Methods of Nonlinear Analysis D-12

Fall 1999

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Homework Assignment 7

Due Monday November 29, 1999.

1. Consider a steady bifurcation to a periodic pattern in a one-dimensional $2\pi-$ periodic system in the presence of a (weak) spatially periodic forcing that breaks the continuous translation symmetry rendering the system D_n-invariant rather than $O(2)-$ invariant. Here D_n acts on $\mathbf{C}$ as

$$\kappa z = z^*, \qquad \zeta z = e^{i2\pi/n} z,$$

where z is the amplitude of the pattern. As shown in class the isotropy subgroups of D_n with this action are $\mathbf{Z}_2(\kappa)$, $\mathbf{Z}_2(\zeta\kappa)$, and $\mathbf{I}$. Show that for n even at most either the solution that is invariant under κ or the solution that is invariant under $\zeta\kappa$ is stable. (Note: large parts of this calculation have been done in class already. This is meant as a suggestion to fill in the blanks.)

2. Given a $\Gamma-$ equivariant dynamical system

$$\frac{dx}{dt} = g(x),$$

i.e.

$$\gamma\, g(x) = g(\gamma\, x) \qquad \text{for all} \quad \gamma \in \Gamma \qquad \text{and} \quad \text{all} \quad x \in V \tag{1}$$

with $V = \mathbf{R}^n$ or $V = \mathbf{C}^n$. Let x_0 be a fixed point of the system,

$$g(x_0) = 0,$$

and consider the linearization $dg(x_0)$ of g around x_0. Expand g suitably and take a suitable derivative of (**??**) to show that if the isotropy subgroup of x_0 is given by Σ_{x_0}, then

$$\sigma\, dg(x_0)x = dg(x_0)\,\sigma x \qquad \text{for} \quad \text{all} \quad \sigma \in \Sigma_{z_0} \qquad \text{and} \quad \text{all} \quad x \in V,$$

i.e. the Jacobian of g at x_0 commutes with all elements of the isotropy subgroup of x_0. As discussed earlier in class, this implies that the Jacobian is block-diagonal with respect to the isotypic decomposition of V.

3. Fill in the missing pieces for the $O(2)-$ Hopf bifurcation. Consider a system that is reflection-symmetric and is translationally invariant undergoing a Hopf bifurcation. Thus, you consider a bifurcation problem in $\mathbf{C}^2$ that is equivariant with $O(2)$ with the action defined in class.

   a) Show that any function $f(A,B)$ that is invariant under this action of $O(2)$ can be written in the form

$$f(A,B) = f(N,\Delta) \qquad \text{with} \qquad N = |A|^2 + |B|^2, \qquad \Delta = (|A|^2 - |B|^2)^2.$$

   Show that any equivariant function $g(A,B)$ can be written as

$$g(A,B) = \left(A\left(p + iq + (r+is)\delta\right), B\left(p + iq - (r+is)\delta\right)\right)$$

   where $\delta = |B|^2 - |A|^2$ and p, q, r, and s are invariant functions of A and B. Use this result to write down the most general coupled differential equations for A and B that are equivariant under this action of $O(2)$. Give also its cubic trunction, i.e. up to terms of $O(A^3, B^3, ...)$.

   b) Determine the stability of the traveling-wave solutions $(a,0)$ and of the standing-wave solutions (a,a) and show that their stability is mutually exclusive, i.e. at most one of the two can be stable. Under which condition are both of them unstable?

   Note: this calculation follows very closely Golubitsky et al. You may therefore find it useful to look at the page I put on the web.

4. In the $O(2)-$Hopf bifurcation the standing-wave and the traveling-wave solutions are not isolated solutions but represent a 2- and a 1-parameter family of solutions, respectively. All the solutions in such a family are on the same group orbit. Use the fact that the respective groups involved are continuous to show that the traveling-wave solutions (as a solution in $\mathbf{C}^2$) has one 0 eigenvalue and the standing-wave solutions has two 0 eigenvalues. To obtain the corresponding eigenvectors consider the derivative of the solution with respect to the continuous parameters characterizing the group elements in this group. Another way to view this situation is to ask about the restoring "force" that would bring back a solution that has been shifted along the group orbit.