Methods of Nonlinear Analysis
Fall 1999
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Take-Home Final

Due Friday December 10, 1999

1. Consider the dynamical system
$$\partial_t \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -x-y-z+\epsilon x - yz \\ -x-y-z+\epsilon y - xz \\ -x-y-z+\epsilon z - xy \end{pmatrix} \tag{1}$$
for $|\epsilon| \ll 1$.
a) Disucss the stability of the fixed point $x=y=z=0$ and determine all eigenvalues whose real part changes sign as ϵ goes through 0. What general type of bifurcation do you expect to occur?
b) Determine a center manifold for (1) to second order in the amplitudes and the evolution equation on that manifold to cubic order. Make use of your expectation for the bifurcation to identify a suitable scaling for ϵ with the amplitudes.
c) To leading order in ϵ, determine all steady-state solutions of the center manifold equations.

2. Consider a general dynamical system the linear part of which is given by
$$\partial_t \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{2}$$
Determine the normal form of the evolution equations on the corresponding center manifold to cubic order in the amplitudes. Note that ω is a fixed parameter, i.e. the bifurcation parameter(s) are chosen exactly at the bifurcation.

3. Consider the modified Swift-Hohenberg model
$$\partial_t \psi = \mathcal{R}\psi - (\partial_x^2 + 1)^2 \psi + \alpha \psi^2 - \psi^3 \tag{3}$$
for convection. The additional quadratic term models the effect of a temperature dependence of the expansion coefficient or of the viscosity (non-Boussinesq effects). In the absence of these effects the fluid equations would lead to the additional

symmetry $\psi \to -\psi$ in (3).
a) For simplicity assume the boundary conditions are periodic and the system has a length of 2π. Identify the symmetries of (3) and the group Γ they form. Consider a weakly nonlinear description of the steady bifurcation occuring at $R = 0$. Give the action of the symmetries on the amplitude of the bifurcating mode.
b) Use symmetry arguments to write down the most general evolution equation for this bifurcation, i.e. give the general amplitude equation that is equivariant under this action of Γ.
c) Determine the coefficients in the cubic truncation of the amplitude equation using multiple (time) scales. Assume in this calculation that α is of the order of the amplitude. Is the bifurcation forward or backward? Does this depend on α?

4. Consider the two-dimensional version of the modified Swift-Hohenberg model of problem 3,

$$\partial_t \psi = \mathcal{R}\psi - (\nabla^2 + 1)^2 \psi + \alpha \psi^2 - \psi^3. \tag{4}$$

The goal of this (long) problem is to study the competition between rolls and hexagons in two-dimensional systems. You may want to consult Golubitsky, Stewart and Schaeffer (case study 4) for this problem.
Consider the weakly nonlinear theory of the steady bifurcation on a hexagonal lattice, i.e. take as the linear part of the expansion

$$\psi = Ae^{ix} + Be^{i(-x/2+y\sqrt{3}/2)} + Ce^{i(-x/2-y\sqrt{3}/2)} + c.c. + h.o.t. \tag{5}$$

with A, B, and C being complex amplitudes. [1]
a) In addition to the translation symmetries the two-dimensional system has rotation symmetry for arbitrary rotation angles. On a hexagonal lattice this induces a symmetry with respect to rotations by 60^o. Show that the (discrete) rotation and reflection symmetries lead to the following action of $\mathbf{D_6}$ on $\mathbf{C^3}$

$$\begin{aligned} R\,(A,B,C) &= (B,C,A), & (6)\\ \kappa\,(A,B,C) &= (A,C,B), & (7)\\ \rho\,(A,B,C) &= (A^*,B^*,C^*). & (8) \end{aligned}$$

b) In order to identify all isotropy subgroups it is useful to identify first all orbit types of the action. Show that for the identification of all isotropy subgroups you only need to consider points in $\mathbf{C^3}$ of the form

$$(a,\,b,\,c\,e^{i\gamma}) \qquad \text{with } a,b,c \in \mathbf{R} \text{ and } \mathbf{0} \le \gamma \le \pi. \tag{9}$$

[1] The boundary conditions are assumed periodic in two dimensions with lengths chosen suitably so as to allow a perfect hexagon pattern.

Calculate all isotropy subgroups that have *one-dimensional* fixed-point subspaces and give their fixed-point subspaces. Show that there are only two of them.
c) Show that the polynomials

$$u_j = |z_j|^2, \qquad q = z_1 z_2 z_3 + c.c. \tag{10}$$

and

$$\sigma_1 = u_1 + u_2 + u_3, \qquad \sigma_2 = u_1 u_2 + u_1 u_3 + u_2 u_3, \qquad \sigma_3 = u_1 u_2 u_3, \tag{11}$$

are invariant under Γ with $z_1 = A$, $z_2 = B$, and $z_3 = C$.
d) Show that the vectors (generators)

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}, \qquad \begin{pmatrix} u_1 z_1 \\ u_2 z_2 \\ u_3 z_3 \end{pmatrix}, \qquad \begin{pmatrix} z_2^* z_3^* \\ z_3^* z_1^* \\ z_1^* z_2^* \end{pmatrix} \tag{12}$$

are equivariant under Γ[2].
e) Use part c) and d) to give the most general amplitude equations to cubic order in the amplitudes. To simplify further calculations you can rescale time and the amplitudes to scale the quadratic and one cubic coefficient to 1. Find the two types of steady solutions that are guaranteed by the equivariant branching lemma. What is the type of bifurcation that gives rise to these solutions? Is it the same for both solutions?
f) Determine the stability of the two types of steady solutions found in e). To do so, in each case perform an isotypic decomposition of $\mathbf{C^3}$ with respect to the isotropy subgroup of the solution. This will allow you to reduce the 6x6 stability problem to solving at most 2x2 matrices. Show that hexagons and rolls are unstable at onset, but that hexagons become stable for larger amplitudes. Sketch a typical bifurcation diagram.
g) In the generic case the equations are not valid for the amplitude at which the hexagons become stable. In order for the equations to capture this stabilization the coefficient of the quadratic term (*before* the rescaling in e)) has to be small (of the order of the amplitudes). Use symmetry arguments to show that for small α in (4) the quadratic term will be small. Thus, one way to obtain the stabilization of the hexagons more rigorously is to consider the bifurcation with additional reflection symmetry $\psi \to -\psi$ and *unfold* that singularity allowing also for small terms that break that reflection symmetry[3].

[2]If amplitude equations beyond the cubic truncation are of interest, additional generators are needed.
[3]Part g) entails almost no work. It is meant to provide more context for the problem.