ESAM 346
Modeling and Computation in Science and Engineering

Winter 2007

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Problem Set 3

February 13 2007 Due Thursday, February 27, 2007

The goal of this homework is to investigate the dynamics of a neuron within the classic Hodgkin-Huxley model (J. Physiol. 117 (1952) 500; pdf-file on class web site). The details of the model are in the class notes. We will use this model to explore the treatment of stiff equations.

1. As a test problem write a code that solves the differential equation

$$\frac{dy}{dt} = F(y,t) \equiv \lambda y - y^3 + A\sin(\omega t) \tag{1}$$

with parameters

$$\lambda = -200 \qquad A = 50,000 \qquad \omega = 0.1 \qquad t_{max} = 50 \qquad y(t=0) = 0. \tag{2}$$

   (a) Use a forward Euler scheme to solve (1) up to $t_{max} = 30$. How small do you have to take the time step Δt to avoid instability? What happens when you increase Δt from $\Delta t = 0.0005$ to $\Delta t = 0.001$ in steps of 0.0001? Plot $y(t)$ for a few representative values of Δt.

   (b) Implement the 2^{nd}-order backward-difference scheme to solve (1). For the first time step use the standard backward Euler scheme. For (1) you have the analytical expression for the Jacobian, which in this case is simply given by the derivative of the function the zero of which you are looking for. As stopping condition for the Newton iteration for the nonlinear equation require that successive approximations for the zero of the function differ by less than tol. Perform a convergence study with $t_{max} = 50$ and decreasing Δt from $\Delta t = 10$ to $\Delta t = 0.01$. Compare the convergence for $tol = 10^{-8}$ with that for $tol = 10^{-3}$.

2. Now consider the Hodgkin-Huxley model, which is discussed in the class notes. There the differential equations for the voltage V, the activation 'particles' n and m as well as for the inactivation 'particle' h are given. On the class web site is a file $F.m$ that defines the differential equations.

   (a) Write a Forward-Euler code with fixed time step to solve the Hodgkin-Huxley model. As initial conditions use

$$V(t=0) = 0.0039 \qquad n(t=0) = 0.32 \qquad m(t=0) = 0.053 \qquad h(t=0) = 0.6.$$

   Run the code to $t_{max} = 50$ with injection current $I_{inj} = 0$ and $I_{inj} = 2.5$. Determine the stability limit for this code (i.e. increase the time step from$\Delta t = 0.05$ until the code becomes unstable).

   (b) Write a Backward-Euler code for the same problem and test its stability with increasing time steps as you did for the Forward-Euler code (for $I_{inj} = 0$ use $t_{max} = 100$. What happens when Δt becomes large?
   For the backward Euler code you need 3 m-functions

i. G_BE.m, which defines the function $G(u) = -u + y_n + \Delta t\, F(u, I_{inj})$, the zero of which you need to find by Newton iteration in each time step.

ii. dG_BE.m, which computes the Jacobian of $G(u)$ by evaluating G with slightly different arguments. That function is available on the class web site.

iii. BE.m, which performs the Newton iteration to find the 0 of $G(u)$ for a given time step Δt. Stop the iteration when the difference between successive iterations $u^{n+1} - u^n$ is smaller than 10^{-12} (do you notice much of a difference when you increase this tolerance to 10^{-3}?).

(c) As you noticed in part 2b, the large time steps that the stability of the backward Euler code allows are only useful when the solution changes slowly, but not during the spike. Therefore incorporate now the adaptive time-stepping from the previous assignment. Make use of the fact that you generate a solution with Δt and with $\Delta t/2$ and use Richardson extrapolation in each time step.
For $t_{max} = 2$ and $I_{inj} = 2.5$ perform a convergence study by decreasing the tolerance tol for the adaptive time step from $tol = 10$ by factors of $\sqrt{10}$ to $tol = 10^{-4.5}$ and plot the magnitude of the difference in $V(t_{max})$ for successive approximations as a function of Δt_{min} and Δt_{max} double-logarithmically. For $tol = 1$ plot the solution $Y \equiv [V(t), n(t), m(t), h(t)]$ as well as the time step $\Delta t(t)$ for $t_{max} = 20$.

(d) Use your adaptive backward Euler code to explore the behavior of this model neuron. As tolerance it is sufficient to take $tol = 10$.

i. Measure the threshold I_{inj}^{thresh} in the injected current for the generation of a spike with an accuracy of $\Delta I_{inj} = \pm 0.01$ by decreasing the injected current starting from $I_{inj} = 2.5$ with $t_{max} = 10$.

ii. Measure the frequency ω of the oscillations, which is given by $\omega = 2\pi/T$ with T being the time between successive spikes (ISI=interspike interval), as a function of the injection current in the range $10 \leq I_{inj} \leq 50$ in steps of 10. Include in addition also the frequency for $I_{inj} = 6.5$ and for $I_{inj} = 7$. For this measurement run the model to $t_{max} = 100$ and read the time between the last and the 4^{th} spike from the end off the graph (use the *data cursor* in the Matlab plot).