two at some length since they have an infinite family of irreducible representations. These geometric ideas are applied in Case Study 4 to the problem of Bénard convection in the plane.

*Algebra (Chapters XIV–XV)*. Chapter XIV sets up equivariant singularity theory, concentrating on the recognition problem, tangent spaces, and intrinsic ideals, by analogy with Chapter II. Similarly Chapter XV develops unfolding theory, by analogy with Chapter III, and includes proofs of the main theorems, promised from Volume I. The ideas are illustrated using the dihedral group $\mathbf{D}_3$ (the symmetry group of an equilateral triangle) and its relation to the spherical Bénard problem via spherical harmonics of order 2. Case Study 5 shows how to apply the algebraic methods to the traction problem for an elastic cube, continuing the analysis outlined previously in §2.

*Hopf Bifurcation (Chapters XVI–XVIII)*. At this stage the theory moves away from static bifurcation and begins to acquire dynamic aspects. Chapter XVI develops a general theory of equivariant Hopf bifurcation, concentrating on existence and stability results. Chapter XVII applies this methodology to Hopf bifurcation with circular symmetry (the group $\mathbf{O}(2)$), considering both the generic case and nonlinear degeneracies (by a trick: reduction to amplitude equations). Quasi-periodic motion on a torus occurs here. The closely related groups $\mathbf{SO}(2)$, $\mathbf{Z}_n$, and $\mathbf{O}(n)$ (acting on $\mathbb{R}^n$) are also discussed. More complicated examples are dealt with in Chapter XVIII. Hopf bifurcation with dihedral group symmetry $\mathbf{D}_n$ is studied in detail and applied to oscillators coupled in a ring. Hopf bifurcation with spherical symmetry and Hopf bifurcation on the hexagonal lattice (relevant to doubly diffusive systems) are sketched.

*Mode Interactions (Chapters XIX–XX)*. Chapter XIX discusses mode interactions without any prescribed symmetry, concentrating on the steady-state/Hopf and Hopf/Hopf cases. Because of the natural $\mathbf{S}^1$ symmetry of Hopf bifurcation, these problems acquire $\mathbf{Z}_2$ and $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ symmetry during the analysis (rendering results from Volume I applicable). Finally Chapter XX considers mode interactions with $\mathbf{O}(2)$ symmetry.

The results are applied to Taylor–Couette flow in long cylinders (i.e., subject to periodic boundary conditions) in Case Study 6, which brings together virtually all of the ideas developed in this volume. The outcome is a coherent description, in symmetry terms, of some of the prechaotic behavior observed in this much-studied experiment.

CHAPTER XII

# Group-Theoretic Preliminaries

## §0. Introduction

The basic theme of this volume is that the symmetries of bifurcating systems impose strong restrictions on the form of their solutions and the way in which the bifurcation may take place. There are two major subthemes, which we might term "geometric" and "algebraic." These lead us to introduce two pieces of mathematical machinery: group representation theory and equivariant singularity theory. The aim of this chapter is to describe, in a fairly concrete fashion, the requisite mathematical background. In this manner we hope to make the methods accessible to a wide audience.

A symmetry of a system $\mathscr{X}$ is a transformation of $\mathscr{X}$ that preserves some particular structure. The set $\Gamma$ of all such transformations has seveal pleasant properties, which can be summarized by saying that $\Gamma$ is a group. In this book $\mathscr{X}$ is a real vector space $\mathbb{R}^n$, the transformations are linear mappings $\gamma: \mathbb{R}^n \to \mathbb{R}^n$, and the structure to be preserved is a particular bifurcation problem. For example, consider the static bifurcation problem

$$g(x, \lambda) = 0 \tag{0.1}$$

where $g: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is a smooth ($C^\infty$) mapping, $\lambda$ being the bifurcation parameter. By "preserved" we mean that for all $\gamma \in \Gamma$

$$g(\gamma x, \lambda) = \gamma g(x, \lambda) \tag{0.2}$$

so that every $\gamma \in \Gamma$ commutes with $g$. It follows that $x$ is a solution if and only if $\gamma x$ is, so the *solution set* to $g = 0$ is preserved by the symmetries $\gamma$. The "geometric" subtheme is the study of how $\Gamma$ transforms $\mathbb{R}^n$; the "algebraic" subtheme deals with the use of (0.2) to restrict the form of $g$.

We therefore begin in §1a with some group theory. We introduce the idea of a Lie group $\Gamma$ acting on a space $\mathbb{R}^n$ and describe fundamental examples including the orthogonal group $\mathbf{O}(n)$, the circle group $\mathbf{S}^1$, the dihedral group $\mathbf{D}_n$ and the $n$-torus $\mathbf{T}^n$. A given abstract group $\Gamma$ can act as transformations of a space in many ways; this is discussed in §1b and leads to the ideas of an *action* and a *representation* of $\Gamma$. These are two slightly different ways of looking at the same basic idea: a group of $n \times n$ matrices that is isomorphic, as an abstract group, to $\Gamma$.

A Lie group has topological as well as algebraic properties, and the important ones for this book are compactness and, to a lesser extent, connectedness. The representation theory of a *compact* Lie group is especially well understood, and we shall confine attention throughout to the compact case. (Every finite group is compact, and so are $\mathbf{O}(n)$, $\mathbf{S}^1$, and $\mathbf{T}^n$.) In §1c we discuss the existence on a compact Lie group of an invariant (Haar) integral, which is important in a number of situations because it allows us to average over the group. For example, it permits us to assume that $\Gamma$ acts by orthogonal transformations of $\mathbb{R}^n$.

In §2 we describe the decomposition of a given representation into simpler ones, called *irreducible* representations. In fact, if $\Gamma$ is a compact Lie group acting on $V = \mathbb{R}^n$, then we can write $V$ as a direct sum

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_s$$

of subspaces $V_j$, each invariant under $\Gamma$, such that $V_j$ has no $\Gamma$-invariant subspaces other than $\{0\}$ and $V_j$. These "irreducible components" of $V$ are the fundamental building blocks of representation theory. The process of decomposing $V$ is in a sense analogous to that of diagonalizing a matrix and is done for the same purpose—to simplify the mathematics.

In §3 we discuss *linear* maps $\mathbb{R}^n \to \mathbb{R}^n$ that commute with an action of $\Gamma$. This discussion has important implications for bifurcation problems (0.1) that satisfy (0.2), because the linearization $(dg)_0$ must commute with $\Gamma$. Two main points are made. The first is that there is a notion stronger than irreducibility, *absolute irreducibility*, which ensures that only scalar multiples of the identity commute with $\Gamma$. The second is that certain uniquely defined subspaces must be invariant under any mapping that commutes with $\Gamma$. We shall use these ideas to restrict the form of $(dg)_0$.

§§4–6 develop the "algebraic" subtheme. In §4 we consider *invariant* functions $f\colon \mathbb{R}^n \to \mathbb{R}$, that is, functions such that

$$f(\gamma x) = f(x), \qquad (x \in \mathbb{R}^n, \gamma \in \Gamma).$$

There are two main results. The first, due to Hilbert and Weyl, states that (when $\Gamma$ is compact) the *polynomial* invariants are generated by a finite set of polynomials $u_1, \ldots, u_s$. The second, due to Schwarz, states that every *smooth* invariant $f$ is of the form $h(u_1, \ldots, u_s)$ for a smooth function $h$. We give examples for the main groups of interest: the proofs are postponed until §6.

In §5 we describe analogous results, due to Poénaru, for *equivariant* mappings, that is, mappings $g\colon \mathbb{R}^n \to \mathbb{R}^n$ that commute with $\Gamma$ as in (0.2). We emphasize the simple but crucial fact that if $f(x)$ is invariant and $k(x)$ is equivariant, then $f(x)k(x)$ is also equivariant. In more abstract language, the space $\vec{\mathscr{E}}(\Gamma)$ of equivariant mappings is a module over the ring $\mathscr{E}(\Gamma)$ of invariant functions. These results are needed in Chapters XIV–XV to set up equivariant singularity theory.

In §6 we discuss the proofs of four theorems from §§4–5: the Hilbert–Weyl theorem, Schwarz's theorem, and their equivariant analogues. This section may be omitted if desired.

In §7 we return to group theory and present three results about torus groups which will be needed in Chapters XIX–XX on mode interactions. This section may be omitted on first reading.



## §1. Group Theory

In order to make precise statements about symmetries, the language and point of view of group theory are indispensable. In this section and the next we present some basic facts about Lie groups. We assume that the reader is familiar with elementary group-theoretic concepts such as subgroups, normal subgroups, conjugacy, homomorphisms, and quotient (or factor) groups. We also assume familiarity with elementary topological concepts in $\mathbb{R}^n$ such as open, compact, and connected sets. See Richtmeyer [1978]. Fortunately we do not require the deeper results from the theory of Lie groups, so the material presented here should prove reasonably tractable. We have adopted a fairly concrete point of view in the hope that this will make the ideas more accessible to readers having only a nodding acquaintance with modern algebra.

We treat three main topics in this section. The first consists of basic definitions and examples. The second is the beginnings of representation theory. The third is the existence of an invariant integral, allowing us to employ averaging arguments which in particular let us identify any representation of a compact Lie group with a group of orthogonal transformations.

### (a) Lie Groups

Let $\mathbf{GL}(n)$ denote the group of all invertible linear transformations of the vector space $\mathbb{R}^n$ into itself, or equivalently the group of nonsingular $n \times n$ matrices over $\mathbb{R}$. For our purposes we shall define a *Lie group* to be a closed subgroup $\Gamma$ of $\mathbf{GL}(n)$. In the literature these are called *linear Lie groups*, and the term *Lie group* is given a more general definition. However, it is a theorem that every compact Lie group in this more general sense is topologically isomorphic to a linear Lie group; see Bourbaki [1960]. By *closed* we mean the following. The space of all $n \times n$ matrices may be identified with $\mathbb{R}^{n^2}$, which contains $\mathbf{GL}(n)$ as an open subset. Then $\Gamma$ is a closed subgroup if it is a closed subset of $\mathbf{GL}(n)$ as well as a subgroup of $\mathbf{GL}(n)$. A *Lie subgroup* of $\Gamma$ is just a closed subgroup in the same sense.

By defining Lie groups as closed groups of matrices we avoid discussing some of their topological and differentiable structure. However, we often wish to refer to a Lie group by the name of its associated abstract group, a practice that is potentially confusing. For example, the two-element group $\mathbf{Z}_2 = \{\pm 1\}$ is isomorphic as an abstract group to the subgroup $\{I_n, -I_n\}$ of $\mathbf{GL}(n)$ for any $n$, where $I_n$ is the $n \times n$ identity matrix. Usually, the precise group of matrices in question will be specified by the context. We often use a phrase such as "$\mathbf{Z}_2$ is the Lie group $\{\pm I_2\}$" rather than the more precise but cumbersome phrase "$\mathbf{Z}_2$ is isomorphic to the Lie group $\{\pm I_2\}$." This practice should not cause confusion.

We now give some examples of Lie groups which will prove useful throughout the book.

EXAMPLES 1.1.

(a) The $n$-dimensional *orthogonal group* $\mathbf{O}(n)$ consists of all $n \times n$ matrices $A$ satisfying

$$AA^t = I_n.$$

Here $A^t$ is the transpose of $A$.

(b) The *special orthogonal group* $\mathbf{SO}(n)$ consists of all $A \in \mathbf{O}(n)$ such that $\det A = 1$. The group $\mathbf{SO}(n)$ is often called the $n$-dimensional *rotation group*. In particular $\mathbf{SO}(2)$ consists precisely of the planar rotations

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}. \tag{1.1}$$

In this way, $\mathbf{SO}(2)$ may be identified with the *circle group* $\mathbf{S}^1$, the identification being $R_\theta \mapsto \theta$. The group $\mathbf{O}(2)$ is generated by $\mathbf{SO}(2)$ together with the *flip*

$$\kappa = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{1.2}$$

(c) Let $\mathbf{Z}_n$ denote the *cyclic group* of order $n$. (Recall that the *order* of a finite group is the number of elements that it contains). We may identify $\mathbf{Z}_n$ with the group of $2 \times 2$ matrices generated by $R_{2\pi/n}$; thus $\mathbf{Z}_n$ is a Lie group.

(d) The *dihedral group* $\mathbf{D}_n$ of order $2n$ is generated by $\mathbf{Z}_n$, together with an element of order 2 that does not commute with $\mathbf{Z}_n$. For definiteness, we identify $\mathbf{D}_n$ with the group of $2 \times 2$ matrices generated by $R_{2\pi/n}$ and the flip $\kappa$ (1.2). This clearly exhibits $\mathbf{D}_n$ as a Lie group. Geometrically $\mathbf{D}_n$ is the symmetry group of the regular $n$-gon, whereas $\mathbf{Z}_n$ is the subgroup of rotational symmetries.

(e) All finite groups are isomorphic to Lie groups; see Exercise 1.2.

(f) The $n$-dimensional torus $\mathbf{T}^n = \mathbf{S}^1 \times \cdots \times \mathbf{S}^1$ ($n$ times) is isomorphic to a Lie group. To show this, identify $\theta \in \mathbf{T}^n$ with the matrix



$$\begin{bmatrix} R_{\theta_1} & 0 & 0 & \dots & 0 \\ 0 & R_{\theta_2} & 0 & \dots & 0 \\ 0 & 0 & R_{\theta_3} & \dots & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot \\ 0 & 0 & 0 & \dots & R_{\theta_n} \end{bmatrix}$$

in $\mathbf{GL}(2n)$.

(g) $\mathbb{R}^n$ is isomorphic to the group of matrices of the form

$$\begin{bmatrix} 1 & a_1 & a_2 & \dots & a_n \\ 0 & 1 & 0 & \dots & 0 \\ & & \dots & & \\ 0 & 0 & \dots & & 1 \end{bmatrix} \in \mathbf{GL}(n+1)$$

where $a_j \in \mathbb{R}$, $j = 1, \dots, n$.

It is important at the outset to eliminate one potential source of confusion. We have already seen that it is possible for a single abstract group to occur in more than one way as a group of matrices. The question that must be addressed is, when should two matrix groups which are isomorphic as abstract groups be considered as essentially the same? This question leads directly into representation theory and is dealt with in subsection (b). To illustrate what is involved, observe on the one hand that changing the basis in $\mathbb{R}^n$ will change the actual matrices that appear in a given Lie group—surely just a cosmetic change. On the other hand, consider the following two groups of matrices isomorphic to $\mathbf{Z}_2$:

$$\{I_2, -I_2\} \tag{1.3}$$

and

$$\left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\right\}. \tag{1.4}$$

There is a definite geometric distinction between (1.3), where the element of order 2 in $\mathbf{Z}_2$ is a rotation, and (1.4), where it is a reflection. Such a distinction is often important in the theory.

Because $\mathbb{R}^{n^2}$ is a topological space, we can talk about topological properties of Lie groups as well as algebraic ones. In particular we say that a Lie group $\Gamma$ is *compact* or *connected* if it is compact or connected as a subset of $\mathbb{R}^{n^2}$. Equivalently, $\Gamma$ is compact if and only if the entries in the matrices defining $\Gamma$ are bounded. It follows that $\mathbf{O}(n)$, $\mathbf{SO}(n)$, $\mathbf{T}^n$, and all finite groups are compact; but $\mathbb{R}^n$ and $\mathbf{GL}(n)$ are not. Compactness is crucial for much of the theory we develop here. It would be of great significance for applications to modify the theory so that it extends to suitable noncompact groups. For an example, see Case Study 4.

The identity element of $\Gamma$ is denoted by $I$. In fact, if $\Gamma \subset \mathbf{GL}(n)$ we must have $I = I_n$, the $n \times n$ identity matrix. The trivial group $\{I\} = \{I_n\}$ is denoted by $\mathbb{1}_n$, or more commonly by $\mathbb{1}$ when the size of matrix is clear from the context.

As a subset of $\mathbb{R}^{n^2}$, the group $\Gamma$ splits into connected components. The connected component that contains $I$ is denoted $\Gamma^0$. For example

$$\mathbf{O}(n)^0 = \mathbf{SO}(n).$$

Being a connected component, $\Gamma^0$ is a closed subset of $\Gamma$. Since $\Gamma$ is closed in $\mathbf{GL}(n)$, so is $\Gamma^0$. Thus $\Gamma^0$ is a Lie subgroup of $\Gamma$ and is compact if $\Gamma$ is. Moreover, $\Gamma^0$ is a normal subgroup of $\Gamma$. To see why, recall that $\Sigma \subset \Gamma$ is *normal* if for each $\gamma \in \Gamma$ we have $\Sigma = \gamma\Sigma\gamma^{-1}$ as a set of matrices. Now $\gamma\Gamma^0\gamma^{-1}$ is a connected component of $\Gamma$ since matrix multiplication is continuous, and it contains $\gamma I \gamma^{-1} = I$. Therefore, $\gamma\Gamma^0\gamma^{-1} = \Gamma^0$, so $\Gamma^0$ is normal.

It is not difficult (Exercise 1.3) to show that a compact Lie group $\Gamma$ has a finite number of connected components, and hence that $\Gamma/\Gamma^0$ is finite.

## (b) Representations and Actions

Let $\Gamma$ be a Lie group and let $V$ be a finite-dimensional real vector space. We say that $\Gamma$ *acts* (*linearly*) *on* $V$ if there is a continuous mapping (the *action*)

$$\begin{aligned} \Gamma \times V &\to V \\ (\gamma, v) &\mapsto \gamma \cdot v \end{aligned} \tag{1.5}$$

such that:

(a) For each $\gamma \in \Gamma$ the mapping $\rho_\gamma \colon V \to V$ defined by

$$\rho_\gamma(v) = \gamma \cdot v \tag{1.6}$$

is linear.

(b) If $\gamma_1, \gamma_2 \in \Gamma$ then

$$\gamma_1 \cdot (\gamma_2 \cdot v) = (\gamma_1\gamma_2) \cdot v. \tag{1.7}$$

The mapping $\rho$ that sends $\gamma$ to $\rho_\gamma \in \mathbf{GL}(V)$ is then called a *representation of* $\Gamma$ on $V$. Here $\mathbf{GL}(V)$ is the group of invertible linear transformations $V \to V$. By abuse of language we will also talk of "the representation $V$." In the sequel we shall often omit the dot and write $\gamma v$ for $\gamma \cdot v$, but for the remainder of this section we retain the dot for clarity. As illustrated shortly, linear actions and representations are essentially identical concepts, differing only in viewpoint. In fact, both (1.5) and $\rho$ must be analytic; see Montgomery and Zippin [1955].

For example, there is an action of the circle group $\mathbf{S}^1$ on $\mathbb{C} \equiv \mathbb{R}^2$ given by

$$\theta \cdot z = e^{i\theta} z \qquad (\theta \in \mathbf{S}^1,\ z \in \mathbb{C}).$$

We verify that this is an action. Clearly (a) holds. To check (b), calculate



$$\theta_1 \cdot (\theta_2 \cdot z) = \theta_1 \cdot (e^{i\theta_2} z) = e^{i\theta_1} e^{i\theta_2} z = e^{i(\theta_1 + \theta_2)} z = (\theta_1 + \theta_2) \cdot z,$$

where by an accident of notation $\theta_1 + \theta_2$ is the "product" in the group $\mathbf{S}^1$. This action gives rise to a representation $\rho$ of $\mathbf{S}^1$ for which $\rho_\theta$ is the rotation matrix

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

on $\mathbb{R}^2 \equiv \mathbb{C}$. The difference in viewpoint is that an *action* tells us how a group element $\gamma$ transforms a given *element* $v \in V$, whereas a *representation* tells us how $\gamma$ transforms the entire *space* $V$. More technically, $\rho$ defines a homomorphism of $\Gamma$ into $\mathbf{GL}(V)$; see Exercise 1.4. An action of $\Gamma$ on $V$ may be defined by specifying (1.5) only on generators of $\Gamma$, as long as this action is consistent in the sense that (1.7) is satisfied.

Examples 1.2.

(a) Every linear Lie group $\Gamma$ is a group of matrices in $\mathbf{GL}(n)$ for some $n$. As such, $\Gamma$ has a natural action on $V = \mathbb{R}^n$ given by matrix multiplication.

(b) Every group $\Gamma$ has a *trivial* action on $V = \mathbb{R}^n$ defined by $\gamma \cdot x = x$ for all $x \in \mathbb{R}^n$, $\gamma \in \Gamma$.

(c) For every integer $k$ the circle group $\mathbf{S}^1$ has an action on $V = \mathbb{C} \equiv \mathbb{R}^2$ defined by

$$\theta \cdot z = e^{ik\theta} z. \tag{1.8}$$

Notice that $k = 0$ corresponds to the trivial action of example (b). The action for $k = 1$ is the one discussed previously in the text.

(d) Each action of $\mathbf{S}^1 = \mathbf{SO}(2)$ defined in (c) extends to an action of $\mathbf{O}(2)$ on $\mathbb{C}$ by letting

$$\kappa \cdot z = \bar{z} \tag{1.9}$$

where $\kappa$ is the flip (1.2).

(e) Each Lie group $\Gamma \subset \mathbf{GL}(n)$ acts on the space of $n \times n$ matrices $A$ by similarity: $\gamma \cdot A = \gamma A \gamma^{-1}$.

It is often possible to give two different descriptions of "the same" action. More precisely, the two actions may be isomorphic in the following sense. Let $V$ and $W$ be $n$-dimensional vector spaces and assume that the Lie group $\Gamma$ acts on both $V$ and $W$. Say that these actions are *isomorphic*, or that the spaces $V$ and $W$ are $\Gamma$-*isomorphic*, if there exists a (linear) isomorphism $A\colon V \to W$ such that

$$A(\gamma \cdot v) = \gamma \cdot (Av) \tag{1.10}$$

for all $v \in V$, $\gamma \in \Gamma$. Note that the action of $\gamma$ on the left-hand side of (1.10) is that on $V$, whereas on the right it is on $W$. Another way to say this is that we get the *same* group of matrices if we identify the *spaces* $V$ and $W$ (via the linear isomorphism $A$). To avoid cumbersome terminology we say that $V$ and $W$ are $\Gamma$-*isomorphic*. It is easy to extend these ideas to the case where $\Gamma$ acts on $V$ and a group $\Delta$, isomorphic to $\Gamma$, acts on $W$.

For example, the actions (1.8, 1.9) of $\mathbf{O}(2)$ for $k$ and $-k$ are isomorphic. To see why, denote the two actions by the symbols $\cdot$ and $*$. Define $A$ by $A(z) = \bar{z}$. Then for $\gamma \in \mathbf{SO}(2)$ we have

$$A(\gamma \cdot z) = \overline{e^{ik\theta}z} = e^{-ik\theta}\bar{z} = e^{-ik\theta}(Az) = \gamma * (Az),$$

and further

$$A(\kappa \cdot z) = \bar{\bar{z}} = z = \kappa * \bar{z} = \kappa * (Az),$$

so (1.10) holds.

In the same way, the groups $\mathbf{SO}(2)$ and $\mathbf{S}^1$ are isomorphic, and the action (1.8) of $\mathbf{S}^1$ on $\mathbb{C}$ with $k = 1$ is isomorphic to the standard action of $\mathbf{SO}(2)$ defined in Example 1.2a.

## (c) Invariant Integration

Every compact Lie group $\Gamma$ in $\mathbf{GL}(n)$ can be identified with a subgroup of the orthogonal group $\mathbf{O}(n)$. Since it is often useful to assume this, we sketch the proof. The identification is made using Haar integration, a form of integration that is invariant under translation by elements of $\Gamma$. In this subsection we define Haar integration, show how its existence leads to the identification of $\Gamma$ with a subgroup of $\mathbf{O}(n)$, and give explicit examples of Haar integration.

*Haar integration* may be defined abstractly as an operation that satisfies three properties. Let $f: \Gamma \to \mathbb{R}$ be a continuous real-valued function. The operation

$$\int_{\gamma \in \Gamma} f(\gamma) \quad \text{or} \quad \int_\Gamma f \quad \text{or} \quad \int_\Gamma f\, d\gamma \in \mathbb{R}$$

is an integral on $\Gamma$ if it satisfies the following two conditions:

(a) *Linearity*. $\int_\Gamma (\lambda f + \mu g) = \lambda \int_\Gamma f + \mu \int_\Gamma g$ where $f, g: \Gamma \to \mathbb{R}$ are continuous and $\lambda, \mu \in \mathbb{R}$. (1.11)
(b) *Positivity*. If $f(\gamma) \geq 0$ for all $\gamma \in \Gamma$ then $\int_\Gamma f \geq 0$.
It is a *Haar integral* if it also has the property
(c) *Translation-Invariance*. $\int_{\gamma \in \Gamma} f(\delta\gamma) = \int_{\gamma \in \Gamma} f(\gamma)$ for any fixed $\delta \in \Gamma$. (1.12)

The Haar integral can be proved to be unique. Because $\Gamma$ is compact, $\int_\Gamma 1$ is finite. We may therefore scale the Haar integral so that $\int_\Gamma 1 = 1$. This yields the *normalized Haar integral*. For compact groups the Haar integral is also invariant under right translations; i.e.,



$$\int_{\gamma \in \Gamma} f(\gamma\delta) = \int_{\gamma \in \Gamma} f(\gamma) \quad \text{for all } \delta \in \Gamma. \tag{1.13}$$

The proof of existence and uniqueness of the Haar integral is in Hochschild [1965], p. 9. Vector-valued mappings may also be integrated, by performing the integration separately on each component.

**Proposition 1.3.** *Let $\Gamma$ be a compact Lie group acting on a vector space $V$ and let $\rho_\gamma$ be the matrix associated with $\gamma \in \Gamma$. Then there exists an inner product on $V$ such that for all $\gamma \in \Gamma$, $\rho_\gamma$ is orthogonal.*

**Remark.** Proposition 1.3 implies that we may identify compact Lie groups in $\mathbf{GL}(n)$ with closed subgroups of $\mathbf{O}(n)$.

Proof. The idea is to use the Haar integral to construct an *invariant* inner product $\langle\ ,\ \rangle_\Gamma$ on $V$, that is, one that satisfies

$$\langle \rho_\delta v, \rho_\delta w \rangle_\Gamma = \langle v, w \rangle_\Gamma \tag{1.14}$$

for all $\delta \in \Gamma$. The construction proceeds as follows. Let $\langle\ ,\ \rangle$ be any inner product on $V$ and define

$$\langle v, w \rangle_\Gamma = \int_\Gamma \langle \rho_\gamma v, \rho_\gamma w \rangle. \tag{1.15}$$

This is also an inner product by (1.11). Invariance of the Haar integral (1.12) shows that the inner product (1.15) satisfies (1.14). □

Examples 1.4.
(a) Let $\Gamma$ be a finite Lie group of order $|\Gamma|$. Then the normalized Haar integral on $\Gamma$ is

$$\int_\Gamma f \equiv \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} f(\gamma). \tag{1.16}$$

(b) Let $\Gamma = \mathbf{SO}(2)$. Every continuous function $f: \mathbf{SO}(2) \to \mathbb{R}$ uniquely determines a continuous $2\pi$-periodic function $\tilde{f}: \mathbb{R} \to \mathbb{R}$ such that

$$\tilde{f}(\theta) = f(R_\theta).$$

The normalized Haar integral on $\mathbf{SO}(2)$ is

$$\int_\Gamma f \equiv \frac{1}{2\pi} \int_0^{2\pi} \tilde{f}(\theta)\, d\theta. \tag{1.17}$$

The abstract definition of the Haar integral that we have given is sufficient for our purposes, because we use it only as a tool to prove abstract results such as Proposition 1.3. There is, however, an explicit definition of the Haar integral that uses the manifold structure of Lie groups; see Exercise 1.8.

## Exercises

1.1. Two elements $\alpha, \beta$ of a Lie group $\Gamma$ are *conjugate* in $\Gamma$ if $\alpha = \gamma^{-1}\beta\gamma$ for some $\gamma \in \Gamma$. Show that all elements of $\mathbf{O}(2) \sim \mathbf{SO}(2)$ are conjugate in $\mathbf{O}(2)$.

1.2. Two subgroups $H, K$ of a Lie group $\Gamma$ are *conjugate* in $\Gamma$ if $H = \gamma^{-1}K\gamma$ for some $\gamma \in \Gamma$.
(a) Show that the closed subgroups of $\mathbf{O}(2)$ are conjugate to $\mathbf{SO}(2)$, $\mathbf{D}_n$, or $\mathbf{Z}_n$.
(b) Find up to conjugacy all subgroups of $\mathbf{D}_n$, $n \geq 3$. (*Hint*: consider separately the cases $n$ even, $n$ odd.)

1.3. Show that every finite group $G$ is isomorphic to a Lie group. (*Hint*: if $\gamma \in G$ then the map $\delta \mapsto \gamma\delta$ is a permutation of $G$. Now consider the corresponding permutation matrix.)

1.4. Show that every compact Lie group has a finite number of connected components.

1.5. Let $\rho: \Gamma \to \mathbf{GL}(V)$ be a representation of the group $\Gamma$ as defined by (1.6, 1.7).
(a) Show that $\rho$ is a group homomorphism.
(b) Show that $\ker \rho$ is a normal subgroup of $\Gamma$.

1.6. Let $\rho$ and $\sigma$ be representations of the Lie group $\Gamma$ on the same space $V$. Show that if $\rho$ and $\sigma$ are isomorphic then $\ker \rho = \ker \sigma$. Conclude that if $\ker \rho \neq \ker \sigma$ then $\rho$ and $\sigma$ are distinct.

1.7. Let $f: V \to \mathbb{R}$ be continuous, and let a compact Lie group $\Gamma$ act on $V$. Show that

$$\hat{f}(x) = \int_{\gamma \in \Gamma} f(\gamma x)$$

has the property that $\hat{f}(\gamma x) = \hat{f}(x)$ for all $\gamma \in \Gamma$.

1.8. (*Warning*: This exercise requires knowledge of the elementary theory of manifolds.) To define the Haar integral explicitly we must use the fact that every Lie group $\Gamma$ is a smooth manifold. Let $U$ be an open neighborhood of 0 in $\mathbb{R}^k$ where $k = \dim \Gamma$ and let $\mathscr{X}: U \to \Gamma$ be a smooth parametrization satisfying $\mathscr{X}(0) = 1$. Let $f: \Gamma \to \mathbb{R}$ be continuous with the support $\operatorname{supp}(f)$ of $f$ contained in $\mathscr{X}(U)$. Define $\int_\Gamma f$ as follows.

Let $L_\delta: \Gamma \to \Gamma$ be left translation by $\delta$; that is, $L_\delta(\gamma) = \delta\gamma$. For $\delta \in \mathscr{X}(U)$ the composition

$$\tilde{L}_\delta = \mathscr{X}^{-1} \circ L_\delta \circ \mathscr{X}$$

is a smooth mapping on a neighborhood of 0 in $\mathbb{R}^k$. Let

$$J(\delta) = \det(d\tilde{L}_\delta)_0.$$

Now define

$$\int_\Gamma f = \int_U f[\mathscr{X}(u)]J(u)^{-1}\,du. \tag{1.18}$$

Suppose that $\sigma \in \Gamma$ and $\sigma(\operatorname{supp}(f)) \subset \mathscr{X}(U)$. Show that

$$\int_U f[\sigma\mathscr{X}(u)]J(\sigma u)^{-1}\,du = \int_\Gamma f.$$

(*Comment*: When the Lie group $\Gamma$ has a single parametrization $\mathscr{X}$ such that

$$\overline{\mathscr{X}(U)} = \Gamma$$

then (1.18) defines a Haar integral on $\Gamma$ since $\Gamma \sim \mathscr{X}(U)$ has "measure zero" and $\int_\Gamma = \int_{\mathscr{X}(U)}$.)



# §2. Irreducibility

The study of a representation of a compact Lie group is often made easier by observing that it decomposes into a direct sum of simpler representations, which are said to be irreducible. We describe the basic properties of this decomposition in this section. The main result, Theorem 2.5, states that the decomposition always exists. In general it is not unique, but the sources of nonuniqueness can be described and controlled.

Let $\Gamma$ be a Lie group acting linearly on the vector space $V$. A subspace $W \subset V$ is called $\Gamma$-*invariant* if $\gamma w \in W$ for all $w \in W$, $\gamma \in \Gamma$. A representation or action of $\Gamma$ on $V$ is *irreducible* if the only $\Gamma$-invariant subspaces of $V$ are $\{0\}$ and $V$. A subspace $W \subset V$ is said to be $\Gamma$-*irreducible* (or *irreducible* if it is clear which group $\Gamma$ is intended) if $W$ is $\Gamma$-invariant and the action of $\Gamma$ on $W$ is irreducible. For example, the actions of $\mathbf{SO}(2)$ and $\mathbf{O}(2)$ on $\mathbb{R}^2$ defined in Example 1.2c, d are irreducible when $k \neq 0$.

One of the fundamental features of actions of compact Lie groups is that invariant subspaces always have invariant complements. More precisely:

**Proposition 2.1.** *Let $\Gamma$ be a compact Lie group acting on $V$. Let $W \subset V$ be a $\Gamma$-invariant subspace. Then there exists a $\Gamma$-invariant complementary subspace $Z \subset V$ such that*

$$V = W \oplus Z.$$

Proof. By Proposition 1.3 there exists a $\Gamma$-invariant inner product $\langle\ ,\ \rangle_\Gamma$ on $V$. Let $Z = W^\perp$ where

$$W^\perp = \{v \in V: \langle w, v\rangle_\Gamma = 0 \text{ for all } w \in W\}.$$

The $\Gamma$-invariance of the inner product implies that $W^\perp$ is a $\Gamma$-invariant complement to $W$. □

It follows directly from this proposition that every representation of a compact Lie group may be written as a direct sum of irreducible subspaces:

**Corollary 2.2** (Theorem of Complete Reducibility). *Let $\Gamma$ be a compact Lie group acting on $V$. Then there exist $\Gamma$-irreducible subspaces $V_1, \ldots, V_s$ of $V$ such that*

$$V = V_1 \oplus \cdots \oplus V_s. \tag{2.1}$$

**Proof.** We may assume $V$ nonzero. Then there exists a nonzero $\Gamma$-irreducible subspace $V_1 \subset V$ (take $V_1$ to be of minimal dimension among the nonzero $\Gamma$-invariant subspaces). By Proposition 2.1 there is a $\Gamma$-invariant complement $Z$ to $V_1$ in $V$. Now repeat the process on $Z$, choosing a nonzero $\Gamma$-invariant subspace $V_2 \subset V$. Since $V$ is finite-dimensional this process must terminate, yielding the desired decomposition (2.1). □

Some specific examples may help to clarify the implications of this result.

**Examples 2.3.**
(a) Define an action of $\mathbf{O}(2)$ on $\mathbb{R}^3$ as follows. Let the rotations $R_\theta \in \mathbf{SO}(2)$ act by rotating the $(x, y)$-plane through angle $2\theta$ and leaving the $z$-axis fixed: that is, define

$$\theta \cdot (x, y, z) = (x \cos 2\theta - y \sin 2\theta, x \sin 2\theta + y \cos 2\theta, z).$$

Let the flip $\kappa \in \mathbf{O}(2)$ act by

$$\kappa \cdot (x, y, z) = (x, -y, -z).$$

Observe that

$$V_1 = \mathbb{R}^2 \times \{0\} = \{(x, y, 0)\}$$
$$V_2 = \{0\} \times \mathbb{R} = \{(0, 0, z)\}$$

are $\mathbf{O}(2)$-invariant subspaces and that $\mathbf{O}(2)$ acts irreducibly on each.

(b) There is a standard irreducible action of $\mathbf{O}(3)$ on $\mathbb{R}^5$. Let $V$ be the vector space of symmetric $3 \times 3$ matrices of trace zero. Such matrices have the form

$$\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & -(a+d) \end{bmatrix}$$

so $\dim V = 5$. Define

$$\gamma \cdot A = \gamma^t A \gamma$$

for $\gamma \in \mathbf{O}(3)$ and $A \in V$. Thus $\mathbf{O}(3)$ acts on $V$ by similarity.

Next view $\mathbf{O}(2) \subset \mathbf{O}(3)$ as follows. Identify the matrix $\delta \in \mathbf{O}(2)$ with

$$\left[\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{$\delta$}} & 0 \\ & & 0 \\ \hline 0 & 0 & 1 \end{array}\right]$$

in $\mathbf{O}(3)$. In this way, we can view $\mathbf{O}(2)$ as acting on $V$. It is a straightforward calculation to show that

$$V_1 = \begin{bmatrix} a & b & 0 \\ b & -a & 0 \\ 0 & 0 & 0 \end{bmatrix},$$



$$V_2 = \begin{bmatrix} 0 & 0 & c \\ 0 & 0 & d \\ c & d & 0 \end{bmatrix}, \text{ and}$$

$$V_3 = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & -2a \end{bmatrix}$$

are invariant irreducible subspaces of $V$ under the action of $\mathbf{O}(2)$. Since $V_1 \oplus V_2 \oplus V_3 = V$ we have decomposed $V$ as in Corollary 2.2.

In general, the decomposition of $V$ in (2.1) is not unique. It will be useful in later sections to understand the sources of nonuniqueness and to find conditions under which the decomposition (2.1) *is* unique. In particular, such a discussion will simplify the computation of linearized asymptotic stability for solutions of differential equations. The remainder of this section is devoted to the issue of nonuniqueness, beginning with an example.

**Example 2.4.** Let $V$ be the four-dimensional space of $2 \times 2$ matrices and let $\mathbf{SO}(2)$ act on $V$ by matrix multiplication on the left. That is,

$$\theta \cdot A = R_\theta A$$

where $\theta \in \mathbf{SO}(2)$ and $A \in V$. Observe that $V = V_1 \oplus V_2$ where

$$V_1 = \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}, \quad V_2 = \begin{bmatrix} 0 & c \\ 0 & d \end{bmatrix},$$

and that $\mathbf{SO}(2)$ acts irreducibly on $V_1$ and $V_2$.

However, we also have $V = V_1 \oplus V_2'$, where (say)

$$V_2' = \begin{bmatrix} 8c & c \\ 8d & d \end{bmatrix}$$

and $\mathbf{SO}(2)$ acts irreducibly on $V_2'$.

It will turn out that the reason for nonuniqueness in the decomposition of Corollary 2.2 is the occurrence in $V$ of two isomorphic irreducible representations. Recall the definition (1.10) of $\Gamma$-isomorphism. We state this more precisely in Corollary 2.6 later. The main result of this section is as follows:

**Theorem 2.5.** *Let $\Gamma$ be a compact Lie group acting on $V$.*
*(a) Up to $\Gamma$-isomorphism there are a finite number of distinct $\Gamma$-irreducible subspaces of $V$. Call these $U_1, \ldots, U_t$.*
*(b) Define $W_k$ to be the sum of all $\Gamma$-irreducible subspaces $W$ of $V$ such that $W$ is $\Gamma$-isomorphic to $U_k$. Then*

$$V = W_1 \oplus \cdots \oplus W_t. \tag{2.2}$$

**Remark.** The subspaces $W_k$ are called the *isotypic components* of $V$, of type $U_k$, for the action of $\Gamma$. The name is chosen to reflect that fact that all irreducible subspaces of $W_k$ have the same isomorphism type. By construction the *isotypic decomposition* (2.2) is unique.

Before proving Theorem 2.5 we show how it implies that the nonuniqueness in the choice of irreducible summands in Proposition 2.1 is directly related to the repetition of irreducible representations among the $V_j$ in (2.1).

**Corollary 2.6.**
(a) *If $W \subset V$ is $\Gamma$-irreducible then $W \subset W_k$ for a unique $k$, namely, that $k$ for which $W$ is $\Gamma$-isomorphic to $U_k$.*
(b) *Let $\Gamma$ be a compact Lie group acting on $V$. Let $V = V_1 \oplus \cdots \oplus V_s$ be a decomposition of $V$ into a direct sum of $\Gamma$-invariant irreducible subspaces. If the representations of $\Gamma$ on the $V_j$ are all distinct (not $\Gamma$-isomorphic) then the only nonzero $\Gamma$-irreducible subspaces of $V$ are $V_1, \ldots, V_s$.*

Proof. Part (a) follows directly from Theorem 2.5 since if $W$ is $\Gamma$-irreducible then it is $\Gamma$-isomorphic to some unique $U_k$, and then by definition $W \subset W_k$. It is useful, however, to have (a) stated explicitly.

For (b), consider the isotypic components $W_k$ of $V$. Each $V_j$ is isomorphic to some $U_k$, hence lies in $W_k$ for some $k$. It follows that the $W_k$ are just the $V_j$, perhaps written in a different order. If $W \neq 0$ is a $\Gamma$-irreducible subspace of $V$ then by part (a) we have $W \subset W_k$ for some $k$. But $W_k = V_j$ for suitable $j$, and irreducibility of $V_j$ implies that $W = V_j$. □

The proof of Theorem 2.5 depends on two lemmas, which we deal with first.

**Lemma 2.7.** *Let $\Gamma$ be a compact Lie group acting on $W$. Suppose that*

$$W = \sum_\alpha U_\alpha$$

*where each $U_\alpha$ is a $\Gamma$-invariant subspace that is $\Gamma$-isomorphic to some fixed irreducible representation $U$ of $\Gamma$. Then every $\Gamma$-irreducible subspace of $W$ is $\Gamma$-isomorphic to $U$.*

**Remark.** Because of nonuniqueness, a $\Gamma$-irreducible subspace of $W$ may not be one of the $U_\alpha$. The lemma says that, provided all the $U_\alpha$ are $\Gamma$-isomorphic, every $\Gamma$-irreducible subspace of $W$ is $\Gamma$-isomorphic to any one of the $U_\alpha$.

Proof. Because of our intended application, the theorem is stated in a way that allows the index $\alpha$ to run over an infinite set. In fact this represents no real increase in generality, since we first show that

$$W = U_{\alpha_1} \oplus \cdots \oplus U_{\alpha_s}, \tag{2.3}$$

a *direct* sum of a finite subset of the $U_\alpha$. The proof is by induction. Suppose



we have found a subspace

$$W' = U_{\alpha_1} \oplus \cdots \oplus U_{\alpha_{t-1}} \subset W.$$

If $W' = W$ we are done. If not, some $U_{\alpha_t}$ is not contained in $W'$. Then $U_{\alpha_t} \cap W' \subset U_{\alpha_t}$ must be $\{0\}$ by irreducibility. Therefore the sum $W' + U_{\alpha_t}$ is direct and we have a subspace

$$W'' = U_{\alpha_1} \oplus \cdots \oplus U_{\alpha_t}.$$

By finiteness of dimension, (2.3) must hold for large enough $s$.

Now let $X$ be a $\Gamma$-irreducible subspace of $W$. There exists $t \leq s$ such that

$$X \not\subset U_{\alpha_1} \oplus \cdots \oplus U_{\alpha_{t-1}} \tag{2.4}$$

$$X \subset U_{\alpha_1} \oplus \cdots \oplus U_{\alpha_t}. \tag{2.5}$$

There is only one such $t$. By irreducibility of $X$,

$$X \cap (U_{\alpha_1} \oplus \cdots \oplus U_{\alpha_{t-1}}) = 0. \tag{2.6}$$

issue: isomorphism: $\pi(x_1 - x_2) = 0$ with $x_1 - x_2 \in X$ $\to x_1 - x_2 = 0$

Let $\pi$ be the projection

$$\pi\colon U_{\alpha_1} \oplus \cdots \oplus U_{\alpha_t} \to U_{\alpha_t}.$$

Then (2.6) implies that $\pi | X$ is a $\Gamma$-isomorphism of $X$ onto $\pi(X)$; and $\pi(X) \subset U_{\alpha_t}$ implies that $\pi(X) = U_{\alpha_t}$ by irreducibility of $U_{\alpha_t}$. Therefore $X$ is $\Gamma$-isomorphic to $U_{\alpha_t}$, hence to $U$. □

**Lemma 2.8.** *Let $\Gamma$ be a compact Lie group acting on $V$. Let $X$, $Y$ be $\Gamma$-invariant subspaces of $V$ such that no two $\Gamma$-irreducible subspaces $W \subset X$, $Z \subset Y$ are $\Gamma$-isomorphic. Then:*

(a) $X \cap Y = \{0\}$,
(b) *If $W \subset X \oplus Y$ is $\Gamma$-irreducible, then $W \subset X$ or $W \subset Y$.*

Proof.
(a) Since $X \cap Y$ is $\Gamma$-invariant, any $\Gamma$-irreducible subspace of $X \cap Y$ would be in both $X$ and $Y$, contrary to the assumptions on $X$ and $Y$. Thus $X \cap Y$ has no nonzero $\Gamma$-irreducible subspaces, and this is possible only when $X \cap Y = \{0\}$; see Corollary 2.2.
(b) The subspaces $W \cap X$ and $W \cap Y$ of $W$ are $\Gamma$-invariant. By the irreducibility of $W$, either $W \cap X = \{0\}$ or $W \subset X$; and similarly for $Y$. If $W \not\subset X$ and $W \not\subset Y$ then $W \cap X = \{0\} = W \cap Y$. Let $\pi_X$ and $\pi_Y$ denote the projections of $X \oplus Y$ onto $X$ and $Y$, respectively. Then $W$ is $\Gamma$-isomorphic to $\pi_X(W)$ and to $\pi_Y(W)$ as in the proof of Lemma 2.7. But this contradicts the hypotheses on $X$ and $Y$. □

Proof of Theorem 2.5. Choose a $\Gamma$-irreducible subspace $U_1 \subset V$. Let $W_1'$ be the sum of all $\Gamma$-invariant subspaces of $V$ that are $\Gamma$-isomorphic to $U_1$. If $W_1' \neq V$, then choose a $\Gamma$-invariant complement $Z$ to $W_1'$ and repeat the process on $Z$ to obtain $W_2'$. By finiteness of dimension this process terminates with

$$V = W_1' \oplus W_2' \oplus \cdots \oplus W_s' \tag{2.7}$$

where each $W_k'$ is the sum of a set of $\Gamma$-isomorphic $\Gamma$-irreducible subspaces of $V$, say isomorphic to $U_k \subset V$; and if $i \neq j$ then $U_i$ is not $\Gamma$-isomorphic to $U_j$. We do not yet know that $W_k'$ is the sum $W_k$ of *all* $\Gamma$-invariant subspaces of $V$ that are $\Gamma$-isomorphic to $U_k$, because we defined $W_k'$ in $Z$, not in $V$. We shall quickly see that in fact $W_k' = W_k$.

Suppose that $U$ is a $\Gamma$-irreducible subspace of $V$. By Lemma 2.8(b) and a simple inductive argument, it follows that

$$U \subset W_k' \tag{2.8}$$

for some $k$. By Lemma 2.7, $U$ is $\Gamma$-isomorphic to $U_k$. This proves part (a). But now we see that

$$W_k' = W_k, \tag{2.9}$$

as defined in the statement of Theorem 2.5b, and (2.7) implies (2.2), proving part (b). □

EXERCISES

2.1. (a) Show that every two-dimensional irreducible representation of $\mathbf{S}^1$ is isomorphic to

$$\rho_\theta^k(z) = e^{ki\theta}z \tag{2.10}$$

for some integer $k > 0$.

(b) Show that the representations $\rho^k$ and $\rho^l$ in (2.10) are not isomorphic if $k > l > 0$. (*Hint*: Use Exercise XII, 1.6.)

(c) Show that the only one-dimensional irreducible representation of $\mathbf{S}^1$ is the trivial representation.

2.2. Let $\mathbf{O}(2)$ act on the four-dimensional space $V$ of $2 \times 2$ matrices by similarity:

$$\gamma \cdot A = \gamma^{-1}A\gamma \qquad (\gamma \in \mathbf{O}(2), A \in V).$$

Show that $V = V_1 \oplus V_2 \oplus V_3$ where

$$V_1 = \left\{\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}\right\}$$

$$V_2 = \left\{\begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix}\right\}$$

$$V_3 = \left\{\begin{bmatrix} c & d \\ d & -c \end{bmatrix}\right\}.$$

Show that

(a) The $\mathbf{O}(2)$-action on $V_1$ is trivial.

(b) The $\mathbf{O}(2)$-action on $V_2$ is the nontrivial one-dimensional representation, in which $\gamma \in \mathbf{O}(2) \sim \mathbf{SO}(2)$ acts as $-I$ and $\gamma \in \mathbf{SO}(2)$ acts as $I$.

(c) The $\mathbf{O}(2)$-action on $V_3$ is isomorphic to the standard action on $\mathbb{R}^2 \equiv \mathbb{C}$.



2.3. In the notation of Exercise 2.2, let $\mathbf{O}(2)$ act on $V$ by matrix multiplication:

$$\gamma \cdot A = \gamma A.$$

Show that $V = V_1 \oplus V_2$, where

$$V_1 = \left\{\begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix}\right\}$$

$$V_2 = \left\{\begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix}\right\},$$

and that the $\mathbf{O}(2)$-action on each of $V_1$, $V_2$ is isomorphic to the standard action. Hence show that $V$ has only one isotypic component, namely $V$ itself. Find an irreducible subspace of $V$ that is not equal to $V_1$ or $V_2$.

## §3. Commuting Linear Mappings and Absolute Irreducibility

In later sections when we compute linearized asymptotic stability of steady-state solutions to ODEs we will need to understand the structure of linear mappings that commute with the action of a compact Lie group. We explore this issue here. The main result is Theorem 3.5, which lets us put commuting linear mappings into a certain block diagonal form.

Let $\Gamma$ be a compact Lie group acting linearly on $V$. A mapping $F: V \to V$ *commutes with* $\Gamma$ or is $\Gamma$-*equivariant* if

$$F(\gamma v) = \gamma F(v) \tag{3.1}$$

for all $\gamma \in \Gamma$, $v \in V$.

EXAMPLES 3.1.

(a) Consider the standard action of $\Gamma = \mathbf{SO}(2)$ on $V = \mathbb{R}^2$ defined by rotation through angle $\theta$. That is,

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

acts on

$$\mathbb{R}^2 = \left\{\begin{bmatrix} x \\ y \end{bmatrix}\right\}$$

by matrix multiplication.

We claim that the linear mappings that commute with this action of $\mathbf{SO}(2)$ all have the form $cR_\theta$ where $c \in \mathbb{R}$ is a scalar; that is, such linear maps have the matrix form

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}. \tag{3.2}$$