Methods of Nonlinear Analysis D-12-1

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Homework Assignment 1

due Monday, October 2, 1999

1. **Eigenspaces of Fixed Points**

   (a) Consider the general dynamical system

   $$\underline{u} = \mathcal{D}\underline{u} + \mathcal{N}(\underline{u}) \tag{1}$$

   where $\underline{u}$, $\mathcal{D}$, and $\mathcal{N}$ are real. If $\mathcal{D}$ has an unrepeated pair of complex eigenvalues λ and λ^*, give a basis for the real eigenspace associated with these eigenvalues in terms of their complex eigenvectors. Show that the vectors $\underline{u}$ in this eigenspace satisfy the equation

   $$(\mathcal{D} - \lambda\mathcal{I})(\mathcal{D} - \lambda^*\mathcal{I})\underline{u} = 0 \tag{2}$$

   where $\mathcal{I}$ is the identify matrix.

   (b) Consider now the three-dimensional dynamical system

   $$\partial_t x = \frac{1}{2}x + (\frac{1}{2} - \alpha)\, y + z - x^3, \tag{3}$$
   $$\partial_t y = (\frac{1}{2} - \alpha)\, x + \frac{1}{2}\, y + z + x^2 - y^3, \tag{4}$$
   $$\partial_t z = -x - y + (1 - \alpha)\, z - z^3. \tag{5}$$

   Determine the stable, unstable, and center eigenspaces of the fixed point $(0, 0, 0)$ as a function of α. (Technical hint: one eigenvalue σ of $\mathcal{D}$ is given by $\sigma = \alpha$.)

2. **Trajectories in a Simple Dynamical System**

   Consider the dynamical system

   $$\partial_t x = -\alpha x - x^2, \tag{6}$$
   $$\partial_t y = -\beta y \tag{7}$$

   with β positive.

   (a) Determine the solution $(x(t), y(t))$ of $(6, 7)$ and the trajectory $y = y(x)$ for $\alpha > 0$.

   (b) Determine the solution $(x(t), y(t))$ of $(6, 7)$ and the trajectory $y = y(x)$ for $\alpha = 0$.

   (c) Compare the trajectories in the two cases. In particular, comment on the effect of the initial conditions on the trajectories as they approach the fixed point $(0, 0)$.

   (d) The case $\alpha = 0$ demonstrates the non-uniqueness of the center manifold. Comment on the distance between the different center manifolds and whether they can be distinguished in a power series expansion around the fixed point $(0, 0)$.

3. **Numerical Exploration**

   Consider the dynamical system (cf. Guckenheimer and Holmes, Chap.7.3)

   $$\partial_t x = y, \tag{8}$$
   $$\partial_t y = \alpha x + \beta y + a x^3 - x^2 y. \tag{9}$$

   Solve this system numerically. To do so you can write your own numerical code or you can use a software package like `dstool`, which provides you with very useful graphical displays.

(a) Determine the stability of the fixed point $(x, y) = (0, 0)$ as a function of α and β. Based on this information, what type of behavior may be expected as α and/or β is varied?

(b) Consider the case $a = -1$. For $\beta = -1$ vary α across the instability identified above. What kind of nonlinear behavior do you find?
For $\alpha = 0$ and $\beta = -1$ identify the center manifold of the attractor graphically by starting from a number of different initial conditions. For a few initial conditions evolve the system until the distance of (x, y) from the fixed point $(0, 0)$ is smaller than 0.01.
Increase now α to 0.05, 0.1, and 1. What happens to the center manifold?

(c) Consider now the case $a = +1$. Choose $\alpha = -1$ and increase β across the instability identified above. What happens?
Measure the amplitude $\mathcal{A}$ and the frequency ω of the oscillations as a function of β, increasing β up to $\beta = 0.2$. Is there a range in β over which the dependence of $\mathcal{A}$ and ω on β agrees with your expectation for such a bifurcation?
What happens for $\beta > 0.2$?