on Z to obtain W_2'. By finiteness of dimension this process terminates with

$$V = W_1' \oplus W_2' \oplus \cdots \oplus W_s' \tag{2.7}$$

where each W_k' is the sum of a set of Γ-isomorphic Γ-irreducible subspaces of V, say isomorphic to $U_k \subset V$; and if $i \neq j$ then U_i is not Γ-isomorphic to U_j. We do not yet know that W_k' is the sum W_k of *all* Γ-invariant subspaces of V that are Γ-isomorphic to U_k, because we defined W_k' in Z, not in V. We shall quickly see that in fact $W_k' = W_k$.

Suppose that U is a Γ-irreducible subspace of V. By Lemma 2.8(b) and a simple inductive argument, it follows that

$$U \subset W_k' \tag{2.8}$$

for some k. By Lemma 2.7, U is Γ-isomorphic to U_k. This proves part (a). But now we see that

$$W_k' = W_k, \tag{2.9}$$

as defined in the statement of Theorem 2.5b, and (2.7) implies (2.2), proving part (b). □

EXERCISES

2.1. (a) Show that every two-dimensional irreducible representation of $\mathbf{S}^1$ is isomorphic to

$$\rho_\theta^k(z) = e^{ki\theta}z \tag{2.10}$$

for some integer $k > 0$.

(b) Show that the representations ρ^k and ρ^l in (2.10) are not isomorphic if $k > l > 0$. (*Hint*: Use Exercise XII, 1.6.)

(c) Show that the only one-dimensional irreducible representation of $\mathbf{S}^1$ is the trivial representation.

2.2. Let $\mathbf{O}(2)$ act on the four-dimensional space V of 2×2 matrices by similarity:

$$\gamma \cdot A = \gamma^{-1}A\gamma \qquad (\gamma \in \mathbf{O}(2), A \in V).$$

Show that $V = V_1 \oplus V_2 \oplus V_3$ where

$$V_1 = \left\{\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}\right\}$$

$$V_2 = \left\{\begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix}\right\}$$

$$V_3 = \left\{\begin{bmatrix} c & d \\ d & -c \end{bmatrix}\right\}.$$

Show that

(a) The $\mathbf{O}(2)$-action on V_1 is trivial.

(b) The $\mathbf{O}(2)$-action on V_2 is the nontrivial one-dimensional representation, in which $\gamma \in \mathbf{O}(2) \sim \mathbf{SO}(2)$ acts as $-I$ and $\gamma \in \mathbf{SO}(2)$ acts as I.

(c) The $\mathbf{O}(2)$-action on V_3 is isomorphic to the standard action on $\mathbb{R}^2 \equiv \mathbb{C}$.

2.3. In the notation of Exercise 2.2, let $\mathbf{O}(2)$ act on V by matrix multiplication:

$$\gamma \cdot A = \gamma A.$$

Show that $V = V_1 \oplus V_2$, where

$$V_1 = \left\{\begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix}\right\}$$

$$V_2 = \left\{\begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix}\right\},$$

and that the $\mathbf{O}(2)$-action on each of V_1, V_2 is isomorphic to the standard action. Hence show that V has only one isotypic component, namely V itself. Find an irreducible subspace of V that is not equal to V_1 or V_2.

§3. Commuting Linear Mappings and Absolute Irreducibility

In later sections when we compute linearized asymptotic stability of steady-state solutions to ODEs we will need to understand the structure of linear mappings that commute with the action of a compact Lie group. We explore this issue here. The main result is Theorem 3.5, which lets us put commuting linear mappings into a certain block diagonal form.

Let Γ be a compact Lie group acting linearly on V. A mapping $F: V \to V$ *commutes with* Γ or is Γ-*equivariant* if

$$F(\gamma v) = \gamma F(v) \tag{3.1}$$

for all $\gamma \in \Gamma$, $v \in V$.

EXAMPLES 3.1.

(a) Consider the standard action of $\Gamma = \mathbf{SO}(2)$ on $V = \mathbb{R}^2$ defined by rotation through angle θ. That is,

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

acts on

$$\mathbb{R}^2 = \left\{\begin{bmatrix} x \\ y \end{bmatrix}\right\}$$

by matrix multiplication.

We claim that the linear mappings that commute with this action of $\mathbf{SO}(2)$ all have the form cR_θ where $c \in \mathbb{R}$ is a scalar; that is, such linear maps have the matrix form

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}. \tag{3.2}$$

Certainly such matrices commute with $\mathbf{SO}(2)$ because $\mathbf{SO}(2)$ is a commutative group, that is $\mathbf{SO}(2)$ satisfies

$$R_\theta R_\varphi = R_\varphi R_\theta.$$

The proof of the converse is a straightforward calculation. Suppose that

$$R_\theta \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} R_\theta \tag{3.3}$$

for all θ. Equate matrix entries on the first row of (3.3) to obtain

$$\begin{aligned} &\text{(a)} \quad a\cos\theta - c\sin\theta = a\cos\theta + b\sin\theta \\ &\text{(b)} \quad b\cos\theta - d\sin\theta = -a\sin\theta + b\cos\theta. \end{aligned} \tag{3.4}$$

Since (3.4) holds for all θ it follows that $b = -c$ and $a = d$. Therefore, the matrix has the desired form.

(b) Now consider the standard action of $\mathbf{O}(2)$ on $\mathbb{R}^2$. We claim that the only linear mappings that commute with $\mathbf{O}(2)$ are cI, $c \in \mathbb{R}$. Note that scalar multiples of the identity commute with any group representation since they commute with any matrix. To prove the claim let M be a matrix commuting with $\mathbf{O}(2)$. Since it commutes with $\mathbf{SO}(2)$ it must have the form (3.2). It is now a simple matter to show that if M commutes with

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

then $b = 0$.

Definition 3.2. A representation of a group Γ on a vector space V is *absolutely irreducible* if the only linear mappings on V that commute with Γ are scalar multiples of the identity.

To justify the terminology we prove:

Lemma 3.3. *Let Γ be a compact Lie group acting on V. If the action of Γ is absolutely irreducible then it is irreducible.*

PROOF. Suppose the action of Γ is not irreducible. Then there is a proper Γ-invariant subspace $W \neq \{0\}$ having a Γ-invariant complement $W^\perp$, by Proposition 2.1. Define $\pi: W \oplus W^\perp \to V$ to be projection onto W with $\ker \pi = W^\perp$. It is easy to check that π commutes with Γ and is not a scalar multiple of the identity. Hence V is not absolutely irreducible. □

Remark. We hasten to point out that if we work with *complex* representations of compact Lie groups then Schur's lemma (Adams [1969], 3.22, p. 40) implies that the complex versions of irreducibility and absolute irreducibility are

equivalent concepts. However, this is not true for real representations, as Example 3.1(a) shows. We provide further discussion at the end of this section.

We now discuss several points about linear maps that commute with nonirreducible representations. The following observation is quite useful.

Lemma 3.4. *Let Γ be a compact Lie group acting on V, let $A: V \to V$ be a linear mapping that commutes with Γ, and let $W \subset V$ be a Γ-irreducible subspace. Then $A(W)$ is Γ-invariant, and either $A(W) = \{0\}$ or the representations of Γ on W and $A(W)$ are isomorphic.*

PROOF. To show that $A(W)$ is Γ-invariant let $z \in A(W)$, so that $z = A(w)$ for $w \in W$. Since A commutes with Γ we have

$$\gamma z = \gamma A(w) = A(\gamma w)$$

so $\gamma z \in A(W)$.

Similarly, $\ker A$ is Γ-invariant since $A(v) = 0$ implies that $A(\gamma v) = \gamma A(v) = \gamma 0 = 0$. Then $\ker A \cap W$ is a Γ-invariant subspace of W, and irreducibility implies that either $W \subset \ker A$ or $\ker A \cap W = \{0\}$. In the first case $A(W) = \{0\}$. In the second, $A(W)$ is isomorphic to W as a vector space, the isomorphism being A; but Γ commutes with A so A is a Γ-isomorphism between A and $A(W)$. □

Lemma 3.4 implies:

Theorem 3.5. *Let Γ be a compact Lie group acting on the vector space V. Decompose V into isotypic components*

$$V = W_1 \oplus \cdots \oplus W_s.$$

Let $A: V \to V$ be a linear mapping commuting with Γ. Then

$$A(W_k) \subset W_k \tag{3.5}$$

for $k = 1, \ldots, s$.

PROOF. Write $W_k = V_1 \oplus \cdots \oplus V_r$, where all V_j are Γ-isomorphic to an irreducible U_k. By Lemma 3.4 either $A(V_j) = \{0\}$ or $A(V_j)$ is also Γ-isomorphic to U_k. In either case $A(V_j) \subset W_k$. By linearity, $A(W_k) \subset W_k$. □

Finally we return to the question of irreducible but not absolutely irreducible representations. Suppose Γ acts irreducibly on V and let

$$\mathscr{D} = \{A: V \to V \mid A \text{ linear}, A\gamma = \gamma A \text{ for all } \gamma \in \Gamma\}$$

be the set of all commuting mappings. The real version of Schur's lemma (Kirillov [1976], Theorem 2, p. 119) states that $\mathscr{D}$ is an associative algebra over $\mathbb{R}$ and is isomorphic to one of $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, where $\mathbb{H}$ is the four-dimensional

algebra of quaternions. The reason is that by Lemma 3.4 $\mathscr{D}$ is a skew field, and skew fields may be classified into the preceding three types. The case $\mathscr{D} \cong \mathbb{R}$ occurs if and only if V is absolutely irreducible. The example of $\mathbf{SO}(2)$ acting on $\mathbb{R}^2$ is a case where $\mathscr{D} \cong \mathbb{C}$. To verify this, recall that the commuting mappings are the matrices

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

The isomorphism $\mathscr{D} \cong \mathbb{C}$ identifies such a matrix with $a + ib \in \mathbb{C}$. Note that

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix}\begin{bmatrix} c & d \\ -d & c \end{bmatrix} = \begin{bmatrix} ac - bd & ad + bc \\ -(ad + bc) & ac - bd \end{bmatrix}$$

and $(a + ib)(c + id) = (ac - bd) + i(ad + bc)$, so this map *is* an isomorphism. The case $\mathscr{D} \cong \mathbb{H}$ can also occur; see Exercise 3.1. The distinction between $\mathbb{C}$ and $\mathbb{H}$ is a basic one when considering nonabsolutely irreducible representations. Most representations discussed in this book will in fact be absolutely irreducible, $\mathscr{D} \cong \mathbb{R}$; but the case $\mathscr{D} \cong \mathbb{C}$ arises repeatedly in the context of Hopf bifurcation. We have found no such natural context for representations with $\mathscr{D} \cong \mathbb{H}$.

EXERCISES

3.1. Let Γ be the group $\mathbf{SU}(2)$ of unit quaternions

$$\{a + bi + cj + dk: a^2 + b^2 + c^2 + d^2 = 1\}.$$

Show that Γ is a compact Lie group. Let Γ act on $\mathbb{R}^4 \equiv \mathbb{H}$ by left multiplication,

$$\gamma \cdot x = \gamma x.$$

Prove that $\mathscr{D}$ consists of mappings δq, $q \in \mathbb{H}$, acting as right multiplication,

$$\delta q(x) = xq.$$

Hence show that $\mathscr{D} \cong \mathbb{H}$.

3.2. Let Γ be a Lie group acting irreducibly on a space V. Let $A: V \to V$ be a nonzero linear map commuting with Γ. Show that A is invertible and that A^{-1} commutes with Γ.

3.3. Let A, B be commuting matrices. Let E be an eigenspace, or a generalized eigenspace, of A. Show that B leaves E invariant.

3.4. If a 2×2 matrix A commutes with $\kappa = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$ then show that A is diagonal. If a diagonal matrix commutes with a rotation matrix R_θ, where θ is not an integer multiple of π, show that it is a scalar multiple of the identity. Hence show that the standard action of $\mathbf{D}_n$, $n \geq 3$, is absolutely irreducible.

3.5. Let Γ act on $V = V_1 \oplus V_2$ where V_1 and V_2 are absolutely irreducible and nonisomorphic. Let $A: V \to V$ commute with Γ. Prove that A has real eigenvalues and that at most two distinct eigenvalues occur.

3.6. Let $\mathbf{O}(3)$ act on the space

$$V = \{3 \times 3 \text{ symmetric trace 0 matrices}\}$$

by similarity:

$$\gamma \cdot A = \gamma^{-1} A \gamma.$$

Show that V is absolutely irreducible. (*Hint*: Let D be the set of diagonal matrices in V.) Observe that

$$D = \{A: \sigma_1 A = A, \sigma_2 A = A\} \tag{3.6}$$

where

$$\sigma_1 = \begin{bmatrix} -1 & & \\ & 1 & \\ & & 1 \end{bmatrix}, \qquad \sigma_2 = \begin{bmatrix} 1 & & \\ & -1 & \\ & & 1 \end{bmatrix}.$$

Let $\alpha: V \to V$ commute with Γ. Use (3.6) to show that $\alpha(D) \subset D$. Since every symmetric matrix can be diagonalized, show that α is uniquely determined by its effect on D. Let $\beta = \alpha|D$. Show that β commutes with the permutation matrices $\mathbf{S}_3$, and that the $\mathbf{S}_3$-action on D is absolutely irreducible. Deduce that the action of $\mathbf{O}(3)$ on V is absolutely irreducible.

3.7. Let Γ act on V and let H be a subgroup of Γ. If V is absolutely irreducible for H, prove that it is absolutely irreducible for Γ.

§4. Invariant Functions

The goal of this section and the next is to present an efficient way of describing *nonlinear* mappings that commute with a group action. We begin with a discussion of invariant functions. There are two main results: the Hilbert–Weyl theorem, which gives a theoretical foundation for describing invariant polynomials, and Schwarz's theorem (Schwarz [1975]), which builds on Hilbert and Weyl's result, yielding a description of invariant C^∞ germs. See II, §1, for a definition and discussion of germs.

Let Γ be a (compact) Lie group acting on a vector space V. Recall that a real-valued function $f: V \to \mathbb{R}$ is *invariant* under Γ if

$$f(\gamma x) = f(x) \tag{4.1}$$

for all $\gamma \in \Gamma$, $x \in V$. An *invariant polynomial* is defined in the obvious way by taking f to be polynomial. Note that it suffices to verify (4.1) for a set of generators of Γ.

EXAMPLES 4.1.

(a) Let $\Gamma = \mathbf{Z}_2$ act nontrivially on $V = \mathbb{R}$. That is, $-1 \cdot x = -x$, where $\mathbf{Z}_2 = \{\pm 1\}$. For this example the invariant functions are just the *even* functions since (4.1) becomes $f(-x) = f(x)$. It is easy to see that if f is an invariant

polynomial then there exists another polynomial h such that

$$f(x) = h(x^2). \tag{4.2}$$

(b) Let $\mathbf{S}^1$ act on $\mathbb{R}^2 \equiv \mathbb{C}$ in the standard way; that is, $\theta z = e^{i\theta}z$ for $\theta \in \mathbf{S}^1$. Equation (4.1) states that $f(e^{i\theta}z) = f(z)$ for every $\theta \in \mathbf{S}^1$. Since $\theta \mapsto e^{i\theta}$ traces out a circle centered at 0 with radius $|z|$ we see that $\mathbf{S}^1$-invariants are functions that are constant on circles. We now show (as is already plausible) that if f is an $\mathbf{S}^1$-invariant polynomial on $\mathbb{C}$ then there exists a polynomial h: $\mathbb{R} \to \mathbb{R}$ such that

$$f(z) = h(z\bar{z}). \tag{4.3}$$

(This observation is contained in the proof of Proposition VIII, 2.3; we give a different proof here.) The proof of (4.3) will be carried out using complex notation, a trick that is often useful. Write f as a polynomial in the "real" coordinates $z, \bar{z}$ on $\mathbb{C}$ in the form

$$f(z) = \sum a_{\alpha\beta} z^\alpha \bar{z}^\beta \tag{4.4}$$

where $a_{\alpha\beta} \in \mathbb{C}$. (They are "real" coordinates in the sense that they coordinatize $\mathbb{C}$ as a real vector space. However, for $z = x + iy$ we have $x = (z + \bar{z})/2$, $y = -i(z - \bar{z})/2$, so the coefficients required may be complex. Thus we have to impose on all polynomials a reality condition: their *values* must be in $\mathbb{R}$.) Here the reality condition is that f is real-valued; that is, $\bar{f} = f$. So the coefficients $a_{\alpha\beta}$ satisfy

$$\bar{a}_{\alpha\beta} = a_{\beta\alpha}. \tag{4.5}$$

Direct computation from (4.4) shows that

$$f(e^{i\theta}z) = \sum a_{\alpha\beta} e^{i\theta(\alpha-\beta)} z^\alpha \bar{z}^\beta. \tag{4.6}$$

Since $f(e^{i\theta}z) \equiv f(z)$ as polynomials, they have identical coefficients. From (4.4, 4.6) we obtain the identity

$$a_{\alpha\beta} = e^{i\theta(\alpha-\beta)} a_{\alpha\beta}. \tag{4.7}$$

Now (4.7) holds for *all* $\theta \in \mathbf{S}^1$ only if $\alpha = \beta$ or $a_{\alpha\beta} = 0$. Thus $\mathbf{S}^1$-invariance implies that

$$f(z) = \sum a_{\alpha\alpha} (z\bar{z})^\alpha$$

where, by (4.5), $a_{\alpha\alpha} \in \mathbb{R}$. If

$$h(x) = \sum a_{\alpha\alpha} x^\alpha$$

then (4.3) is satisfied.

(c) Let $\Gamma = \mathbf{D}_n$ in its standard action on $\mathbb{C}$. We claim that for every $\mathbf{D}_n$-invariant polynomial $f(z)$ there exists a polynomial g: $\mathbb{R}^2 \to \mathbb{R}$ such that

$$f(z) = g(z\bar{z}, z^n + \bar{z}^n). \tag{4.8}$$

We verify (4.8) in a similar way to (4.3). We may again assume f has the form (4.4) and satisfies the reality condition (4.5). Since the action of $\mathbf{D}_n$ is generated by

$$\theta z = e^{i\theta}z \quad (\theta = 2\pi/n) \quad \text{and} \quad \kappa z = \bar{z}$$

we need verify (4.1) only for these elements. The restriction placed on f by the first generator is (4.7) when $\theta = 2\pi/n$. The restriction placed by κ is

$$a_{\alpha\beta} = a_{\beta\alpha}, \tag{4.9}$$

and from (4.5, 4.9) we conclude that $a_{\alpha\beta} \in \mathbb{R}$. In summary, we require

$$\begin{aligned} &\text{(a)} \quad a_{\alpha\beta} \in \mathbb{R} \\ &\text{(b)} \quad a_{\alpha\beta} = a_{\beta\alpha} \\ &\text{(c)} \quad a_{\alpha\beta} = 0 \quad \text{unless} \quad \alpha \equiv \beta (\text{mod } n). \end{aligned} \tag{4.10}$$

Using (4.10) we may rewrite (4.4) as

$$f(z) = \sum_{\alpha \le \beta} A_{\alpha\beta} (z^\alpha \bar{z}^\beta + \bar{z}^\alpha z^\beta)$$

where

$$A_{\alpha\beta} = \begin{cases} a_{\alpha\beta} & \text{if } \alpha \neq \beta, \\ a_{\alpha\beta}/2 & \text{if } \alpha = \beta. \end{cases}$$

Next, we factor out the largest powers of $z\bar{z}$ and use (4.10c) to arrive at the form

$$f(z) = \sum_{j,k} B_{jk} (z\bar{z})^j (z^{kn} + \bar{z}^{kn}) \tag{4.11}$$

for certain coefficients B_{jk}. Finally we use the identity

$$z^{kn} + \bar{z}^{kn} = (z^n + \bar{z}^n)(z^{(k-1)n} + \bar{z}^{(k-1)n}) - z\bar{z}(z^{(k-2)n} + \bar{z}^{(k-2)n})$$

inductively, to write the polynomial in the form

$$f(z) = \sum C_{lm} (z\bar{z})^l (z^n + \bar{z}^n)^m$$

for certain real coefficients C_{lm}. Now define

$$h(x, y) = \sum C_{lm} x^l y^m.$$

We make one very important observation about the invariant polynomials in Examples 4.1. There is a *finite* subset of invariant polynomials $u_1, \ldots, u_s$ such that *every* invariant polynomial may be written as a polynomial function of $u_1, \ldots, u_s$. This finite set of invariants (which is not unique) is said to *generate* the set of invariants, or to form a *Hilbert basis*. We denote the set of invariant polynomials by $\mathscr{P}(\Gamma)$. Note that $\mathscr{P}(\Gamma)$ is a ring since sums and products of Γ-invariant polynomials are again Γ-invariant. The existence of this finite set of generators is a general phenomenon. The main theoretical result, initiated by Hilbert and proved by Weyl [1946], is as follows:

Theorem 4.2 (Hilbert–Weyl Theorem). *Let Γ be a compact Lie group acting on V. Then there exists a finite Hilbert basis for the ring $\mathscr{P}(\Gamma)$.*

Remarks.
(a) The actual computation of a generating set for $\mathscr{P}(\Gamma)$ can be extremely difficult. In many cases, such as those in Examples 4.1, a set of invariant generators may be obtained by a combination of tricks and direct calculation.
(b) Since Γ is a compact Lie group, we may assume it is a subgroup of the orthogonal group $\mathbf{O}(n)$ by Proposition 1.3. In this case, the norm

$$\|x\|^2 = x_1^2 + \cdots + x_n^2$$

is always Γ-invariant.

We prove this theorem in §6; similar proofs are given in Weyl [1946] and Poénaru [1976]. In individual examples, such as those of Examples 4.1, we may verify Theorem 4.2 explicitly by exhibiting a finite set of invariant generators.

It is not surprising that a similar result to Theorem 4.2 holds for real analytic functions. It is perhaps more surprising, however, that this sort of result remains true for C^∞ germs, and it is in this category that we wish to work. Although a finitude theorem for C^∞ germs was known in special cases (see Whitney [1943] for $\mathbf{Z}_2$ acting on $\mathbb{R}$, and Glaeser [1963] for the symmetric group $\mathbf{S}_n$ acting as permutations on $\mathbb{R}^n$) it was not until Schwarz [1975] that the C^∞ germ result was proved for general compact Lie groups. We state Schwarz's theorem here and sketch its proof in §6. We use the notation $\mathscr{E}(\Gamma)$ for the ring of Γ-invariant germs $V \to \mathbb{R}$.

Theorem 4.3 (Schwarz [1975]). *Let Γ be a compact Lie group acting on V. Let $u_1, \ldots, u_s$ be a Hilbert basis for the Γ-invariant polynomials $\mathscr{P}(\Gamma)$. Let $f \in \mathscr{E}(\Gamma)$. Then there exists a smooth germ $h \in \mathscr{E}_s$ such that*

$$f(x) = h(u_1(x), \ldots, u_s(x)). \tag{4.12}$$

Here $\mathscr{E}_s$ is the ring of C^∞ germs $\mathbb{R}^s \to \mathbb{R}$.

We conclude this section with a discussion of some special structure often found in the ring $\mathscr{P}(\Gamma)$, which is quite useful when making explicit calculations. It implies in particular that when f in (4.12) is polynomial then there is a unique choice of the polynomial h.

More precisely, say that a set of Γ-invariant polynomials has a *relation* if there exists a nonzero polynomial $r(y_1, \ldots, y_s)$ such that

$$r(u_1(x), \ldots, u_s(x)) \equiv 0. \tag{4.13}$$

The ring $\mathscr{P}(\Gamma)$ is a *polynomial ring* if it has a Hilbert basis without relations. (*Warning*: A polynomial ring is *not* just a ring of polynomials.)

An example of a group action for which $\mathscr{P}(\Gamma)$ is not a polynomial ring is given by $\Gamma = \mathbf{Z}_2$ acting on $\mathbb{R}^2$, where the action of $-1 \in \mathbf{Z}_2$ is defined by $x \mapsto -x$. It is easy to see that $\mathscr{P}(\mathbf{Z}_2)$ is generated by all monomials of even total degree. The polynomials

$$u_1 = x_1^2, \qquad u_2 = x_1 x_2, \qquad u_3 = x_2^2$$

form a Hilbert basis for $\mathscr{P}(\mathbf{Z}_2)$, but there is a relation

$$u_1 u_3 - u_2^2 \equiv 0.$$

Indeed it can be shown that no choice of Hilbert basis can eliminate all relations, so $\mathscr{P}(\mathbf{Z}_2)$ is *not* a polynomial ring.

There is a simple test to determine whether a given Hilbert basis $u_1, \ldots, u_s$ for $\mathscr{P}(\Gamma)$ makes it into a polynomial ring. Define the mapping $\rho: V \to \mathbb{R}^s$, called the *discriminant* of Γ, by

$$\rho(x) = (u_1(x), \ldots, u_s(x)). \tag{4.14}$$

Lemma 4.4. *If the Jacobian $(d\rho)_x$ is onto for some x, then $\mathscr{P}(\Gamma)$ is a polynomial ring.*

PROOF. If $(d\rho)_x$ is onto, then by the implicit function theorem $\rho(V)$ contains an open subset of $\mathbb{R}^s$. Hence any polynomial mapping $r: \mathbb{R}^s \to \mathbb{R}$ is uniquely determined by $r|\rho(V)$. Now suppose r satisfies (4.13); that is, $r|\rho(V) \equiv 0$. It follows that $r \equiv 0$ and that there are no nontrivial relations. □

Note that in the preceding example of $\mathbf{Z}_2$ $\rho(x_1, x_2) = (x_1^2, x_1 x_2, x_2^2)$ and $(d\rho)_x: \mathbb{R}^2 \to \mathbb{R}^3$. Hence it is impossible for $(d\rho)_x$ to be onto. (However, the converse of Lemma 4.4 has not been proved, so this does not show that $\mathscr{P}(\Gamma)$ is not a polynomial ring.)

We may use Lemma 4.4 to check that for Examples 4.1, $\mathscr{P}(\Gamma)$ is a polynomial ring. For instance, consider Example 4.1c, where $\Gamma = \mathbf{D}_n$ acts on $\mathbb{C}$. Recall from (4.8) that

$$u_1(z, \bar{z}) = z\bar{z}, \qquad u_2(z, \bar{z}) = z^n + \bar{z}^n$$

is a Hilbert basis. Then

$$\rho(z, \bar{z}) = (z\bar{z}, z^n + \bar{z}^n)$$

so that

$$d\rho = \begin{bmatrix} z & \bar{z} \\ nz^{n-1} & n\bar{z}^{n-1} \end{bmatrix}.$$

It follows that $\det d\rho = n(z^n - \bar{z}^n)$, which is (often) nonzero. By Lemma 4.4, $\mathscr{P}(\mathbf{D}_n)$ is a polynomial ring.

Remarks.
(a) When $\mathscr{P}(\Gamma)$ is a polynomial ring in the Hilbert basis $u_1, \ldots, u_s$, then every invariant polynomial f has *uniquely* the form

$$f(x) = h(u_1(x), \ldots, u_s(x)).$$

To prove this, suppose not. Then also $f = k(u_1(x), \ldots, u_s(x))$. If $r = h - k$ then $r(u_1(x), \ldots, u_s(x)) \equiv 0$, so r is a relation. This is a contradiction.

(b) Even when $\mathscr{P}(\Gamma)$ is a polynomial ring, uniqueness need not hold in (4.12) for C^∞ germs. For example, let $\mathbf{Z}_2$ act on $\mathbb{R}$ in the standard way. Then $u_1(x) = x^2$ is a Hilbert basis. By Theorem 4.3 every invariant germ $f \in \mathscr{E}(\Gamma)$ has the form $f(x) = h(x^2)$ for some $h \in \mathscr{E}_x$. However, define

$$k(x) = \begin{cases} e^{-1/x} & \text{if } x < 0 \\ 0 & \text{if } x \geq 0. \end{cases}$$

Then k is smooth, and

$$f(x) = h(x^2) + k(x^2)$$

so uniqueness fails. More generally, if $\operatorname{Im} \rho$ in Lemma 4.4 does not contain a neighborhood of the origin in $\mathbb{R}^s$ then uniqueness in (4.12) fails in $\mathscr{E}(\Gamma)$.

(c) It is, however, true that if $\mathscr{P}(\Gamma)$ is a polynomial ring then the Taylor expansion of h in (4.12) at the origin is uniquely defined. Since in our analysis of bifurcation problems we consider only finitely determined situations (that is, those in which the problem may be reduced to a finite part of the Taylor expansion of f), it follows that to all intents and purposes uniqueness in $\mathscr{E}(\Gamma)$ does hold.

(d) Another test to show that $\mathscr{P}(\Gamma)$ is a polynomial ring, even simpler than Lemma 4.4, is give in XIII, §1.

EXERCISES

4.1. Let $\mathbf{S}^1$ act on $\mathbb{C}^n$ by $(z_1, \ldots, z_n) \mapsto (e^{i\theta} z_1, \ldots, e^{i\theta} z_n)$. Show that a Hilbert basis is $\{\operatorname{Re}(z_j \bar{z}_k), \operatorname{Im}(z_j \bar{z}_k)\}$.

4.2. Let $\mathbf{S}^1$ act on $\mathbb{C}^2$ by $(z_1, z_2) \mapsto (e^{ki\theta} z_1, e^{li\theta} z_2)$ where k, l are coprime. Show that a Hilbert basis is $\{\operatorname{Re}(z_1^l \bar{z}_2^k), \operatorname{Im}(z_1^l \bar{z}_2^k), |z_1|^2, |z_2|^2\}$.

4.3. Let $(\theta, \varphi) \in \mathbf{T}^2$ act on $\mathbb{C}^2$ by $(z_1, z_2) \mapsto (e^{k_1 i\theta + k_2 i\varphi} z_1, e^{l_1 i\theta + l_2 i\varphi} z_2)$ where k_1, l_1 and k_2, l_2 are coprime. Find a Hilbert basis. (*Hint*: Apply Exercise 4.2 to the θ-action and observe the action of φ on a Hilbert basis. Or use brute force on monomials $z_1^a \bar{z}_1^b z_2^c \bar{z}_2^d$.)

4.4. Which of the preceding rings of invariants are polynomial rings?

4.5. Let the symmetric group $\mathbf{S}_3$, consisting of all permutations of $\{1, 2, 3\}$, act on $\mathbb{R}^3$ by permuting a basis. Show that the invariant functions are generated by $s_1 = x_1 + x_2 + x_3$, $s_2 = x_1 x_2 + x_2 x_3 + x_1 x_3$, and $s_3 = x_1 x_2 x_3$. Prove that the ring of invariants $\mathscr{E}(\mathbf{S}_3)$ is a polynomial ring.

4.6. Prove results analogous to the preceding for $\mathbf{S}_n$ acting on $\mathbb{R}^n$.

4.7. Let Γ be the group of all symmetries, including reflections, of a cube centered at the origin of $\mathbb{R}^3 = \{(x, y, z)\}$ with edges parallel to the axes. (In the notation of XIII, §9, this is the group $\mathbf{O} \oplus \mathbf{Z}_2^c$.) Prove that the ring of Γ-invariants is generated by

$$u = x^2 + y^2 + z^2$$
$$v = x^2 y^2 + y^2 z^2 + x^2 z^2$$
$$w = x^2 y^2 z^2$$

and that it is a polynomial ring.

The next group of exercises investigates conditions under which a function $\operatorname{Fix}(\Sigma) \to \mathbb{R}$ extends to a Γ-invariant function.

4.8. Let Γ act on V and let Σ be an isotropy subgroup. Let $f: V \to \mathbb{R}$ be Γ-invariant and let $\varphi = f|\operatorname{Fix}(\Sigma)$. Let $N = N_\Gamma(\Sigma)$.
(a) Show that φ is N-invariant. Hence a *necessary* condition for a function $\psi: \operatorname{Fix}(\Sigma) \to \mathbb{R}$ to extend to a Γ-invariant function on V is that ψ be N-invariant.
(b) φ has the following more general *hidden symmetry property*: If there exist $\gamma \in \Gamma \sim N$ and $v \in V$ such that $v, \gamma v \in \operatorname{Fix}(\Sigma)$ then $\varphi(v) = \varphi(\gamma v)$.
(c) If $\psi: \operatorname{Fix}(\Sigma) \to \mathbb{R}$ is N-invariant, then a necessary condition that ψ should extend to a Γ-invariant function $\hat{\psi}: V \to \mathbb{R}$ is that ψ satisfies the hidden symmetry condition.

4.9. Find an example where the hidden symmetry condition is violated, showing that the condition in Exercise 4.8(a) is not sufficient for an extension to exist.

4.10. If $\psi: \operatorname{Fix}(\Sigma) \to \mathbb{R}$ is N-invariant and satisfies the hidden symmetry condition prove that there exists a *continuous* Γ-invariant extension $\hat{\psi}: V \to \mathbb{R}$. (*Hint*: Work inside a suitable closed ball center 0. Define $\mathscr{X} = \bigcup_{\gamma \in \Gamma} \gamma \operatorname{Fix}(\Sigma)$. Prove that $\mathscr{X}$ is a closed subset of V and that ψ extends uniquely to a Γ-invariant function $\tilde{\psi}$ on $\mathscr{X}$. Use the Tietze extension theorem to extend $\tilde{\psi}$ from $\mathscr{X}$ to V, and average over Γ by Haar integration.)

4.11. Let $\Gamma = \mathbf{D}_5$ acting on $\mathbb{C}$, $\Sigma = \mathbf{Z}_2(\kappa)$, and $\psi(x) = x^3$ $(x \in \operatorname{Fix}(\Sigma) = \mathbb{R})$. Observe that ψ trivially satisfies the hidden symmetry condition and is N-invariant. By considering 3-jets (Taylor expansions to degree 3) show that ψ has no *smooth* extension to a Γ-invariant function $\mathbb{C} \to \mathbb{R}$.

4.12. Investigate analogous results to Exercises 4.8–4.11 for the extension of N-equivariant mappings on $\operatorname{Fix}(\Sigma)$ to Γ-equivariant mappings on V.

§5. Nonlinear Commuting Mappings

As usual we let Γ be a compact Lie group acting on a vector space V. Recall that a mapping $g: V \to V$ *commutes with* Γ or is Γ-*equivariant* if

$$g(\gamma x) = \gamma g(x) \tag{5.1}$$

for all $\gamma \in \Gamma$, $x \in V$. In §3 we discussed some of the restrictions placed on linear mappings g by (5.1). In this section we describe the restrictions placed on nonlinear g.

The main observation is that the product of an equivariant mapping and an invariant function is another equivariant mapping.

Lemma 5.1. *Let $f: V \to \mathbb{R}$ be a Γ-invariant function and let $g: V \to V$ be a Γ-equivariant mapping. Then $fg: V \to V$ is Γ-equivariant.*

PROOF. This follows from an easy calculation. For all $\gamma \in \Gamma$ and $x \in V$ we have:

$$\begin{aligned}(fg)(\gamma x) &= f(\gamma x)g(\gamma x)\\ &= f(x)\cdot \gamma g(x)\\ &= \gamma f(x)g(x)\\ &= \gamma fg(x).\end{aligned} \tag{5.2}$$

The first and fourth equalities in (5.2) use the definition of fg; the second equality follows by Γ-invariance and Γ-equivariance; and the third follows because γ acts linearly on V and $f(x)$ is a scalar. □

For example, when $\Gamma = \mathbf{Z}_2$ acts on $\mathbb{R}$ by $-1\cdot x = -x$, then the $\mathbf{Z}_2$-equivariant mappings are just the odd functions; that is, they satisfy $g(-x) = -g(x)$. It is well known that every odd function may be written as an even function times x. This was proved in Corollary VI, 2.2; nevertheless we reproduce the argument here. Since $g(0) = 0$ we use Taylor's theorem to write $g(x) = f(x)x$. Since g is odd,

$$f(-x)x = f(x)x,$$

so f is even. Moreover, we know that $f(x) = h(x^2)$ for a suitably chosen smooth h, by (4.3) and Theorem 4.2 (or by Lemma VI.2.1). Hence

$$g(x) = h(x^2)x. \tag{5.3}$$

We now abstract some general principles from the preceding observations. Let $\vec{\mathscr{P}}(\Gamma)$ be the space of Γ-equivariant polynomial mappings of V into V, and let $\vec{\mathscr{E}}(\Gamma)$ be the space of Γ-equivariant germs (at the origin) of C^∞ mappings of V into V. Lemma 5.1 implies that $\vec{\mathscr{P}}(\Gamma)$ is a module over the ring of invariant polynomials $\mathscr{P}(\Gamma)$, and equally that $\vec{\mathscr{E}}(\Gamma)$ is a module over the ring of invariant function germs $\mathscr{E}(\Gamma)$. This means that if $f \in \mathscr{P}(\Gamma)$ and $g \in \vec{\mathscr{P}}(\Gamma)$ then $fg \in \vec{\mathscr{P}}(\Gamma)$, with a similar statement for $\mathscr{E}$, and this is the content of Lemma 5.1.

The results for $\Gamma = \mathbf{Z}_2$ can be stated in symbols:

$$\begin{aligned}&\text{(a)}\quad \vec{\mathscr{P}}(\mathbf{Z}_2) = \mathscr{P}(\mathbf{Z}_2)\{x\},\\ &\text{(b)}\quad \vec{\mathscr{E}}(\mathbf{Z}_2) = \mathscr{E}(\mathbf{Z}_2)\{x\}.\end{aligned} \tag{5.4}$$

In words, the module $\vec{\mathscr{E}}(\mathbf{Z}_2)$ (or $\vec{\mathscr{P}}(\mathbf{Z}_2)$) is generated over the ring $\mathscr{E}(\mathbf{Z}_2)$ (or $\mathscr{P}(\mathbf{Z}_2)$) by the single $\mathbf{Z}_2$-equivariant mapping x. In general, we say that the equivariant polynomial mapping $g_1, \dots, g_r$ *generate* the module $\vec{\mathscr{P}}(\Gamma)$ over the ring $\mathscr{P}(\Gamma)$ if every Γ-equivariant g may be written as

$$g = f_1 g_1 + \cdots + f_r g_r \tag{5.5}$$

for invariant polynomials $f_1, \dots, f_r$. A similar definition may be made for $\vec{\mathscr{E}}(\Gamma)$. The next theorem follows from, and is similar in spirit to, the Hilbert–Weyl theorem. A proof is given in §6.

Theorem 5.2. *Let Γ be a compact Lie group acting on V. Then there exists a finite set of Γ-equivariant polynomials $g_1, \dots, g_r$ that generates the module $\vec{\mathscr{P}}(\Gamma)$.*

The Γ-equivariant version of Schwarz's theorem (Theorem 4.3) is proved in Poénaru [1976]. We present this proof in §6 too.

Theorem 5.3 (Poénaru [1976]). *Let Γ be a compact Lie group and let $g_1, \dots, g_r$ generate the module $\vec{\mathscr{P}}(\Gamma)$ of Γ-equivariant polynomials over the ring $\mathscr{P}(\Gamma)$. Then $g_1, \dots, g_r$ generate the module $\vec{\mathscr{E}}(\Gamma)$ over the ring $\mathscr{E}(\Gamma)$.*

The implications of Theorems 5.2 and 5.3 are illustrated by the following examples.

EXAMPLES 5.4.

(a) Let $\Gamma = \mathbf{S}^1$ in its standard action on $V = \mathbb{C}$. We claim that every $\mathbf{S}^1$-equivariant mapping $g \in \vec{\mathscr{E}}(\mathbf{S}^1)$ has the form

$$g(z) = p(z\bar{z})z + q(z\bar{z})iz \tag{5.6}$$

where p and q are real-valued $\mathbf{S}^1$-invariant functions. This has already been proved in Proposition VIII, 2.5 in slightly different notation; we give a different proof here.

Let $g: \mathbb{C} \to \mathbb{C}$ be an $\mathbf{S}^1$-equivariant polynomial. In the coordinates $z, \bar{z}$ it has the form

$$g = \sum b_{jk} z^j \bar{z}^k \tag{5.7}$$

where $b_{jk} \in \mathbb{C}$. The equivariance condition (5.1) can be restated as an invariance condition

$$g(x) = \gamma^{-1} g(\gamma x), \tag{5.8}$$

which is often more convenient to use. In the case $\Gamma = \mathbf{S}^1$ we have

$$g = e^{-i\theta} \sum b_{jk} e^{(j-k)i\theta} z^j \bar{z}^k = \sum b_{jk} e^{(j-k-1)i\theta} z^j \bar{z}^k. \tag{5.9}$$

Hence $b_{jk} = 0$ unless $j = k + 1$. Thus

$$g(z) = \sum b_{k+1,k} (z\bar{z})^k z$$

and g has the form (5.6), where

$$\begin{aligned}p(y) &= \sum \mathrm{Re}(b_{k+1,k}) y^k,\\ q(y) &= \sum \mathrm{Im}(b_{k+1,k}) y^k.\end{aligned}$$

Now apply Theorem 5.3.

(b) Let $\Gamma = \mathbf{O}(2)$ in its standard action on $\mathbb{C}$. We claim that every $\mathbf{O}(2)$-equivariant mapping $g \in \bar{\mathscr{E}}(\mathbf{O}(2))$ has the form

$$g(z) = p(z\bar{z})z. \tag{5.10}$$

To prove this, observe that g is in particular $\mathbf{S}^1$-equivariant, hence has the form (5.6). But $\mathbf{O}(2)$ is generated by $\mathbf{S}^1 \cong \mathbf{SO}(2)$ and the flip κ, which acts by $\kappa z = \bar{z}$. Now compute

$$\overline{g(\bar{z})} = p(z\bar{z})z - q(z\bar{z})iz. \tag{5.11}$$

The only way that $\overline{g(\bar{z})}$ can equal $g(z)$ is if $q(z\bar{z}) = 0$, thus proving the claim.

(c) Let $\Gamma = \mathbf{D}_n$ in its standard action on $\mathbb{C}$. We claim that every $\mathbf{D}_n$-equivariant germ $g \in \bar{\mathscr{E}}(\mathbf{D}_n)$ has the form

$$g(z) = p(u, v)z + q(u, v)\bar{z}^{n-1} \tag{5.12}$$

where $u = z\bar{z}$ and $v = z^n + \bar{z}^n$.

We begin again with a $\mathbf{D}_n$-equivariant polynomial g of the form

$$g(z) = \sum b_{jk} z^j \bar{z}^k \tag{5.13}$$

where $b_{jk} \in \mathbb{C}$. We first obtain restrictions on the b_{jk} by using the equivariance of g with respect to κ, where $\kappa z = \bar{z}$. Now

$$\overline{g(\bar{z})} = \sum \bar{b}_{jk} z^j \bar{z}^k.$$

Hence $\overline{g(\bar{z})} = g(z)$ implies that b_{jk} is real.

Recall that $\mathbf{D}_n$ is generated by κ and $\zeta = 2\pi/n$, which acts as multiplication by $e^{i\zeta}$. Now equivariance with respect to ζ implies that

$$\begin{aligned} g(z) &= e^{-i\zeta} g(e^{i\zeta} z) \\ &= \sum b_{jk} e^{(j-k-1)i\zeta} z^j \bar{z}^k. \end{aligned} \tag{5.14}$$

Hence $b_{jk} = 0$ unless $j \equiv k + 1 (\mathrm{mod}\, n)$. (It is here that the analysis begins to differ from the case $\Gamma = \mathbf{S}^1$.)

We now show that z and $\bar{z}^{n-1}$ generate the module $\bar{\mathscr{P}}(\mathbf{D}_n)$ over $\mathscr{P}(\mathbf{D}_n)$. In individual terms in (5.13) we can factor out powers of $z\bar{z}$, which are $\mathbf{D}_n$-invariants, until we are left either with $j = 0$ or $k = 0$. Since $j \equiv k + 1 \ (\mathrm{mod}\ n)$ the terms z^{ln+1} and $\bar{z}^{(l+1)n-1}$, $l = 0, 1, 2, \ldots$, generate the module $\bar{\mathscr{P}}(\mathbf{D}_n)$. However, the identities

$$\begin{aligned} &\text{(a)} \quad z^{(l+2)n+1} = (z^n + \bar{z}^n) z^{(l+1)n+1} - (z\bar{z})^n z^{ln+1} \\ &\text{(b)} \quad \bar{z}^{(l+3)n-1} = (z^n + \bar{z}^n)\bar{z}^{(l+2)n-1} - (z\bar{z})^n \bar{z}^{(l+1)n-1} \end{aligned} \tag{5.15}$$

show that the generators z^{ln+1}, $\bar{z}^{(l+1)n-1}$ are redundant for $l \geq 2$. Similarly

$$\begin{aligned} &\text{(c)} \quad z^{n+1} = (z^n + \bar{z}^n) z - (z\bar{z})\bar{z}^{n-1}, \\ &\text{(d)} \quad \bar{z}^{2n-1} = (z^n + \bar{z}^n)\bar{z}^{n-1} - (z\bar{z})^{n-1} z. \end{aligned}$$

Hence the generators z^{n+1} and $\bar{z}^{2n-1}$ are redundant. This proves the claim.

To end this section we discuss when the representation of a Γ-equivariant g in (5.5) in terms of given generators $g_1, \ldots, g_r$ is unique. We say that $g_1, \ldots, g_r$ *freely generate* the module $\bar{\mathscr{E}}(\Gamma)$ over $\mathscr{E}(\Gamma)$ if the relation

$$f_1 g_1 + \cdots + f_r g_r \equiv 0, \tag{5.16}$$

where $f_j \in \mathscr{E}(\Gamma)$, implies that

$$f_1 \equiv \cdots \equiv f_r \equiv 0. \tag{5.17}$$

We also say that $\bar{\mathscr{E}}(\Gamma)$ is a *free* module over $\mathscr{E}(\Gamma)$. (This definition is the module version of linear independence in vector spaces.) It is clear that if $g_1, \ldots, g_r$ freely generate $\bar{\mathscr{E}}(\Gamma)$ then every $g \in \bar{\mathscr{E}}(\Gamma)$ may be written uniquely as $g = f_1 g_1 + \cdots + f_r g_r$ where $f_j \in \mathscr{E}(\Gamma)$.

Each module discussed in the preceding examples is free. We show this for Example 5.3(c), where $\Gamma = \mathbf{D}_n$. Suppose that

$$p(z\bar{z}, z^n + \bar{z}^n)z + q(z\bar{z}, z^n + \bar{z}^n)\bar{z}^{n-1} \equiv 0. \tag{5.18}$$

Suppose there exists $z \in \mathbb{C}$ at which $q(z\bar{z}, z^n + \bar{z}^n) \neq 0$. By continuity $q \neq 0$ in a neighbourhood of z. Multiply (5.18) by $\bar{z}$ and solve for

$$\bar{z}^n = p(z\bar{z}, z^n + \bar{z}^n) z\bar{z} / q(z\bar{z}, z^n + \bar{z}^n). \tag{5.19}$$

The right-hand side of (5.18) is real, but $\bar{z}^n$ is never real-valued on an open set (or else it would be everywhere real) so we have a contradiction. Hence $q \equiv 0$. But (5.18) now implies $p \equiv 0$. Hence $\bar{\mathscr{E}}(\mathbf{D}_n)$ is a free module over the ring $\mathscr{E}(\mathbf{D}_n)$ with free generators z and $\bar{z}^{n-1}$.

Exercises

5.1. Let $\mathbf{S}^1$ act on $\mathbb{C}^n$ as in Exercise 4.1. Prove that the equivariants are generated as a module over the invariants by the mappings $z \mapsto z_k$, $z \mapsto iz_k$, for $k = 1, \ldots, n$.

5.2. Let $\mathbf{S}^1$ act on $\mathbb{C}^2$ as in Exercise 4.2. Prove that the equivariants are generated as a module over the invariants by the mappings $(z_1, z_2) \mapsto$

$$(z_1, 0), (iz_1, 0), (\bar{z}_1^{l-1} z_2^k, 0), (i\bar{z}_1^{l-1} z_2^k, 0),$$

$$(0, z_2), (0, iz_2), (0, z_1^l \bar{z}_2^{k-1}), (0, iz_1^l \bar{z}_2^{k-1}).$$

5.3. Let $\mathbf{T}^2$ act on $\mathbb{C}^2$ as in Exercise 4.3. Find generators for the equivariants.

5.4. Which of the preceding modules of equivariants are free?

5.5. Let $\Gamma = \mathbb{O} \oplus \mathbf{Z}_2^c$ be the symmetry group of a cube acting on $\mathbb{R}^3$ as in Exercise 4.7. Prove that the module of Γ-equivariants is generated by the mappings

$$X_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad X_2 = \begin{bmatrix} x^3 \\ y^3 \\ z^3 \end{bmatrix} \quad X_3 = \begin{bmatrix} y^2 z^2 x \\ z^2 x^2 y \\ x^2 y^2 z \end{bmatrix},$$

and is a free module over the ring of invariants.