Methods of Nonlinear Analysis D-12-1

Fall 1999

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Homework Assignment 4

due Monday, November 1, 1999

1. **Normal-Form Transformation for the Brusselator**
   Consider again the Brusselator model for a chemical oscillatory reaction

$$\partial_t u = \partial_x^2 u + A - (B+1)\,u + u^2 v, \tag{1}$$
$$\partial_t v = D\partial_x^2 v + Bu - u^2 v \tag{2}$$

   for the concentrations u and v.

   (a) Perform a center manifold reduction of this system in the case of a Hopf bifurcation (at vanishing wavenumber).

   (b) For $A = 1$ use a near-identity transformation of the complex amplitude of the oscillations to bring the equation that describes the dynamics on the center manifold into its normal form. How does the result compare with that of the multiple-scale approach?

2. **Normal Form for Takens-Bogdanov Bifurcation**
   Consider a dynamical system of the form

$$\partial_t x = y + F(x,y), \tag{3}$$
$$\partial_t y = G(x,y) \tag{4}$$

   Thus, at the bifurcation point the linear part $\mathbf{L}$ consists of a Jordan block. Assume that the (strictly) nonlinear terms $F(x,y)$ and $G(x,y)$ are odd under reflections in x and in y, i.e. their lowest order terms are of cubic order (cf. problem 3 in HW 1, Guckenheimer & Holmes Ch. 7.2, Wiggins Example 2.2.2).

   (a) Determine the range of $\mathcal{L}$ in $\mathcal{H}^{(3)} \equiv \{(u(x,y), v(x,y))^t | (u(ax,ay), v(ax,ay))^t = a^3 (u(x,y), v(x,y))^t\}$ and use it to identify the nonlinear terms that can be removed by a near-identity transformation. Give a basis for a vector space that is complementary to the range of $\mathcal{L}$, i.e. determine which type of terms cannot be removed. Note that this complementary subspace is not unique: by adding suitable multiples of vectors from the range of $\mathcal{L}$ to the basis of the complementary space, other complementary subspaces can be obtained. Show that one such choice leads to

$$\partial_t x = y, \tag{5}$$
$$\partial_t y = ax^3 + bx^2 y. \tag{6}$$

   (b) Find a choice for the complementary subspace that is invariant under the transformation $\exp(\mathbf{L}^t s)$, i.e. each of the vectors $\mathbf{V} \equiv (u,v)^t$ in this subspace satisfies

$$e^{-\mathbf{L}^t s}\,\mathbf{V}(e^{\mathbf{L}^t s}(x,y)^t) = \mathbf{V}((x,y)^t). \tag{7}$$

3. **Squares and Stripes in a SH-Model**

Consider the modified Swift-Hohenberg model

$$\partial_t \psi = R\psi - (\nabla^2 + 1)^2 \psi - \frac{a}{3}\psi^3 + b\nabla \cdot \left(\nabla\psi\, (\nabla\psi)^2\right). \tag{8}$$

It is a modification of a model derived by Gertsberg and Sivashinsky [**?**] for the description of convection with poorly conducting boundary. With such boundary conditions the critical wavenumber for the onset of convection becomes very small and a systematic expansion in the small wavenumber can be performed. A detailed analysis of the possible planforms in an equation of this type has been performed in [**?**].

(a) Use the method of your choice to derive coupled amplitude equations for the amplitudes of rolls in the x- and in the y-direction. Allow the wavevector (q, p) of the modes to differ slightly from the critical wavevector, $(q, p) = (1 + \epsilon Q, 0)$ and $(q, p) = (0, 1 + \epsilon P)$, respectively.

(b) Calculate all fixed points of the resulting amplitude equations. Determine their stability. What kind of patterns do the fixed points represent?

(c) As a function of a and b, determine all possible phase diagrams, i.e. delineate the regions of existence and of stability of all fixed points in the R-Q-plane with $P = 1$ fixed.

(d) Sketch the flow in the phase plane in each of the regions identified in 3c).

(e) Can any persistent dynamics arise in eq.(8)?