \sin \mu L] = 0$$

$$\boxed{\mu + \tan \mu L = 0}$$

Another unpleasant transcendental equation!

This one has an infinite number of roots, which cannot be expressed in a closed form, but can be indicated graphically as on the attached plots. The eigenfunctions are

$$\boxed{X_n(x) = c_n e^x \sin \mu_n x \quad | \text{with } \mu_n \text{ as shown}}$$

