(a) Two one-dimensional irreducibles, the trivial representation and the representation in which $\gamma \in \mathbf{O}(2)$ acts as multiplication by $\det \gamma$.
(b) A countably infinite family of two-dimensional irreducibles defined by
   (i) $z \mapsto e^{ki\theta} z$
   (ii) $z \mapsto \bar{z}$
   where $z \in \mathbb{C} \equiv \mathbb{R}^2$ and $k = 1, 2, 3, \ldots$.
   (*Hint*: If $\mathbf{O}(2)$ acts irreducibly on a vector space V, show that the subgroup $\mathbf{SO}(2)$ also acts irreducibly on V.)

7.3. Show that all irreducibles for $\mathbf{Z}_n$ and $\mathbf{D}_n$ are of dimension 1 or 2.

CHAPTER XIII

Symmetry-Breaking in Steady-State Bifurcation

§0. Introduction

In this chapter we begin to study the structure of bifurcations of steady-state solutions to systems of ODEs

$$\frac{dx}{dt} + g(x, \lambda) = 0 \tag{0.1}$$

where $g: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ commutes with the action of a compact Lie group Γ on $V = \mathbb{R}^n$. Steady-state solutions satisfy $dx/dt = 0$; that is,

$$g(x, \lambda) = 0. \tag{0.2}$$

We focus here on the symmetries that a solution x may possess and in particular define some simple "geometric" notions that will prove to be of central importance.

In §1 we note that since Γ commutes with g, if x is a solution then so is γx for all $\gamma \in \Gamma$. The set of all γx for $\gamma \in \Gamma$ is the *orbit* of x under Γ. The amount of symmetry present in a solution x is measured by its *isotropy subgroup*

$$\Sigma = \Sigma_x = \{\sigma \in \Gamma: \sigma x = x\}.$$

The smaller Σ is, the larger is the orbit of x.

In §2 we introduce the *fixed-point subspace*

$$\text{Fix}(\Sigma) = \{v \in V | \sigma v = v \text{ for all } \sigma \in \Sigma\}.$$

It is a linear subspace of V and, remarkably, is invariant under g (even when g is *nonlinear*). This leads to a strategy for finding solutions to (0.2) with preassigned isotropy subgroups Σ: restrict g to $\text{Fix}(\Sigma)$ and solve there. This strategy will be used repeatedly in the sequel. It is important to be able to compute dim $\mathrm{Fix}(\Sigma)$, and we prove a trace formula for this.

The main result of this chapter, proved in §3, is the equivariant branching lemma (Theorem 3.2) due to Vanderbauwhede [1980] and Cicogna [1981]. This states that, with certain conditions on Σ, a unique branch of solutions to (0.2) with isotropy subgroup Σ exists. The main hypothesis is that the fixed-point subspace $\mathrm{Fix}(\Sigma)$ is one-dimensional. Thus the point of view is to prescribe in advance the symmetries required of x and to reduce the problem to a study of $g|\mathrm{Fix}(\Sigma)$.

The restriction $\dim \mathrm{Fix}(\Sigma) = 1$ is not as arbitrary as it may appear, and this condition is often satisfied. The problem (0.2) is connected with "spontaneous symmetry-breaking" as follows. Suppose that (0.1) has for each λ a trivial solution $x = 0$ (which manifestly has isotropy subgroup Γ). Suppose it to be asymptotically stable for $\lambda < 0$ and to lose stability at $\lambda = 0$. Usually such a loss of stability is associated with the occurrence of new branches of solutions $x \neq 0$ to (0.2), emanating from the trivial branch at $\lambda = 0$. Such solutions often have isotropy subgroups Σ smaller than Γ. We may ask, Which Σ typically arise in this way? In the language of symmetry-breaking, one says that the solution spontaneously breaks symmetry from Γ to Σ. "Spontaneously" here means that the *equation* $g = 0$ still commutes with all of Γ. Instead of a unique solution $x = 0$ with all of Γ as its symmetries, we see a set of symmetrically related solutions (orbits under Γ modulo Σ) each with symmetry group (conjugate to) Σ. In many examples it turns out that the subgroups Σ are maximal isotropy subgroups—not contained in any larger isotropy subgroup other than Γ. (Exceptions to this statement do occur; see §10.) If $\dim \mathrm{Fix}(\Sigma) = 1$ then Σ is maximal, and such Σ are the most tractable maximal isotropy subgroups.

Thus the equivariant branching lemma yields a set of solution branches in a relatively simple way. It is important to decide whether the solutions associated with any of these branches can be asymptotically stable. In §4 we show that for some group actions Γ on $\mathbb{R}^n$, *all* such branches are unstable. This means that in some problems it is essential to consider degeneracies; this leads to problems that can be solved using singularity theory. See Chapters XIV and XV.

In §5 we discuss in more detail how to represent Γ-equivariant bifurcations by a (schematic) bifurcation diagram. Such diagrams are very convenient, but we make their schematic nature explicit to avoid misunderstandings.

§§6–9 apply the theory thus developed to two classes of examples: the groups $\mathbf{SO}(3)$ and $\mathbf{O}(3)$ acting in *any* irreducible representation. The proofs may be omitted if so desired. These representations are obtained in §7, which links them to the classical idea of "spherical harmonics."

Finally in §10 we discuss to what extent we may expect spontaneous symmetry-breaking to occur to maximal isotropy subgroups. This section is optional. Although many questions remain unanswered, it is possible to establish a number of facts. In particular there are three distinct types of

maximal isotropy subgroup, which we call real, complex, and quaternionic. A theorem due to Dancer [1980a] effectively rules out all but the real maximal isotropy subgroups. On the other hand, Chossat [1983] and Lauterbach [1986] give examples in which submaximal isotropy subgroups arise generically, and we outline their results. We also describe two contexts in which solutions occur for *all* maximal isotropy subgroups. These contexts are variational equations (Michel [1972]) and periodic solutions near equilibria of Hamiltonian systems (Montaldi, Roberts, and Stewart [1986]).

§1. Orbits and Isotropy Subgroups

Let Γ be a Lie group acting on the vector space V. There are two simple notions used in describing aspects of a group action, which are intimately related to the way we think of bifurcation problems with symmetry. We explain these ideas and relations in the following discussion.

The *orbit* of the action of Γ on $x \in V$ is the set

$$\Gamma x = \{\gamma x : \gamma \in \Gamma\}. \tag{1.1}$$

Suppose that $f: V \to V$ is Γ-equivariant; then when f vanishes, it vanishes on orbits of Γ. For if $f(x) = 0$, then

$$f(\gamma x) = \gamma f(x) = \gamma 0 = 0.$$

In other words, this calculation shows that symmetric equations (Γ-equivariants) cannot distinguish between points (solutions) on the same orbit.

The *isotropy subgroup* of $x \in V$ is

$$\Sigma_x = \{\gamma \in \Gamma : \gamma x = x\}. \tag{1.2}$$

See the following for an example. We think of isotropy subgroups as giving the symmetries of the point x (under the action of Γ). In later sections we shall attempt to find solutions to $f = 0$, for some unspecified Γ-equivariant mapping f, by specifying required symmetries for the solution x, that is, by specifying the isotropy subgroup of x.

It is natural to ask how the isotropy subgroups of two points on the same orbit compare. The answer is as follows:

Lemma 1.1. *Points on the same orbit of Γ have conjugate isotropy subgroups. More precisely,*

$$\Sigma_{\gamma x} = \gamma \Sigma_x \gamma^{-1}. \tag{1.3}$$

Remarks.

(a) Let $\Sigma \subset \Gamma$ be a subgroup and let $\gamma \in \Gamma$. Then

$$\gamma \Sigma \gamma^{-1} = \{\gamma \sigma \gamma^{-1} : \sigma \in \Sigma\}$$

is a subgroup of Γ, said to be *conjugate* to Σ.

(b) The *conjugacy class* of Σ consists of all subgroups of Γ that are conjugate to Σ.

PROOF. Let $x \in V$ and $\gamma \in \Gamma$. Suppose that $\sigma \in \Sigma_x$. We claim that $\gamma\sigma\gamma^{-1} \in \Sigma_{\gamma x}$. We may check this directly:

$$\gamma\sigma\gamma^{-1}(\gamma x) = \gamma\sigma(\gamma^{-1}\gamma)x = \gamma\sigma x = \gamma x,$$

the last equality holding since $\sigma \in \Sigma_x$. It follows that

$$\Sigma_{\gamma x} \supset \gamma\Sigma_x\gamma^{-1}.$$

Replacing x by γx and γ by γ^{-1} yields $\Sigma_x \supset \gamma^{-1}\Sigma_{\gamma x}\gamma$, which proves the lemma. □

A convenient method for describing geometrically the group action of Γ on V is to lump together in a set W all points of V that have conjugate isotropy subgroups. We say that W is an *orbit type* of the action.

We illustrate these ideas by considering the action of the dihedral group $\mathbf{D}_n$ on $\mathbb{C}$ generated by

$$\kappa: z \mapsto \bar{z} \quad \text{and} \quad \zeta: z \mapsto e^{2\pi i/n}z.$$

Geometrically we picture the action of $\mathbf{D}_n$ as the symmetries of a regular n-gon centered at the origin in the plane. This n-gon is shown in Figure 1.1 by dashed lines, when $n = 5$. We derive in the following the orbit types of this group action. The result depends on whether n is odd or even, and for simplicity we consider only the case when n is odd. The complete results may be found in §5. The vertices on the n-gon, shown as □ in Figure 1.1, are mapped into each other by Γ. More precisely, these vertices constitute a single orbit of the action of Γ. The isotropy subgroup of a vertex on the real axis (not at the origin) is the group $\mathbf{Z}_2$ generated by κ. The other vertices have isotropy subgroups conjugate to $\mathbf{Z}_2$, by Lemma 1.1. Finally, if $t \neq 0$ then linearity of the action implies that $\Sigma_{tz} = \Sigma_z$. So all points on the lines joining the origin to a vertex have conjugate isotropy subgroups and belong to the same orbit type.

Next we consider a point near, but not on, the real axis, indicated by a • in Figure 1.1. By reflection and rotation we see that its orbit contains $2n$ points,

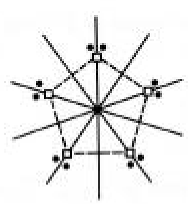

Figure 1.1. Orbits of the action of $\mathbf{D}_5$ on $\mathbb{C}$.

Table 1.1. Orbit Types and Isotropy Subgroups for $\mathbf{D}_n$ on $\mathbb{C}$, n odd

Orbit Type	Isotropy Subgroup	Size of Orbit
$\{0\}$	$\mathbf{D}_n$	1
$\{z \in \mathbb{C} \mid \mathrm{Im}(z^n) = 0, z \neq 0\}$	$\mathbf{Z}_2$	n
$\{z \in \mathbb{C} \mid \mathrm{Im}(z^n) \neq 0\}$	$\mathbb{1}$	$2n$

and the only group element that fixes one of these points is the identity in $\mathbf{D}_n$. Hence all points in the wedges between the vertex-origin lines belong to the same orbit type.

Finally, of course, the origin forms an orbit on its own and is fixed by the whole group $\mathbf{D}_n$. Thus there are three orbit types. We list these, along with their (conjugacy class of) isotropy subgroups, in Table 1.1. (In the case n even, points on the lines joining the origin to midpoints of edges of the n-gon have nontrivial isotropy subgroups not conjugate to those listed in Table 1.1; see §5.)

In this example "almost all" points—an open dense set—have trivial isotropy subgroup. It is a general theorem (Bredon [1972], p. 179) that there exists a unique minimal isotropy subgroup $\Sigma_{\min}$ for any linear action of a Lie group Γ on a vector space V and that points with this isotropy form an open dense subset of V. Since $\mathrm{Fix}(\Sigma_{\min})$ contains an open dense subset of V and is a vector space, it must be the whole of V; therefore, $\Sigma_{\min}$ is the *kernel* of the action—the subgroup of all elements of Γ that act on V as the identity. The points with isotropy group $\Sigma_{\min}$ are said to have *principal orbit type*.

We see that in this example, the larger the orbit, the smaller the isotropy subgroup. We formalize this observation as follows:

Proposition 1.2. *Let Γ be a compact Lie group acting on V. Then*
(a) *If $|\Gamma| < \infty$, then $|\Gamma| = |\Sigma_x||\Gamma x|$.*
(b) $\dim \Gamma = \dim \Sigma_x + \dim \Gamma x$.

Remarks.
(a) Proposition 1.2(a) states that the order of the group Γ is the product of the order of Σ_x and the size of the orbit of x. This formula may be checked for $\Gamma = \mathbf{D}_n$ from Table 1.1, using the fact that $|\mathbf{D}_n| = 2n$.
(b) Lie groups are always smooth manifolds and have well-defined dimensions. Since isotropy subgroups are always Lie subgroups both $\dim \Gamma$ and $\dim \Sigma_x$ make sense. Similarly orbits of Lie groups are always submanifolds and have well-defined dimensions. Thus $\dim \Gamma x$ makes sense.

SKETCH OF PROOF. There is a natural map $\varphi: \Gamma \to \Gamma x$ defined by

$$\varphi(\gamma) = \gamma x. \tag{1.4}$$

By definition φ maps onto the orbit Γx, and $\varphi^{-1}(x) = \Sigma_x$. Define the *coset space* of a subgroup Σ of Γ to be

$$\Gamma/\Sigma = \{\gamma\Sigma \mid \gamma \in \Gamma\}$$

where we recall that the cosets of Σ in Γ are the sets

$$\gamma\Sigma = \{\gamma\sigma \mid \sigma \in \Sigma\}.$$

Then φ induces a map

$$\begin{aligned} \psi&: \Gamma/\Sigma_x \to \Gamma x \\ \psi(\gamma) &= \gamma x \end{aligned} \tag{1.5}$$

which is both one-to-one and onto. In the case that Γ is finite, a simple counting of the cosets in Γ/Σ_x verifies part (a). In general, both φ and ψ are smooth mappings and $(d\psi)_0$ is invertible. It follows from the inverse function theorem that

$$\dim \Gamma x = \dim(\Gamma/\Sigma_x)$$

from which part (b) is immediate. □

Remark (d) of XII, §4, promised a simple criterion for $\mathscr{P}(\Gamma)$ to be a polynomial ring. We have now defined the concepts needed to state this; we omit the proof. Suppose that Γ acts on V with minimal isotropy subgroup $\Sigma_{\min}$. Let $\{u_1(x), \ldots, u_s(x)\}$ be a Hilbert basis for $\mathscr{P}(\Gamma)$. If

$$s = \dim V - \dim \Gamma + \dim \Sigma_{\min} \tag{1.6}$$

then $\mathscr{P}(\Gamma)$ is a polynomial ring.

In particular, if Γ is finite then (1.6) reduces to

$$s = \dim V. \tag{1.7}$$

For example, (1.7) trivially implies that $\mathscr{P}(\mathbf{D}_n)$ is a polynomial ring whenever $\mathbf{D}_n$ acts irreducibly on $\mathbb{C}$.

Exercises

1.1. Let $\mathbf{O}(n)$ act on $\mathbb{R}^n$ in its standard representation. Find the orbits and the corresponding isotropy subgroups.

1.2. Let Γ be the group of all symmetries, including reflections, of a cube center the origin of $\mathbb{R}^3$ with edges parallel to the axes. (In the notation of XIII, §9, Γ is the group $\mathbb{O} \oplus \mathbf{Z}_2^c$.) Show that

(a) $|\Gamma| = 48$ and Γ is generated by

$$\kappa_x = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \quad R_y = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

(b) Show that the orbit data for Γ are as follows:

Orbit Representative	Isotropy Subgroup
$(0, 0, 0)$	Γ
$(x, 0, 0)$	$\mathbf{D}_4$
$(x, x, 0)$	$\mathbf{Z}_2' \oplus \mathbf{Z}_2'$
(x, x, x)	$\mathbf{S}_3$
$(x, y, 0)$	$\mathbf{Z}_2'$
(x, x, z)	$\mathbf{Z}_2'$
(x, y, z) $\lvert x\rvert, \lvert y\rvert, \lvert z\rvert$ distinct and $\neq 0$	$\mathbb{1}$

where $\mathbf{D}_4$ is generated by R_x and κ_z (in obvious notation, compare (a)), $\mathbf{Z}_2'$ by κ_z, and $\mathbf{Z}_2'$ by $(x, y, z) \mapsto (x, z, y)$.

(c) Verify Proposition 1.2(a) directly for this example.

1.3. Find the orbits and isotropy subgroups for $\mathbf{O}(3)$ in its five-dimensional representation (as in Exercise XII, 3.6). Verify Proposition 1.2(b) for this example. (*Hint*: Every symmetric matrix can be diagonalized.)

1.4. (a) Show that in $\mathbf{D}_{2n+1}$ all reflections are conjugate.

(b) In $\mathbf{D}_{2n}$ show that there are two geometrically distinct types of reflection: those through lines joining the origin to a vertex and those through lines joining the origin to the midpoint of an edge. Prove that all reflections of the same type are conjugate in $\mathbf{D}_{2n}$, but that different types of reflection are not conjugate.

(c) Prove that all reflections in $\mathbf{D}_{2n}$ are conjugate in $\mathbf{D}_{4n}$.

1.5. Let $\mathbf{Z}_2$ act on $\mathbb{R}^2$ so that $-1 \in \mathbf{Z}_2$ acts as $(x, y) \mapsto (-x, y)$. Prove that $\mathscr{P}(\mathbf{Z}_2)$ is not a polynomial ring.

§2. Fixed-Point Subspaces and the Trace Formula

This section divides into three subsections, devoted to the following topics:

(a) The existence of invariant subspaces for *nonlinear* equivariant mappings: the fixed-point subspaces,

(b) A method for computing the dimensions of fixed-point subspaces: the trace formula,

(c) Ways to use the dimensions of fixed-point subspaces to find an important class of isotropy subgroups: the maximal isotropy subgroups.

(a) Fixed-Point Subspaces

One of the most remarkable as well as one of the simplest features of nonlinear Γ-equivariant mappings is that their equivariance forces them to have invariant linear subspaces. Moreover, these invariant subspaces correspond naturally to certain subgroups of Γ.

Let $\Sigma \subset \Gamma$ be a subgroup. The *fixed-point subspace* of Σ is

$$\mathrm{Fix}(\Sigma) = \{x \in V: \sigma x = x \text{ for all } \sigma \in \Sigma\}. \tag{2.1}$$

If it is important to display the space V explicitly we write $\mathrm{Fix}_V(\Sigma)$. Observe that $\mathrm{Fix}(\Sigma)$ is always a linear subspace of V since

$$\mathrm{Fix}(\Sigma) = \bigcap_{\sigma \in \Sigma} \ker(\sigma - Id)$$

and each kernel is a linear subspace.

Note that the simplest fixed-point subspaces are $\mathrm{Fix}(\mathbb{1})$ and $\mathrm{Fix}(\Gamma)$. Since the identity subgroup $\mathbb{1}$ fixed every point, we have $\mathrm{Fix}(\mathbb{1}) = V$. At the other extreme, $\mathrm{Fix}(\Gamma)$ consists of all vectors in V that are fixed by every element in Γ. Thus $\mathrm{Fix}(\Gamma)$ is the subspace of V on which Γ acts trivially. We shall often adopt the hypothesis that $\mathrm{Fix}(\Gamma) = \{0\}$.

We now show that the fixed-point subspaces have the invariance property asserted earlier.

Lemma 2.1. *Let $f: V \to V$ be Γ-equivariant. Let $\Sigma \subset \Gamma$ be a subgroup. Then*

$$f(\mathrm{Fix}(\Sigma)) \subset \mathrm{Fix}(\Sigma). \tag{2.2}$$

PROOF. Let $\sigma \in \Sigma$, $x \in \mathrm{Fix}(\Sigma)$. Then

$$f(x) = f(\sigma x) = \sigma f(x) \tag{2.3}$$

where the first equality follows from the definition of $\mathrm{Fix}(\Sigma)$, and the second from equivariance. From (2.3) we see that σ fixed $f(x)$. Therefore, $f(x) \in \mathrm{Fix}(\Sigma)$. □

Remark. In Lemma 2.1 we do not require Σ to be an isotropy subgroup. However, for any subgroup Σ, $\mathrm{Fix}(\Sigma)$ is equal to the sum W of all subspaces $\mathrm{Fix}(\Delta)$ where $\Delta \supset \Sigma$ is an isotropy subgroup. To prove this, first let $v \in \mathrm{Fix}(\Sigma)$. Then $\Sigma_v \supset \Sigma$ and $v \in \mathrm{Fix}(\Sigma_v)$. Hence we may take $\Delta = \Sigma_v$ to show that $v \in W$, so $\mathrm{Fix}(\Sigma) \subset W$. On the other hand, if $w \in W$ the $w = w_1 + \cdots + w_k$ where $w_j \in \mathrm{Fix}(\Delta_j)$, for an isotropy subgroup $\Delta_j \supset \Sigma$. But this means that $\sigma w_j = w_j$ for all $\sigma \in \Sigma$, so $w_j \in \mathrm{Fix}(\Sigma)$; therefore, $w \in \mathrm{Fix}(\Sigma)$ and so $W \subset \mathrm{Fix}(\Sigma)$. Hence $W = \mathrm{Fix}(\Sigma)$.

Thus in theory there is no real loss of generality if we let Σ run through just the isotropy subgroups of Γ. However, it may sometimes be convenient *not* to require Σ to be an isotropy subgroup, since this condition may not be easy to check.

For an example where we can check Lemma 2.1 directly, consider once more $\Gamma = \mathbf{D}_n$ in its standard action on $\mathbb{C}$. We find $\mathrm{Fix}(\Sigma)$ for the isotropy subgroups Σ. Obviously if $\Sigma = \mathbb{1}$ then $\mathrm{Fix}(\Sigma) = V$; and if $\Sigma = \mathbf{D}_n$ then $\mathrm{Fix}(\Sigma) = \{0\}$. If $\Sigma = \mathbf{Z}_2$ then $\mathrm{Fix}(\Sigma)$ is the real axis; and if Σ is a conjugate of $\mathbf{Z}_2$ then $\mathrm{Fix}(\Sigma)$ is the image of the real axis under an element of $\mathbf{D}_n$, that is, one of the lines through the origin and a vertex.

Taking $\Sigma = \mathbf{Z}_2$ in Lemma 2.1 it follows that every $\mathbf{D}_n$-equivariant mapping f must leave the real axis invariant. By (XII, 5.12) the general f has the form

$$f(z) = p(u, v)z + q(u, v)\bar{z}^{n-1}$$

where $u = z\bar{z}$, $v = z^n + \bar{z}^n$. If $z = x$ is real, then

$$f(x) = p(x^2, 2x^n)x + q(x^2, 2x^n)x^{n-1}$$

is also real. So $\mathrm{Fix}(\mathbf{Z}_2)$ is invariant under f as predicted.

An immediate consequence of Lemma 2.1 is the existence of trivial solutions for Γ-equivariant mappings f. More precisely, if $\mathrm{Fix}(\Gamma) = \{0\}$ then $\{0\}$ must be invariant under f, so that $f(0) = 0$. In fact, we have three equivalent properties:

Proposition 2.2. *Let Γ be a compact Lie group acting on V. The following are equivalent:*

(a) $\mathrm{Fix}(\Gamma) = \{0\}$.

(b) *Every Γ-equivariant map $f: V \to V$ satisfies $f(0) = 0$ (there always exist trivial solutions).*

(c) *The only Γ-invariant linear function is the zero function.*

Remark. The most important implication (a) $\Rightarrow$ (b) we showed previously, using Lemma 2.1.

PROOF. The converse (b) $\Rightarrow$ (a) is proved easily as follows. We claim that for every $v \in \mathrm{Fix}(\Gamma)$, the constant mapping $f(x) = v$ is Γ-equivariant. If so, (b) will imply that $v = f(0) = 0$, proving (a). To verify the claim, compute

$$\gamma f(x) = \gamma v = v = f(\gamma x).$$

The first equality is by definition of $f(x)$, the second follows since $v \in \mathrm{Fix}(\Gamma)$, and the third holds since f is constant.

Next we show that (a) implies (c). Let $L: V \to \mathbb{R}$ be linear and invariant. We may write L in the form

$$L(x) = \langle v, x \rangle$$

for some $v \in V$. We claim that $v \in \mathrm{Fix}(\Gamma)$, whence (a) implies (c). Since L is Γ-invariant, $L(x) = L(\gamma^{-1}x)$ for all $\gamma \in \Gamma$. Since Γ acts orthogonally $\gamma^{-1} = \gamma^t$. Thus

$$\langle v, x \rangle = \langle v, \gamma^{-1}x \rangle = \langle v, \gamma^t x \rangle = \langle \gamma v, x \rangle$$

for all x. Hence $\gamma v = v$ for all γ and $v \in \mathrm{Fix}(\Gamma)$, as claimed.

Finally we prove that (c) $\Rightarrow$ (b). Let $f: V \to V$ be Γ-equivariant. We must show that $f(0) = 0$. To do this, define

$$L(x) = \langle f(0), x \rangle$$

where $\langle\ ,\ \rangle$ is a Γ-invariant inner product on V. We claim that the linear function L is Γ-invariant. If so, then $L \equiv 0$ and $f(0) = 0$. To verify the claim, compute

$$L(\gamma x) = \langle f(0), \gamma x \rangle = \langle \gamma^{-1} f(0), x \rangle = \langle f(0), x \rangle = L(x). \qquad \square$$

(b) The Trace Formula

In later sections we shall want to compute the dimension of $\mathrm{Fix}(\Sigma)$. There is an elegant formula for this, which depends only on the trace $\mathrm{tr}(\sigma)$ for $\sigma \in \Sigma$. Because Γ acts linearly on V we may think of $\gamma \in \Gamma$ as acting by the linear mapping $\rho_\gamma: x \mapsto \gamma x$. By $\mathrm{tr}(\sigma)$ we mean the trace of ρ_σ on V.

Theorem 2.3 (Trace Formula). *Let Γ be a compact Lie group acting on V and let $\Sigma \subset \Gamma$ be a Lie subgroup. Then*

$$\dim \mathrm{Fix}(\Sigma) = \int_\Sigma \mathrm{tr}(\sigma) \tag{2.4}$$

where $\int$ denotes the normalized Haar integral on Σ.

Remark. If Σ is finite then (2.4) can be rephrased as

$$\dim \mathrm{Fix}(\Sigma) = \frac{1}{|\Sigma|} \sum_{\sigma \in \Sigma} \mathrm{tr}(\sigma). \tag{2.5}$$

See Example XIII, 1.4.

Proof. Define the linear transformation $A: V \to V$ by

$$A = \int_\Sigma \sigma. \tag{2.6}$$

Because the Haar integral is Σ-invariant, we see that

$$A = \int_\Sigma \sigma'\sigma$$

where σ' is any fixed element of Σ. It follows that

$$A^2 = A; \tag{2.7}$$

that is, A is a linear projection. To check (2.7), compute

$$\begin{aligned} A^2 = A \circ A = A\left(\int_{\sigma \in \Sigma} \sigma\right) &= \int_{\sigma' \in \Sigma} \sigma' \left(\int_{\sigma \in \Sigma} \sigma\right) \\ &= \int_{\sigma' \in \Sigma} \left(\int_{\sigma \in \Sigma} \sigma'\sigma\right) \\ &= \int_{\sigma' \in \Sigma} A \\ &= A. \end{aligned}$$

By (2.7)

$$\begin{aligned} &\text{(a)} \quad V = \ker A \oplus \operatorname{Im} A \\ &\text{(b)} \quad A | \operatorname{Im} A = Id. \end{aligned} \tag{2.8}$$

We verify (2.8b) first. Suppose $x \in \operatorname{Im} A$, so $x = Ay$. Using (2.7) we have

$$Ax = A^2 y = Ay = x,$$

proving (2.8b). To verify (2.8a) observe that $\dim \ker A + \dim \operatorname{Im} A = \dim V$, since A is linear. Thus it suffices to show that $\ker A \cap \operatorname{Im} A = \{0\}$. However, if $x \in \ker A \cap \operatorname{Im} A$ then $x = Ax$ by (2.8b), and $Ax = 0$.

It follows directly from (2.8) that

$$\mathrm{tr}(A) = \dim \operatorname{Im} A. \tag{2.9}$$

We claim that $\operatorname{Im} A = \mathrm{Fix}(\Sigma)$. The theorem will then follow since $\dim \operatorname{Im} A = \dim \mathrm{Fix}(\Sigma)$ and

$$\mathrm{tr}(A) = \int_{\sigma \in \Sigma} \mathrm{tr}(\sigma).$$

To prove the claim, observe that $\mathrm{Fix}(\Sigma) \supset \operatorname{Im} A$ by (2.8(b)). Conversely, $\mathrm{Fix}(\Sigma) \subset \operatorname{Im} A$ by (2.8(a)). More precisely, suppose $x \in \mathrm{Fix}(\Sigma)$. Write $x = k + y$ where $k \in \ker A$ and $y \in \operatorname{Im} A$. Then $x = Ax = Ak + Ay = y$. This can happen only if $k = 0$ and $x \in \operatorname{Im} A$. $\square$

In certain cases it is possible to use the trace formula to reduce the calculation of $\dim \mathrm{Fix}(\Sigma)$ to finding the dimensions of fixed-point spaces $\mathrm{Fix}(\Delta)$ for certain subgroups Δ of Σ. This reduction, stated in Lemma 2.5 later, will be of particular use when we discuss the fixed-point subspaces for subgroups of $\mathbf{SO}(3)$ and $\mathbf{O}(3)$ in §§6–9.

Definition 2.4. Let $H_1, \ldots, H_k$ be subgroups of a group Σ. We say that Σ is the *disjoint union* of $H_1, \ldots, H_k$ if

(a) $\Sigma = H_1 \cup \cdots \cup H_k$

(b) $H_i \cap H_j = \mathbb{1}$ for all $i \neq j$.

We use the notation $\Sigma = H_1 \dot{\cup} \cdots \dot{\cup} H_k$ to denote disjoint unions.

When Σ has a disjoint union decomposition then we can compute $\dim \mathrm{Fix}(\Sigma)$ in terms of the numbers $\dim \mathrm{Fix}(H_j)$:

Lemma 2.5. *Let $\Sigma = H_1 \dot{\cup} \cdots \dot{\cup} H_k$ be a finite subgroup of Γ, with Γ acting on V. Then*

$$\dim \mathrm{Fix}(\Sigma) = \frac{1}{|\Sigma|}\left[\sum_{i=1}^{k} |H_i| \dim \mathrm{Fix}(H_i) - (k-1)\dim V\right]. \tag{2.10}$$

Proof. From (2.5) we see that

$$\begin{aligned} \dim \mathrm{Fix}(\Sigma) &= \frac{1}{|\Sigma|}\sum_{\sigma\in\Sigma} \mathrm{tr}(\sigma) \\ &= \frac{1}{|\Sigma|}\left[\sum_{i=1}^{k}\sum_{h\in H_i} \mathrm{tr}(h) - (k-1)\mathrm{tr}(I)\right] \end{aligned} \tag{2.11}$$

where the second equality is obtained by splitting the sum over Σ into a sum over the H_i. Since Σ is a disjoint union of the H_i we must add $\mathrm{tr}(I)$ (k times) for the overlap on the identity element. Since we want to count $\mathrm{tr}(I)$ only once we subtract the overenumeration, obtaining (2.11).

To derive (2.10) from (2.11) we make two observations. First, $\mathrm{tr}(I) = \dim V$. Second, we apply the trace formula (2.5) directly to each H_i, obtaining

$$\dim \mathrm{Fix}(H_i) = \frac{1}{|H_i|}\sum_{h\in H_i} \mathrm{tr}(h).$$

Substitute this in (2.11) to yield the desired result. □

(c) Maximal Isotropy Subgroups

It is important to be able to determine, in as simple a manner as possible, whether a given closed subgroup is an isotropy subgroup. That is, we wish to do this without knowing the orbit structure of Γ. We now consider a distinguished class of isotropy subgroups for which this question may be answered using the dimensions of fixed-point subspaces.

Definition 2.6. Let Γ be a Lie group acting on V. An isotropy subgroup $\Sigma \subseteq \Gamma$ is *maximal* if there does not exist an isotropy subgroup Δ of Γ satisfying $\Sigma \subsetneq \Delta \subsetneq \Gamma$.

Lemma 2.7. *Let $\mathrm{Fix}(\Gamma) = \{0\}$, and let Σ be a subgroup of Γ. Then Σ is a maximal isotropy subgroup of Γ if and only if*:

$$\begin{aligned} &\text{(a)}\quad \dim \mathrm{Fix}(\Sigma) > 0 \\ &\text{(b)}\quad \dim \mathrm{Fix}(\Delta) = 0 \textit{ for every closed subgroup } \Delta \supsetneq \Sigma. \end{aligned} \tag{2.12}$$

Proof. Suppose Σ is a maximal isotropy subgroup of Γ. Then $\dim \mathrm{Fix}(\Sigma) > 0$ since Σ must fix some nonzero vector, by the definition of an isotropy subgroup. Suppose $\Delta \supsetneq \Sigma$ and suppose there is a vector $x \in V$ fixed by Δ. Then the isotropy subgroup Σ_x of x satisfies $\Sigma_x \supset \Delta \supsetneq \Sigma$. Since Σ is a maximal isotropy subgroup we must have $\Sigma_x = \Gamma$. But $\mathrm{Fix}(\Gamma) = \{0\}$, so $x = 0$. Therefore, $\dim \mathrm{Fix}(\Delta) = 0$.

Conversely, suppose that Σ satisfies (2.12). Then some nonzero vector $x \in V$ is fixed by Σ, so Σ_x contains Σ. Since Σ_x is an isotropy subgroup, it is closed. If $\Sigma_x \neq \Sigma$ then (2.12(b)) implies that $\dim \mathrm{Fix}(\Sigma_x) = 0$, contrary to Σ_x being an isotropy subgroup. Therefore $\Sigma = \Sigma_x$, so Σ is an isotropy subgroup. The same argument now proves that Σ is maximal. □

Lemma 2.7 provides a strategy for finding the maximal isotropy subgroups of Γ if we know enough about the dimensions of fixed-point spaces of subgroups of Γ. Namely, we find the largest closed subgroups with nonzero fixed-point subspaces. We use this strategy in §§6–9 to compute the maximal isotropy subgroups of $\mathbf{SO}(3)$ and $\mathbf{O}(3)$.

Exercises

2.1. Find the fixed-point subspaces for the isotropy subgroups of Exercises 1.1 and 1.3.

2.2. Let Σ be an isotropy subgroup of Γ. Show that the largest subgroup of Γ that leaves $\mathrm{Fix}(\Sigma)$ setwise invariant is $N = N_\Gamma(\Sigma)$. If $\dim \mathrm{Fix}(\Sigma) = 1$ show that N/Σ is either $\mathbb{1}$ or $\mathbf{Z}_2$. If it is $\mathbf{Z}_2$ show that the corresponding bifurcation is of pitchfork type.

2.3. Show that for the group $\mathbf{O} \oplus \mathbf{Z}_2^c$ of Exercise 1.2, the fixed-point subspaces are as follows:

Isotropy Subgroup	Fixed-Point Subspace	Dimension
Γ	$\{(0,0,0)\}$	0
$\mathbf{D}_4$	$\{(x,0,0)\}$	1
$\mathbf{Z}_2^c \oplus \mathbf{Z}_2^c$	$\{(x,x,0)\}$	1
$\mathbf{S}_3$	$\{(x,x,x)\}$	1
$\mathbf{Z}_2^c$	$\{(x,y,0)\}$	2
$\mathbf{Z}_2^c$	$\{(x,x,z)\}$	2
$\mathbb{1}$	$\mathbb{R}^3$	3

2.4. Let $\Gamma = \mathbf{Z}_2 \oplus \mathbf{Z}_2$ act on $\mathbb{R}^2$ by $(x,y) \mapsto (\pm x, \pm y)$ as in X, §1(a). Show that the action of Γ is not irreducible, but that $\mathrm{Fix}(\Gamma) = \{0\}$; that is, Γ-equivariant bifurcation problems have a trivial solution.

2.5. Show that the group $\mathbf{O}$ of rotational symmetries of a cube has a disjoint union decomposition into cyclic subgroups.

2.6. Verify Theorem 2.3 directly for the three maximal isotropy subgroups of $\mathbf{O} \oplus \mathbf{Z}_2^c$ listed in Exercise 2.2.

2.7. Let Γ act on V and let Σ be an isotropy subgroup. It is clear that $\dim \mathrm{Fix}(\Sigma)$ is the dimension of the trivial part of the isotypic decomposition of V for Σ, that is, the multiplicity with which the trivial representation of Σ occurs on V. If instead we ask the multiplicity of some other representation, then there is an analogous formula to Theorem 2.3 which may be deduced from the orthogonality relations for characters (see XIII, §7(f)). This exercise asks for a bare-hands proof of a special case.

Let $\Sigma = \mathbf{O}(2)$, and let ρ be the representation on $\mathbb{R}$ in which $\mathbf{SO}(2)$ acts trivially and κ acts as -1. Show that the dimension of the isotypic component corresponding to ρ is

$$\int_{\sigma \in \mathbf{SO}(2)} \operatorname{tr} \sigma - \int_{\sigma \in \mathbf{O}(2) \sim \mathbf{SO}(2)} \operatorname{tr} \sigma.$$

(*Hint*: Let

$$A = \int_{\sigma \in \mathbf{SO}(2)} \sigma - \int_{\sigma \in \mathbf{O}(2) \sim \mathbf{SO}(2)} \sigma$$

and mimic the proof of Theorem 2.3.)

§3. The Equivariant Branching Lemma

In this section we prove a simple but useful theorem of Vanderbauwhede [1980] and Cicogna [1981] to the effect that isotropy subgroups with one-dimensional fixed-point subspaces lead to solutions of bifurcation problems with symmetry.

Definition 3.1. Let Γ be a Lie group acting on a vector space V. A *bifurcation problem with symmetry group* Γ is a germ $g \in \vec{\mathscr{E}}_{x,\lambda}(\Gamma)$ satisfying $g(0,0) = 0$ and $(dg)_{0,0} = 0$.

Here we recall notation used earlier in this volume as well as in Volume I. A germ $g \in \vec{\mathscr{E}}_{x,\lambda}(\Gamma)$ is the germ of a Γ-equivariant mapping, which by abuse of notation we also denote by g. Here $g\colon V \times \mathbb{R} \to V$ satisfies

$$g(\gamma x, \lambda) = \gamma g(x, \lambda) \tag{3.1}$$

for all $\gamma \in \Gamma$. By convention our germs are based at the origin $(x, \lambda) = (0, 0)$.

In Definition 3.1 we require that $g(0,0) = 0$ to avoid trivial complications. If $\mathrm{Fix}(\Gamma) = \{0\}$ then Proposition 2.2 implies that $g(0, \lambda) \equiv 0$, and hence $g(0,0) = 0$. However, in general $g(0,0)$ need not vanish.

We also require that $(dg)_{0,0} = 0$. Recall that dg is the $n \times n$ Jacobian matrix obtained by differentiating g *in the* V-directions. Here $n = \dim V$. If $(dg)_{0,0}$ is nonzero, then we can use the Liapunov–Schmidt reduction with symmetries (see VIII, §3) to reduce g to the case where the Jacobian vanishes. Of course, this process will change n to a smaller value n' and will also change the representation of Γ. Nevertheless, we assume that this reduction *has already been performed* and we therefore assume $(dg)_{0,0} = 0$.

We claim that generically we may assume the action of Γ on $V = \mathbb{R}^n$ to be absolutely irreducible. Before stating the result more precisely, we must discuss the term *generic*. A rigorous definition is somewhat technical, and we try instead to convey the underlying idea.

Recall from Chapter II that a bifurcation problem $g(x, \lambda)$ is equivalent to a limit point singularity $\pm x^2 \pm \lambda$ precisely when the defining conditions

$$g(0,0) = 0, \qquad g_x(0,0) = 0 \tag{3.2}$$

and the nondegeneracy conditions

$$g_{xx}(0,0) \neq 0, \qquad g_\lambda(0,0) \neq 0 \tag{3.3}$$

are satisfied. We say that among those bifurcation problems g in one state variable having a singularity at the origin (i.e., those g satisfying (3.2)) it is generic for the singularity to be a limit point. More succinctly, we say that the "generic singularity" is a limit point.

We abstract this process as follows. Let g be a germ satisfying some property $\mathscr{P}$, where the defining conditions for $\mathscr{P}$ consist of a finite number of equalities involving a finite number of derivatives of g evaluated at the origin. The equalities in (3.2) provide an example, with $\mathscr{P}$ being the property "g has a singularity at the origin." A set S of germs is *generic* for property $\mathscr{P}$ if there exists a finite number of inequalities Q involving a finite number of derivatives of g at the origin, such that $g \in S$ if and only if g has property $\mathscr{P}$ and g satisfies the inequalities in Q. Thus, in the example, Q is given by (3.3) and limit points—those germs satisfying (3.2, 3.3)—are generic singularities.

Actually, even this definition must be qualified. The inequalities Q must not *contradict* any of the defining equalities of $\mathscr{P}$. For example, if $\mathscr{P}$ is defined by $g_x(0,0) = 0$ then Q should not include the inequality $g_x(0,0) \neq 0$. We do not intend that the empty set S be considered generic.

We find it convenient to use the word *generic* when we do not wish to specify the inequalities Q explicitly. The important point is that a "typical" germ with property $\mathscr{P}$ will be generic, where by *typical* we mean "not satisfying any additional constraints" (e.g., on derivatives). This follows since an atypical germ must violate an inequality in Q, that is, satisfy a further *equality*.

For example, in applications one expects to see only limit point singularities in steady-state bifurcation problems $g(x, \lambda)$, *unless* some other constraint such as symmetry is placed on g. (The effect of symmetry is to constrain certain terms of the Taylor series of g, so symmetry effectively imposes conditions on derivatives of g at the origin.) In Volume I we focused on nongeneric or degenerate singularities, since these are expected to occur "generically" in *multiparameter* systems. A major theme of this volume is to identify a "generic" class of one-parameter bifurcation problems with symmetry.

The following proposition, whose proof will be sketched at the end of this section, is a first step in that direction.

Proposition 3.2. *Let $G: \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}^N$ be a one-parameter family of Γ-equivariant mappings with $G(0,0) = 0$. Let $V = \ker(dG)_{0,0}$. Then generically the action of Γ on V is absolutely irreducible.*

Remark. When one is interpreting this proposition in the preceding framework, $\mathscr{P}$ is defined as follows. A germ G has property $\mathscr{P}$ if it is a germ of a one-parameter family of Γ-equivariant mappings, and $G(0,0) = 0$. The inequalities Q which imply that the action of Γ on $\ker(dG)_{0,0}$ is absolutely irreducible are left unstated.

Proposition 3.2 supports our assumption later that Γ acts absolutely irreducibly on $\mathbb{R}^n$ and that $g: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is a Γ-equivariant bifurcation problem. We use the assumption of absolute irreducibility as follows. Apply the chain rule to the identity $g(\gamma x, \lambda) = \gamma g(x, \lambda)$ to obtain

$$(dg)_{0,\lambda}\gamma = \gamma(dg)_{0,\lambda}. \tag{3.4}$$

Absolute irreducibility states that the only matrices commuting with all $\gamma \in \Gamma$ are scalar multiples of the identity. Therefore $(dg)_{0,\lambda} = c(\lambda)I$. Since $(dg)_{0,0} = 0$ by Definition 3.1, we have $c(0) = 0$. We now assume the hypothesis

$$c'(0) \neq 0, \tag{3.5}$$

which is valid generically.

We next state the result of Vanderbauwhede and Cicogna, which—despite the simplicity of its proof—forms the basis of many bifurcation results for symmetric problems.

Theorem 3.3 (Equivariant Branching Lemma). *Let Γ be a Lie group acting absolutely irreducibly on V and let $g \in \vec{\mathscr{E}}_{x,\lambda}(\Gamma)$ be a Γ-equivariant bifurcation problem satisfying* (3.5). *Let Σ be an isotropy subgroup satisfying*

$$\dim \mathrm{Fix}(\Sigma) = 1. \tag{3.6}$$

Then there exists a unique smooth solution branch to $g = 0$ such that the isotropy subgroup of each solution is Σ.

Remarks 3.4.

(a) We may restate the equivariant branching lemma as follows: Generically, bifurcation problems with symmetry group Γ have solutions corresponding to all isotropy subgroups with one-dimensional fixed-point subspaces. Since Σ is an isotropy subgroup satisfying (3.6) it follows that Σ is a maximal isotropy subgroup. Thus the equivariant branching lemma gives us a method for finding solutions corresponding to a special class of maximal isotropy subgroups. To see that Σ is maximal, suppose $\Delta \supsetneq \Sigma$ is an isotropy subgroup. Then $\mathrm{Fix}(\Delta) \subsetneq \mathrm{Fix}(\Sigma)$, whence $\mathrm{Fix}(\Delta) = \{0\}$, which is impossible.

(b) Cicogna [1981] generalizes Theorem 3.3 to the case in which $\dim \mathrm{Fix}(\Sigma)$ is odd, using a topological degree argument. However, to obtain effective information in this case we must also assume that Σ is a maximal isotropy subgroup. Otherwise, the solutions in $\mathrm{Fix}(\Sigma)$ whose existence is being asserted might actually have a larger isotropy subgroup than Σ.

In fact, we prove a slightly more general result than Theorem 3.3:

Theorem 3.5. *Let Γ be a Lie group acting on V. Assume*

(a) $\mathrm{Fix}(\Gamma) = \{0\}$,

(b) $\Sigma \subset \Gamma$ *is an isotropy subgroup satisfying* (3.6),

(c) $g: V \times \mathbb{R} \to V$ *is a Γ-equivariant bifurcation problem satisfying*

$$(dg_\lambda)_{0,0}(v_0) \neq 0 \tag{3.7}$$

where $v_0 \in \mathrm{Fix}(\Sigma)$ is nonzero.

Then there exists a smooth branch of solutions $(tv_0, \lambda(t))$ to the equation $g(t, \lambda) = 0$.

Two remarks make it clear why Theorem 3.3 follows from Theorem 3.5. First, it is easy to show that nontrivial irreducible actions satisfy $\mathrm{Fix}(\Gamma) = \{0\}$, since by Lemma 2.1 $\mathrm{Fix}(\Gamma)$ is an invariant subspace. Second, when Γ acts absolutely irreducibly,

$$(dg_\lambda)_{0,0}(v_0) = Kc'(0)$$

for some nonzero constant K. Hence (3.5) is equivalent to (3.7).

Remarks.

(a) The advantage of hypothesis (3.5) over (3.7) is that it holds simultaneously for all subgroups Σ of Γ.

(b) The advantage of Theorem 3.5 is that it does not require that Γ act irreducibly on V. However, a separate nondegeneracy condition (3.7) is required for each subgroup Σ satisfying (3.6).

(c) Since the solution branch $(tv_0, \lambda(t))$ lies in $\mathrm{Fix}(\Sigma) \times \mathbb{R}$, each solution for $t \neq 0$ has as its symmetries the isotropy subgroup Σ.

Proof of Theorem 3.5. It follows from Lemma 2.1 that

$$g: \mathrm{Fix}(\Sigma) \times \mathbb{R} \to \mathrm{Fix}(\Sigma).$$

Since $\dim \mathrm{Fix}(\Sigma) = 1$ we have

$$g(tv_0, \lambda) = h(t, \lambda)v_0.$$

Moreover, the assumption that $\mathrm{Fix}(\Gamma) = \{0\}$ implies by Corollary 2.2 that g has a trivial solution. So $h(0, \lambda) = 0$. Applying Taylor's theorem to h yields

$$g(tv_0, \lambda) = k(t, \lambda)tv_0.$$

By Definition 3.1

$$k(0, 0)v_0 = (dg)_{0,0}(v_0) = 0$$

and further

$$k_\lambda(0, 0)v_0 = (dg_\lambda)_{0,0}(v_0) \neq 0,$$

by assumption. Apply the implicit function theorem to solve $k(t, \lambda) = 0$ for $\lambda = \lambda(t)$ as required. □

EXAMPLE 3.6. $\Gamma = \mathbf{D}_n$ acting on $V = \mathbb{C}$. We know that the isotropy subgroup of every point on the real axis is a two-element subgroup $\mathbf{Z}_2$ generated by the reflection $\kappa: z \mapsto \bar{z}$. See Table 1.1. Moreover, the only complex numbers fixed by κ are the reals. Thus $\mathrm{Fix}(\mathbf{Z}_2) = \mathbb{R}$ and $\dim \mathrm{Fix}(\mathbf{Z}_2) = 1$. We conclude, using the equivariant branching lemma, that generically $\mathbf{D}_n$-equivariant bifurcation problems have solution branches consisting of solutions with $\mathbf{Z}_2$ symmetry.

We end this section with the following, as promised.

SKETCH OF PROOF OF PROPOSITION 3.2. In this sketch we show only that there exist small perturbations G_ε of G such that Γ acts absolutely irreducibly on $\ker(dG_\varepsilon)_{0,0}$. This argument can be expanded, with some effort, to give a proof of genericity.

We begin by claiming that the action of Γ on V may be assumed irreducible. Write $\mathbb{R}^N = V \oplus W$ where W is Γ-invariant and write

$$V = V_1 \oplus \cdots \oplus V_k$$

where each V_j is irreducible. In fact we can take W to be the sum of the generalized eigenspaces corresponding to nonzero eigenvalues of $(dG)_{0,0}$. Define $M: \mathbb{R}^N \to \mathbb{R}^N$ to be the unique linear mapping such that

$$M|W = 0$$

$$M|V_1 = 0$$

$$M|V_j = Id_{V_j}.$$

Let $\varepsilon \in \mathbb{R}$ and consider the Γ-equivariant perturbation

$$G_\varepsilon(x, \lambda) = G(x, \lambda) + \varepsilon Mx.$$

The eigenvalues of $(dG_\varepsilon)_{0,0}$ are 0 on V_1, and nonzero on W. Apply a Liapunov–Schmidt reduction to G_ε near $(0, 0)$ to obtain a bifurcation problem on V_1. Since Γ acts irreducibly on V_1, we have verified the claim.

We now assume that $g: V \times \mathbb{R} \to V$ is a bifurcation problem with symmetry group Γ and that Γ acts irreducibly but not absolutely irreducibly on V. We claim that in these circumstances there exist small perturbations of g which have *no* steady-state bifurcations near the origin. Let $\mathscr{D}$ be the vector space of linear mappings on V that commute with Γ. Recall from XII, §3, that $\mathscr{D}$ is isomorphic to one of $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, and that $\mathscr{D} \cong \mathbb{R}$ means that Γ acts absolutely irreducibly on V. Now Γ acts irreducibly on V, so $g(0, \lambda) \equiv 0$. The linear maps $L_\lambda = (dg)_{0,\lambda}$ commute with Γ and form a curve in $\mathscr{D}$. Since g is a bifurcation problem, $L_0 = 0$, so the curve passes through the origin. Generically we may assume that $\rho = (d/d\lambda)L_\lambda|_{\lambda=0} \neq 0$; that is, the curve L_λ has a nonzero tangent vector at $\lambda = 0$.

Assume that $\dim_{\mathbb{R}} \mathscr{D} > 1$, so that Γ does not act absolutely irreducibly on V. We can choose $0 \neq \delta \in \mathscr{D}$ such that ρ and δ are linearly independent. For $\varepsilon \in \mathbb{R}$ define the Γ-equivariant perturbation

$$g_\varepsilon(x) = g(x, \lambda) + \varepsilon\delta x.$$

When $\varepsilon = 0$, the curve

$$(dg_\varepsilon)_{0,\lambda} = (dg)_{0,\lambda} + \varepsilon\delta = L_\lambda + \varepsilon\delta$$

in $\mathscr{D}$ misses the origin entirely for λ near 0.

Thus $L_\lambda + \varepsilon\delta$ is not zero. A general argument now shows that it has no zero eigenvalues. Indeed, if $\alpha \in \mathscr{D}$ has a zero eigenvalue then $\alpha = 0$. To see this, suppose that $\alpha v = 0$ where $\alpha \neq 0$, $v \neq 0$. Since $\mathscr{D}$ is a division algebra, α^{-1} exists, and $v = 1v = \alpha^{-1}\alpha v = \alpha^{-1}0 = 0$. This contradiction forces $\alpha = 0$ as claimed.

Thus when g is a bifurcation problem whose symmetry group Γ acts irreducibly but not absolutely irreducibly, small perturbations of g have no steady-state bifurcation whatsoever. □

Remark. Proposition 3.2 does not exclude the possibility that Γ-invariant equilibria can lose stability by having center subspaces with irreducible but not absolutely irreducible representations of Γ. This can happen generically with Hopf bifurcation, but *not* with steady-state bifurcation. See the definition of Γ-simple in XVI, §1.

EXERCISES

3.1. Use the results of Exercises 1.2 and 2.3 to investigate steady-state bifurcation with the symmetry $\mathbf{O} \oplus \mathbf{Z}_2^c$ of the cube (see Melbourne [1987a]). Prove that
   (a) Generically three branches of solutions bifurcate, with isotropy subgroups $\mathbf{D}_4$, $\mathbf{S}_3$, and $\mathbf{Z}_2^c \oplus \mathbf{Z}_2'$.
   (b) Generically there are no solution branches corresponding to the isotropy subgroups $\mathbf{Z}_2^c$, $\mathbf{Z}_2'$, and 1.

3.2. Let $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ act on $\mathbb{R}^2$ by $(\pm x, \pm y)$ as in Chapter X. Show that the existence of the pure mode solutions (X, 1.11(b), (c)) can be obtained by applying the equivari-