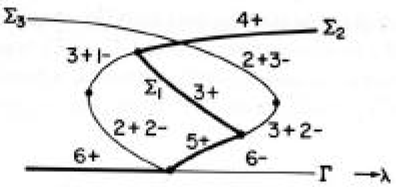

Figure 5.1. Fictitious bifurcation diagram with symmetry.

has branches of nontrivial solutions corresponding to isotropy subgroups Σ_j ($j = 1, 2, 3$) where $\dim \Sigma_j = j$. Consider the fictitious bifurcation diagram in Figure 5.1. Each point on branch Σ_j corresponds to a group orbit of dimension $\dim \Gamma - \dim \Sigma_j = 4 - j$. For example, a point on the Σ_2 branch actually corresponds to a two-dimensional manifold of solutions, and where the branches corresponding to Σ_2 and Σ_1 intersect we actually have a 3-manifold of solutions in $\mathbb{R}^6$ merging into a 2-manifold. Obviously such intersections can be very complicated, but fortunately, for most aspects of bifurcation theory, the detailed geometric picture of how such transitions take place is not particularly relevant.

Because our schematic bifurcation diagrams are "projections" into $\mathbb{R}^2$, branches of solutions may appear to intersect, even though they do not actually intersect in $V \times \mathbb{R}$. We "solve" this problem by placing bold dots at genuine intersection points. Thus Figure 5.1 illustrates a situation where branch Σ_1 does intersect branches Σ_2 and Σ_3, but branches Σ_2 and Σ_3 intersect only at the origin.

We now turn to the question of orbital stability. Recall that equivariance under Γ forces several eigenvalues of dg at $g = 0$ to zero. The number of these zero eigenvalues is equal to the dimension of the orbit of solutions. When making stability assignments we employ two conventions. First, we indicate eigenvalues of dg with positive real part by "+" and those with negative real part by "$-$". Thus, along the Σ_2 branch, the annotation $3 + 1 -$ indicates solutions where dg has three eigenvalues with positive real part and one with negative real part. Second, eigenvalues forced to zero by the group action are *not* included. Along each branch the total number of eigenvalues must equal $\dim V$, which is 6 in this case. Indeed along branch Σ_2 the number of eigenvalues forced by the group action to be zero is 2, so that $2 + 4 = 6$ gives the correct number of eigenvalues altogether. Note that at limit points the stabilities of solutions change. For this reason limit points are also indicated by bold dots.

To end the discussion of stabilities, note that we are seeking equilibrium solutions to a system of ODEs written in the form

$$\frac{dx}{dt} + g(x, \lambda) = 0.$$

Using this form, eigenvalues with positive real part indicate (linearized) stability, whereas those with negative real part indicate instability. Thus a solution is orbitally stable when no "$-$" signs appear in the stability assignments. Orbitally stable solutions are shown by heavy lines. Note that there are orbitally stable solutions on part of the branches Σ_1, Σ_2, Σ_3, and Γ (with 3, 4, 5, and 6 positive eigenvalues, respectively).

The most important information preserved in these schematic diagrams is the answers to the following two questions:

(a) For each λ, how many orbits of solutions are there to the equation $g = 0$, and which are stable?
(b) For which values of λ do transitions in the number of solutions, or their stability, occur?

The answers to these questions are preserved by projection onto the λ-axis, allowing us to keep track of smooth bifurcations, jump transitions (when solutions cease to exist or change stability as λ varies), and hysteretic phenomena.

We end this section by discussing the simplest bifurcation diagrams for problems with $\mathbf{D}_n$ symmetry. Not all features of the diagram in Figure 5.1 appear, but all of these features will be important in later sections. For example, see Case Study 4.

(b) Bifurcation Diagrams for $\mathbf{D}_n$ Symmetry

We begin by describing the isotropy subgroups of $\mathbf{D}_n$ in its standard action on $\mathbb{C} \equiv \mathbb{R}^2$, generated by

$$\begin{aligned} &\text{(a)} \quad \kappa z = \bar{z} \\ &\text{(b)} \quad \zeta z = e^{2\pi i/n} z. \end{aligned} \tag{5.1}$$

By computing the isotropy subgroups and applying the equivariant branching lemma we will be able to determine the expected number of solution branches. The actual bifurcation diagrams are given at the end of this section in Figures 5.3, and 5.4. In this way it should become apparent just how much information is contained in one of these pictures.

The lattice of isotropy subgroups, Figure 5.2, depends on whether n is odd or even. We compute the isotropy subgroups by choosing representative points on the group orbits. Recall (Lemma 1.1) that points on the same orbit have conjugate isotropy subgroups. Moreover, any two points on the same line through (but not including) the origin have the same isotropy subgroup. Thus it sufficies to compute Σ_z for points $z = e^{i\theta}$ on the unit circle.

We claim that z is on the same orbit as a point $e^{i\theta}$ with

$$0 \le \theta \le \pi/n. \tag{5.2}$$

It is easy to arrange for $0 \le \theta \le 2\pi/n$ by sending z to $(\zeta^l)z = e^{(\theta + (2\pi l/n))i}$ for

$$\begin{tikzcd} \mathbf{D}_n \\ \mathbf{Z}_2(\kappa) \arrow[u] \\ \mathbb{1} \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} & \mathbf{D}_n & \\ \mathbf{Z}_2(\kappa) \arrow[ur] & & \mathbf{Z}_2(\kappa\zeta) \arrow[ul] \\ & \mathbb{1} \arrow[ul] \arrow[ur] & \end{tikzcd}$$

n odd $\qquad\qquad$ n even

Figure 5.2. Lattice of isotropy subgroups of $\mathbf{D}_n$ ($n \geq 3$).

some appropriately chosen l. Now if $\pi/n < \theta \leq 2\pi/n$, we have

$$\zeta\kappa z = e^{i((2\pi/n)-\theta)i} \tag{5.3}$$

and $0 < 2\pi/n - \theta \leq \pi/n$, as promised.

Next observe that if n is odd, we can assume that

$$0 \leq \theta < \pi/n. \tag{5.4}$$

To see this, let $n = 2m + 1$. Then $(\zeta^m)e^{i\pi/n} = e^{i\pi} = -1$, and this has the same isotropy subgroup as $z = 1$, i.e., $\theta = 0$.

We now leave it to the reader to check that the isotropy subgroup of $z = e^{i\theta}$ is $\mathbb{1}$ if $0 < \theta < \pi/n$; $\mathbf{Z}_2(\kappa)$ if $\theta = 0$; and $\mathbf{Z}_2(\zeta\kappa)$ if n is even and $\theta = \pi/n$ (see (5.3)). We emphasize that for n even the two $\mathbf{Z}_2$ isotropy subgroups, $\mathbf{Z}_2(\kappa)$ and $\mathbf{Z}_2(\zeta\kappa)$, are *not* conjugate in $\mathbf{D}_n$. This is clear on geometrical grounds, since vertices and midpoints of edges lie on different lines through the origin. Alternatively, a simple calculation shows that the conjugates of κ are the elements $\zeta^s\kappa$ where s is *even*. For even n, these do not include $\zeta\kappa$.

Since the fixed-point subspaces of both $\mathbf{Z}_2(\kappa)$ and $\mathbf{Z}_2(\zeta\kappa)$ are clearly one-dimensional, the equivariant branching lemma implies that generically there are unique branches of solutions to bifurcation problems with $\mathbf{D}_n$ symmetry, corresponding to these isotropy subgroups. In general a given "branch" of solutions, defined say for all $x \in \mathbb{R}$, may correspond to either one orbit, or two distinct orbits, of solutions, depending on whether or not x and $-x$ lie in the same orbit of Γ. See Exercise 4.1. In particular for $\mathbf{D}_n$ with n odd there is one branch, and when n is even there are two. We show later that when n is odd the one branch splits into two orbits of solutions, whereas when n is even each branch corresponds to a unique orbit.

We can determine more complete information about these branches by analyzing the general form of $\mathbf{D}_n$-equivariant mappings. Recall from Chapter XII, §5, that if $g: \mathbb{C} \times \mathbb{R} \to \mathbb{C}$ commutes with $\mathbf{D}_n$, then

$$g(z, \lambda) = p(u, v, \lambda)z + q(u, v, \lambda)\bar{z}^{n-1}, \tag{5.5}$$

where $u = z\bar{z}$ and $v = z^n + \bar{z}^n$. In order for g to be a bifurcation problem, the linear terms in (5.5) must vanish. Hence

$$p(0, 0, 0) = 0. \tag{5.6}$$

Table 5.1. Solution of $g = 0$ for $\mathbf{D}_n$-Equivariant g, $n \geq 3$

Isotropy Subgroup	Fixed-Point Subspace	Equations
$\mathbf{D}_n$	$\{0\}$	$z = 0$
$\mathbf{Z}_2(\kappa)$	$\mathbb{R}$	$p(x^2, 2x^n, \lambda) + x^{n-2}q(x^2, 2x^n, \lambda) = 0$ $x \neq 0$ [n odd], $\quad x > 0$ [n even]
$\mathbf{Z}_2(\zeta\kappa)$ [n even]	$\mathbb{R}\{e^{i\pi/n}\}$	$p(x^2, -2x^n, \lambda) - x^{n-2}q(x^2, -2x^n, \lambda) = 0$ $x > 0$
$\mathbb{1}$	$\mathbb{C}$	$p = q = 0$ $\operatorname{Im}(z^n) \neq 0$

In addition, the genericity hypothesis of the equivariant branching lemma requires

$$p_\lambda(0, 0, 0) \neq 0. \tag{5.7}$$

We now prove that a second nondegeneracy hypothesis, namely

$$q(0, 0, 0) \neq 0, \tag{5.8}$$

implies that generically the *only* (local) solution branches to $g = 0$ are those obtained using the equivariant branching lemma.

Observe that z and $\bar{z}^{n-1}$ are collinear only when $\operatorname{Im}(z^n) = 0$. Thus when $\operatorname{Im}(z^n) \neq 0$, solving $g = 0$ is equivalent to solving

$$p = q = 0. \tag{5.9}$$

Thus, under the genericity hypothesis (5.8), it is not possible to find solutions to (5.9) near the origin. Now $\operatorname{Im}(z^n) \neq 0$ precisely when the isotropy subgroup of z is $\mathbb{1}$. Thus the only solutions to $g = 0$ are those corresponding to the maximal isotropy subgroups. The full solution to $g = 0$ is given in Table 5.1.

When n is even, $\zeta^{n/2} = -1$, so that the points z and $-z$ are on the same orbit. Thus when n is even we may assume $x > 0$ (not just $x \neq 0$) in Table 5.1.

In the remainder of this section we discuss the direction of branching and the asymptotic stability of the solutions we have found. In this discussion we restrict attention to $n \geq 5$ since $n = 3$ and $n = 4$ are exceptional. See Chapter XV, §4, for a discussion of the case $n = 3$ and Chapter XVII, §6, for $n = 4$.

We first explain why $n = 3$ and 4 are special. When $n = 3$ there is a nontrivial $\mathbf{D}_3$-equivariant quadratic $\bar{z}^2$. Then Theorem 4.4 implies that generically the branch of $\mathbf{Z}_2(\kappa)$ solutions is unstable. Therefore, in order to find asymptotically stable solutions to a $\mathbf{D}_3$-equivariant bifurcation problem by a local analysis, we must consider the degeneracy $q(0, 0, 0) = 0$ and apply unfolding theory. We return to this point in the discussion of the traction problem in Case Study 5 and the spherical Bénard problem in Chapter XV, §5.

In the case $n = 4$ the term $\bar{z}^{n-1}$ is cubic, and $q(0, 0, 0)$ enters nontrivially into the branching equations. See Table 5.1, Exercise 5.1, and Chapter XVII, §6.

We now restrict attention to $n \geq 5$. Observe from Table 5.1 that the lowest

Table 5.2. Data on Solutions of Generic $\mathbf{D}_n$-Equivariant Bifurcation Problems, $n \geq 5$

Isotropy	Branching Equation	Signs of Eigenvalues
$\mathbf{D}_n$	$z = 0$	$p_\lambda(0,0,0)\lambda$ (twice)
$\mathbf{Z}_2(\kappa)$	$\lambda = -\dfrac{p_u(0,0,0)}{p_\lambda(0,0,0)}x^2 + \cdots$	$p_u(0,0,0)$
	$x > 0$ [n even]	$-q(0,0,0)$ [n even]
	$x \neq 0$ [n odd]	$-q(0,0,0)x$ [n odd]
$\mathbf{Z}_2(\zeta\kappa)$	$\lambda = -\dfrac{p_u(0,0,0)}{p_\lambda(0,0,0)}x^2 + \cdots$	$p_u(0,0,0)$
[n even]	$x > 0$	$q(0,0,0)$

order terms in the equation for both $\mathbf{Z}_2$ solutions are

$$p_u(0,0,0)x^2 + p_\lambda(0,0,0)\lambda + \cdots. \tag{5.10}$$

Thus the $\mathbf{Z}_2$ branches are supercritical when $p_u(0,0,0)p_\lambda(0,0,0) < 0$ and subcritical when $p_u(0,0,0)p_\lambda(0,0,0) > 0$. Generically we may assume that

$$p_u(0,0,0) \neq 0 \tag{5.11}$$

so that the direction of branching is determined.

We now discuss stabilities. Both κ and $\zeta\kappa$ are reflections, having 1 and -1 as distinct eigenvalues. Therefore, from the restrictions imposed by isotropy (see (4.8)) dg leaves the corresponding one-dimensional eigenspaces invariant. Hence the eigenvalues of dg must be real.

A straightforward calculation shows that if we think of g as a function of real coordinates $z, \bar{z}$, then

$$(dg)(w) = g_z w + g_{\bar{z}}\bar{w}. \tag{5.12}$$

(The method for computing dg in (5.12) is typically the most efficient when g is defined using complex variables.) Compute (5.12) to obtain:

$$\begin{aligned} &\text{(a)} \quad g_z = p + p_u z\bar{z} + np_v z^n + (q_u\bar{z} + nq_v z^{n-1})\bar{z}^{n-1} \\ &\text{(b)} \quad g_{\bar{z}} = p_u z^2 + np_v z\bar{z}^{n-1} + (n-1)q\bar{z}^{n-2} + (q_u z + nq_v\bar{z}^{n-1})\bar{z}^{n-1}. \end{aligned} \tag{5.13}$$

We list the branching and eigenvalue information in Table 5.2 and now verify those data. Along the trivial solution $z = 0$ we have

$$(dg)_{0,\lambda}(w) = p(0,0,\lambda)w = (p_\lambda(0,0,0)\lambda + \cdots)w.$$

Thus dg is a multiple of the identity, having a repeated eigenvalue whose sign is $\operatorname{sgn}(p_\lambda(0,0,0)\lambda)$ since $p_\lambda(0,0,0)$ is assumed nonzero.

Next we consider the $\mathbf{Z}_2(\kappa)$ solutions. The fixed-point subspace is the real axis ($w = \bar{w}$) and the -1 eigenspace is the imaginary subspace ($w = -\bar{w}$). Since these subspaces are invariant under dg we can find the eigenvalues directly from (5.12). They are

$$g_z + g_{\bar{z}} \quad \text{and} \quad g_z - g_{\bar{z}}. \tag{5.14}$$

Using (5.13) and (5.14) we compute these eigenvalues to lowest order. They are

$$\begin{aligned} &\text{(a)} \quad g_z + g_{\bar{z}} = 2p_u(0,0,0)x^2 + \cdots \\ &\text{(b)} \quad g_z - g_{\bar{z}} = -(n-1)q(0,0,0)x^{n-2} + \cdots, \end{aligned} \tag{5.15}$$

assuming that $n \geq 5$. It follows that the signs of the eigenvalues are determined by $p_u(0,0,0)$ and $-q(0,0,0)x$, as recorded in Table 5.2.

Finally we consider the $\mathbf{Z}_2(\zeta\kappa)$ solutions which appear as a distinct orbit of solutions only when n is even. The eigenspaces of $\zeta\kappa$ corresponding to the eigenvalues 1 and -1 are, respectively, $\mathbb{R}\{e^{i\pi/n}\}$ and $\mathbb{R}\{ie^{i\pi/n}\}$. Using (5.13), the eigenvalues of $(dg)_z$ where $z = xe^{i\pi/n}$ are

$$\begin{aligned} &\text{(a)} \quad g_z|_{z=xe^{i\pi/n}} = p + x^2p_u - np_v x^n - q_u x^n + nq_v x^{2n-2} \\ &\text{(b)} \quad g_{\bar{z}}|_{z=xe^{i\pi/n}} = [p_u x^2 - np_v x^n - (n-1)qx^{n-2} - q_u x^n + nq_v x^{2n-2}]e^{2\pi i/n}. \end{aligned} \tag{5.16}$$

Using (5.12) we compute

$$\begin{aligned} &\text{(a)} \quad (dg)_z(e^{i\pi/n}) \\ &\qquad = [p - (n-1)qx^{n-2} + 2x^2p_u - 2np_v x^n - 2q_u x^n + 2nq_v x^{2n-2}]e^{i\pi/2}, \\ &\text{(b)} \quad (dg)_z(ie^{i\pi/n}) = [p + (n-1)qx^{n-2}]ie^{i\pi/n}. \end{aligned} \tag{5.17}$$

Since $p = x^{n-2}q$ along the $\mathbf{Z}_2(\zeta\kappa)$ solution branch (Table 5.1) and $n \geq 5$, the eigenvalues of $(dg)_z$ are

$$\begin{aligned} &\text{(a)} \quad 2p_u(0,0,0)x^2 + \cdots \\ &\text{(b)} \quad nq(0,0,0)x^{n-2} + \cdots, \end{aligned} \tag{5.18}$$

giving the last entry in Table 5.2.

Thus we have shown that for $n \geq 5$, if we assume

$$p_\lambda(0) \neq 0, \qquad p_u(0) \neq 0, \quad \text{and} \quad q(0) \neq 0 \tag{5.19}$$

then the bifurcation diagram of $g = 0$ is determined. For each n there are eight possible diagrams, depending on the signs of the terms in (5.19). To reduce the complexity we assume that the trivial solution $z = 0$ is stable subcritically, that is, that $p_\lambda(0) < 0$. The remaining possibilities for the bifurcation diagrams are drawn in Figures 5.3 [n odd] and 5.4 [n even]. These diagrams are constructed from the data in Table 5.2, along with a final observation. When n is even, $\zeta^{n/2} \in \mathbf{D}_n$ acts as -1 on $\mathbb{C}$. Hence solutions z and $-z$ to $g = 0$ lie on the same orbit of solutions. This fact accounts for the restriction $x > 0$ when n is even.

Remarks 5.1.

(a) There is a remarkable parallel between the generic bifurcation diagrams with $\mathbf{D}_n$ symmetry when n is odd and when n is even (as long as $n \geq 5$), despite

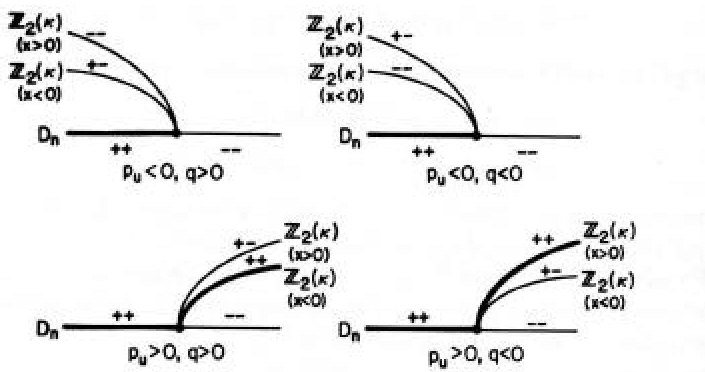

Figure 5.3. Bifurcation diagrams for $\mathbf{D}_n$ symmetry when $p_\lambda(0) < 0$, n odd, $n \geq 5$.

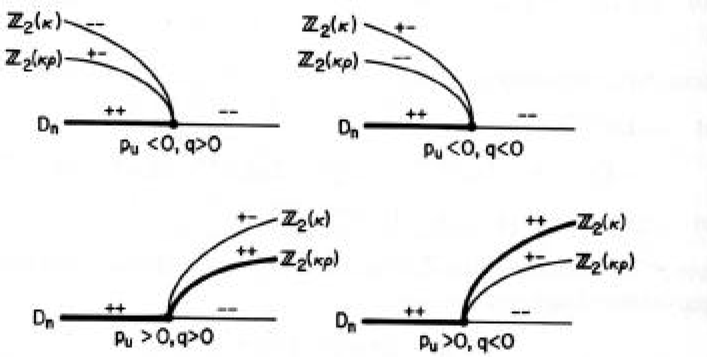

Figure 5.4. Bifurcation diagrams for $\mathbf{D}_n$ symmetry when $p_\lambda(0) < 0$, n even, $n \geq 6$.

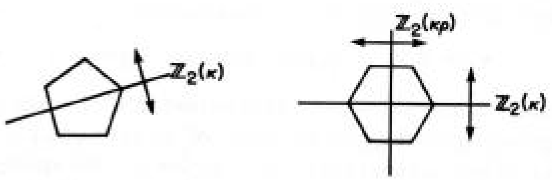

Figure 5.5. Geometric differences in solutions to a PDE with $\mathbf{D}_n$ symmetry.

the differences in the calculation. In particular, the diagrams for n even may be obtained from those for n odd by replacing $\mathbf{Z}_2(\kappa)$ $(x > 0)$ by $\mathbf{Z}_2(\kappa)$, and $\mathbf{Z}_2(\kappa)$ $(x < 0)$ by $\mathbf{Z}_2(\zeta\kappa)$. In spite of this apparent similarity, there is a subtle difference which is captured by the isotropy subgroups. Imagine a solution to a $\mathbf{D}_n$-equivariant PDE posed on the interior of a regular n-gon in $\mathbb{R}^2$. Suppose that there is a $\mathbf{D}_n$-invariant steady state which bifurcates to a solution which breaks the $\mathbf{D}_n$ symmetry. As we have shown, generically we expect exactly two different orbits of solutions to bifurcate, and both should be super- or sub-critical together. However, when n is odd we expect both orbits of solutions to be invariant under a reflection in a line joining a vertex of the n-gon to the midpoint of the opposite edge; see Figure 5.5. On the other hand, when n is even, we expect one orbit of solutions to be invariant under reflection in a line joining opposite vertices, and another to be invariant under reflection in a line joining opposite midpoints of edges. See Figure 5.5.

EXERCISE

5.1. Compute the branching equations for generic $\mathbf{D}_4$ bifurcation.

§6.† Subgroups of SO(3)

We now embark upon an extended example, applying techniques developed previously to the orthogonal group $\mathbf{O}(3)$ and the special orthogonal group $\mathbf{SO}(3)$. Recall that $\mathbf{SO}(3)$ is the group of all orthogonal 3×3 matrices over $\mathbb{R}$ of determinant 1. In this section we discuss the closed subgroups of $\mathbf{SO}(3)$. We classify them, describe specific realizations, show that the finite ones have disjoint union decompositions, and list containment relations between them. In §7 we discuss the irreducible representations of $\mathbf{SO}(3)$. In §8 we find (for all irreducible representations) the dimensions of the fixed-point subspaces of closed subgroups of $\mathbf{SO}(3)$, and list the isotropy subgroups of $\mathbf{SO}(3)$ with one-dimensional fixed-point subspaces. These results may be found in Michel [1980], although we follow the presentation by Ihrig and Golubitsky [1984]. In §9 we extend the results to $\mathbf{O}(3)$. We do not prove everything in detail. In particular, results from Lie theory that would require substantial development—and yet are well known—are just stated along with appropriate references.

(a) Classification

We now describe the closed subgroups of $\mathbf{SO}(3)$. Geometrically, $\mathbf{SO}(3)$ is the group of orientation-preserving rigid motions of $\mathbb{R}^3$ that fix the origin. Its closed subgroups have nice geometric interpretations in terms of symmetries of subsets of $\mathbb{R}^3$. Choose a plane P in $\mathbb{R}^3$ and an axis A orthogonal to P. The subgroup of transformations leaving P invariant consists of rotations about A together with reflections through lines in P (combined with reversals in the sense of A to yield elements of $\mathbf{SO}(3)$; see the following discussion). This group is isomorphic to $\mathbf{O}(2)$. If we require the sense of A to be preserved this is reduced to the special orthogonal group in two dimensions, or the circle group, $\mathbf{SO}(2)$.

The symmetries of a regular n-gon lying in P yield a subgroup of $\mathbf{O}(2)$ isomorphic to the dihedral group $\mathbf{D}_n$. This consists of rotations through $2k\pi/n$

CHAPTER XVII

Hopf Bifurcation with $\mathbf{O}(2)$ Symmetry

§0. Introduction

The object of this chapter is to study Hopf bifurcation with $\mathbf{O}(2)$ symmetry in some depth, including a formal analysis—that is, assuming Birkhoff normal form—of nonlinear degeneracies. The most important case, to which most others reduce, is the standard action of $\mathbf{O}(2)$ on $\mathbb{R}^2$. Since this representation is absolutely irreducible the corresponding Hopf bifurcation occurs on $\mathbb{R}^2 \oplus \mathbb{R}^2$. We repeat the calculations of XVI, §7(c), in a more convenient coordinate system and in greater detail. In §1 we find that there are two maximal isotropy subgroups, corresponding to standing and rotating waves, as in the example of a circular hosepipe. We also give a brief discussion of nonstandard actions of $\mathbf{O}(2)$, for which the standing and rotating waves acquire extra spatial symmetry. In §2 we derive the generators for the invariants and equivariants of $\mathbf{O}(2) \times \mathbf{S}^1$ acting on $\mathbb{R}^2 \oplus \mathbb{R}^2$. In §3 we apply these results to analyze the branching directions of these solutions in terms of the Taylor expansion of the vector field.

In §4 we reformulate Hopf bifurcation with $\mathbf{O}(2)$ symmetry in terms of phase/amplitude equations. To do this we assume that the vector field is in Birkhoff normal form to all orders and introduce suitable polar coordinates. Eliminating the phases leads to amplitude equations on $\mathbb{R}^2$ which turn out to be the most general equations that commute with the standard action of the dihedral group $\mathbf{D}_4$. We apply the results to obtain the stabilities of the rotating and standing wave solutions, showing that one of these two branches is stable only if both are supercritical, in which case (generically) precisely one branch is stable. Which branch is stable is determined by cubic terms in the vector field.

In §5 we show that these results generalize without difficulty to Hopf bifurcation for $\mathbf{O}(n)$ in its standard action on $\mathbb{R}^n$.

We return to $\mathbf{O}(2)$ Hopf bifurcation in §6 but now allow the bifurcation to be degenerate. The amplitude equation method reduces this problem to degenerate static bifurcation with $\mathbf{D}_4$ symmetry, and we use the methods of Chapters XIV and XV to classify such bifurcations in low codimension. An interesting feature is that modal parameters enter at topological codimension 0. The corresponding bifurcation diagrams are described in §7. In codimension 1 we find 2-tori with linear flow, and in codimension 2 we find 3-tori.

Finally in §8 we consider the simpler cases of Hopf bifurcation with $\mathbf{SO}(2)$ or $\mathbf{Z}_n$ symmetry, acting on $\mathbb{R}^2$. These illustrate the way the analysis works for non-absolutely irreducible representations. Here the *existence* of solutions follows from the usual Hopf theorem, but the analysis of their *symmetries* is more naturally carried out within the framework set up in Chapter XVI. In particular we show that for nonstandard actions the generic branches are rotating waves with additional spatial symmetry.

§1. The Action of $\mathbf{O}(2) \times \mathbf{S}^1$

We begin by investigating the group-theoretic and invariant-theoretic generalities of $\mathbf{O}(2)$ Hopf bifurcation. In subsection (a) we summarize the results of this section and §§2 and 3. In (b) we write the group action in convenient coordinates. In (c) we find the isotropy subgroups and related data, and in (d) we consider how to modify the results for nonstandard actions of $\mathbf{O}(2)$. The results include the verification, in the new coordinate system, of the results described without proof in XVI, §7c.

(a) The Main Results

The first example we consider is the *standard* action of $\Gamma = \mathbf{O}(2)$ on $\mathbb{R}^2 \equiv \mathbb{C}$. The other actions (in which $\mathbf{SO}(2)$ acts by k-fold rotations, $k > 1$) reduce to this case, but some care is needed when interpreting the results; see subsection (d). In the philosophy of Chapter XVI, the first step in studying nondegenerate Hopf bifurcation is to find the conjugacy classes of isotropy subgroups of $\mathbf{O}(2) \times \mathbf{S}^1$ acting on $\mathbb{C} \oplus \mathbb{C}$. We will show that the isotropy lattice consists of four conjugacy classes as in Figure 1.1. Since the fixed-point subspaces of $\widetilde{\mathbf{SO}}(2)$ and $\mathbf{Z}_2 \oplus \mathbf{Z}_2^c$ are each two-dimensional, Theorem 4.1 implies that there exist two distinct branches of periodic solutions corresponding to these isotropy subgroups, provided the usual transversality condition, that eigenvalues cross the imaginary axis with nonzero speed, holds.

This group-theoretic structure has several implications for Hopf bifurcation

$$\begin{tikzcd}[column sep=small]
& \mathbf{O}(2) \times \mathbf{S}^1 & \\
\widetilde{\mathbf{SO}}(2) = \{(\theta, \theta) \in \mathbf{SO}(2) \times \mathbf{S}^1\} \arrow[ur] & & \mathbf{Z}_2 \oplus \mathbf{Z}_2^c = \langle \kappa, (\pi, \pi) \rangle \arrow[ul] \\
& \mathbf{Z}_2^c = \langle (\pi, \pi) \rangle \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Figure 1.1. The isotropy lattice of $\mathbf{O}(2) \times \mathbf{S}^1$.

to periodic solutions in a system of ODEs with $\mathbf{O}(2)$ symmetry, which we now list.

Since the element $(\pi, \pi) \in \mathbf{O}(2) \times \mathbf{S}^1$ acts trivially on $\mathbb{C} \oplus \mathbb{C}$, every periodic solution $x(t)$ obtained in this analysis has the following property:

> Spatial rotation of $x(t)$ through the angle π has the same effect as shifting the phase of $x(t)$ by half a period.

That is,

$$R_\pi x(t) = x(t + \pi). \tag{1.1}$$

Periodic solutions $x(t)$ with $\widetilde{\mathbf{SO}}(2)$ symmetry have a more general property:

> Spatial rotation of $x(t)$ through any angle θ has the same effect as shifting the phase of $x(t)$ by θ.

That is,

$$R_\theta x(t) = x(t + \theta).$$

In particular, such solutions satisfy

$$x(\theta) = R_\theta x(0). \tag{1.2}$$

Periodic solutions satisfying (1.2) are called *rotating waves*. In rotating waves there is a coupling between spatial and temporal symmetries. There have been numerous studies of bifurcation to (and from) rotating waves. We mention Renardy [1982] and, in the context of reaction–diffusion equations, Erneux and Herschkowitz-Kaufman [1977] and Auchmuty [1979].

Solutions with $\mathbf{Z}_2 \oplus \mathbf{Z}_2^c$ isotropy possess a purely spatial symmetry in addition to (1.1). Let $\kappa \in \mathbf{O}(2)$ be the flip, acting by complex conjugation on $\mathbb{C}$. Then periodic solutions $x(t)$ with isotropy subgroup $\mathbf{Z}_2 \oplus \mathbf{Z}_2^c$ satisfy $\kappa x(t) = x(t)$. We call this solution a *standing wave*. As we have already seen in XVI, §7, these solutions correspond to the two modes of an oscillating hosepipe mentioned in XI, §1.

The submaximal isotropy subgroup $\mathbf{Z}_2^c$ acts trivially and thus has a four-

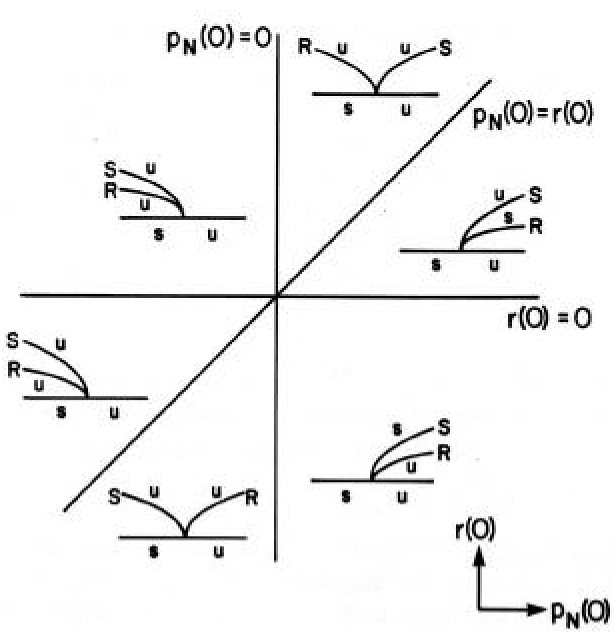

Figure 1.2. Branching and stability in nondegenerate Hopf bifurcation with $\mathbf{O}(2)$ symmetry. S = standing wave; R = rotating wave.

dimensional fixed-point subspace. Solutions of this type occur only in *degenerate* $\mathbf{O}(2)$ Hopf bifurcation and correspond (for vector fields in Birkhoff normal form) to invariant tori with linear flow. We discuss such degeneracies in §6.

Further, by using the invariant theory of $\mathbf{O}(2) \times \mathbf{S}^1$, the stabilities of the two primary branches can be determined. The stabilities depend on the coefficients of two third order terms, which we call $p_N(0)$ and $r(0)$, in the Birkhoff normal form. The possibilities are shown in Figure 1.2, on the assumption that the trivial state is stable subcritically and unstable supercritically. Observe that for either branch to consist of (orbitally) asymptotically stable periodic solutions, both branches must be supercritical. If so, then one branch is stable and the other unstable. This relationship contrasts dramatically with the usual exchange of stability in standard Hopf bifurcation, see XIII, §4.

(b) The Group Action

We start by rewriting the action of $\mathbf{O}(2) \times \mathbf{S}^1$ in a more convenient form. We claim that there exist coordinates (z_1, z_2) on $\mathbb{C}^2$ for which the action is given by

$$\begin{aligned}
&\text{(a)} \quad \theta(z_1, z_2) = (e^{i\theta}z_1, e^{i\theta}z_2) \qquad (\theta \in \mathbf{S}^1)\\
&\text{(b)} \quad \varphi(z_1, z_2) = (e^{-i\varphi}z_1, e^{i\varphi}z_2) \qquad (\varphi \in \mathbf{SO}(2)) \qquad (1.3)\\
&\text{(c)} \quad \kappa(z_1, z_2) = (z_2, z_1) \qquad (\kappa = \text{flip in } \mathbf{O}(2)).
\end{aligned}$$

These coordinates were used by van Gils [1984] and pointed out to us by A. Vanderbauwhede. We derive them from those given by the general theory. There $\Gamma = \mathbf{O}(2)$ acts diagonally on $\mathbb{C} \oplus \mathbb{C}$; that is, we have

$$\begin{aligned}
&\text{(a)} \quad \varphi(w_1, w_2) = (e^{i\varphi}w_1, e^{i\varphi}w_2) \qquad (\varphi \in \mathbf{SO}(2))\\
&\text{(b)} \quad \kappa(w_1, w_2) = (\bar{w}_1, \bar{w}_2)
\end{aligned} \qquad (1.4)$$

in the usual coordinates (w_1, w_2). We construct new coordinates (z_1, z_2) by finding a two-dimensional subspace $Z_1 \subset \mathbb{C}^2$ such that

$$\begin{aligned}
&\text{(a)} \quad \mathbf{SO}(2) \times \mathbf{S}^1 \quad \text{leaves } Z_1 \text{ invariant},\\
&\text{(b)} \quad \mathbb{C}^2 = Z_1 \oplus Z_2 \quad \text{where } Z_2 = \kappa Z_1.
\end{aligned} \qquad (1.5)$$

Concretely, we have $Z_1 = \mathbb{C}\{(1, i)\}$. However, it is instructive to deduce the existence of Z_1 abstractly: we "diagonalize" the action of $\mathbf{SO}(2) \times \mathbf{S}^1$. This is a torus group and it acts nontrivially. Theorem XII, 7.1, states that nontrivial irreducible representations of tori are two-dimensional. Let Z_1 be a two-dimensional irreducible subspace; then Z_1 satisfies (1.5(a)). We claim that $Z_2 = \kappa Z_1$ is also invariant under $\mathbf{SO}(2) \times \mathbf{S}^1$. Suppose that $\theta \in \mathbf{S}^1$. Then $\theta\kappa = \kappa(\theta)$, whence $\theta Z_2 \subset Z_2$. Similarly for $\varphi \in \mathbf{SO}(2)$ we have $\varphi\kappa = \kappa(-\varphi)$, which implies that $\varphi Z_2 \subset Z_2$. Now $Z_1 \cap Z_2 = \{0\}$ since it is invariant under $\mathbf{O}(2) \times \mathbf{S}^1$ and this acts irreducibly on $\mathbb{C}^2$; recall Lemma XVI, 3.4(b). Thus (1.5(b)) holds.

If we choose a complex coordinate z_1 on Z_1 and let $z_2 = \kappa z_1$ then (1.3(c)) holds. Since $\mathbf{S}^1$ acts on $\mathbb{C}^2$ without fixed points and commutes with κ, we can choose z_1 so that (1.3(a)) holds. Since $\varphi \in \mathbf{SO}(2)$ acts standardly on $\mathbb{C}$ and diagonally on $\mathbb{C}^2$, it acts on Z_1 by $e^{\pm i\varphi}$. Since $\kappa\varphi\kappa = -\varphi$ the action is by $e^{\mp i\varphi}$ on Z_2. Interchanging Z_1 and Z_2 if necessary, we have (1.3(b)). Alternatively, a concrete calculation yields the same result. Namely, in the coordinates $z_2(1, i) + \bar{z}_1(1, -i)$,

$$\begin{aligned}
&\varphi(z_1, z_2) = (e^{-i\varphi}z_1, e^{i\varphi}z_2) \qquad (\varphi \in \mathbf{SO}(2))\\
&\kappa(z_1, z_2) = (z_2, z_1).
\end{aligned} \qquad (1.6)$$

(c) The Isotropy Lattice

We can now compute the (conjugacy classes of) isotropy subgroups for the preceding action of $\mathbf{O}(2) \times \mathbf{S}^1$.

Proposition 1.1. *There are four conjugacy classes of isotropy subgroups for the standard action of* $\mathbf{O}(2) \times \mathbf{S}^1$ *on* $\mathbb{C}^2$. *They are listed, together with their orbit representatives and fixed-point subspaces, in Table* 1.1.

Table 1.1. Group-Theoretic Data for the Standard Action of $\mathbf{O}(2) \times \mathbf{S}^1$ on $\mathbb{C}^2$

Orbit Representative	Isotropy Subgroup	Fixed-Point Subspace	Dimension
$(0, 0)$	$\mathbf{O}(2) \times \mathbf{S}^1$	$\{0\}$	0
$(a, 0)$, $a > 0$	$\widetilde{\mathbf{SO}}(2) = \{(\theta, \theta)\}$	$\{(z_1, 0)\}$	2
(a, a), $a > 0$	$\mathbf{Z}_2 \oplus \mathbf{Z}_2^c = \{(0, 0), \kappa, (\pi, \pi), \kappa(\pi, \pi)\}$	$\{(z_1, z_1)\}$	2
(a, b), $a > b > 0$	$\mathbf{Z}_2^c = \{(0, 0), (\pi, \pi)\}$	$\{(z_1, z_2)\}$	4

Remarks 1.2.
(a) The manifold $(\mathbf{O}(2) \times \mathbf{S}^1)/\widetilde{\mathbf{SO}}(2)$ has two connected components, each a circle (see Remark XIV, 1.3). The orbit representatives for these two components are $(a, 0)$ and $(0, a)$. The isotropy subgroup for $(0, a)$ is the conjugate $\{(\theta, -\theta)\}$ of $\widetilde{\mathbf{SO}}(2)$. Physically, the solutions corresponding to these points are both rotating waves; however, they correspond to the two possible senses of rotation—"clockwise" and "counterclockwise."
(b) The manifold $(\mathbf{O}(2) \times \mathbf{S}^1)/(\mathbf{Z}_2 \oplus \mathbf{Z}_2^c)$ is a connected 2-torus, foliated by circular trajectories. Each circle corresponds to a particular choice of axis for the reflectional "flip" symmetry, which can be any *conjugate* $\kappa\theta$ of κ. That is, in the fixed-point subspace for each isotropy subgroup conjugate to $\mathbf{Z}_2 \oplus \mathbf{Z}_2^c$ we find a periodic solution. All such solutions lie in the same group orbit. They glue together to form the 2-torus.
(c) We consider (a) and (b) for the "hosepipe" example of XI, §1. As already remarked, the rotating wave oscillations correspond to the isotropy subgroup $\widetilde{\mathbf{SO}}(2)$, as in Remark (a). There are two distinct senses of rotation, and there is a unique solution (up to choice of phase) in each sense. The standing wave solutions have isotropy subgroup $\mathbf{Z}_2 \oplus \mathbf{Z}_2^c$. The first $\mathbf{Z}_2$, generated by the flip, imposes mirror symmetry in some axis (and confines the oscillation to a fixed vertical plane). The $\mathbf{Z}_2^c$ symmetry means that, just as for a pendulum, the oscillations to left and right are identical except for a phase shift of π. The axis of reflection for a flip symmetry can be any radial line throught the origin: different radial lines correspond to different (conjugate) choices of flip. There is a circle's worth of radial lines corresponding to the circle's worth of (circular) trajectories foliating the 2-torus.

Proof of Proposition 1.1. Elements in the same orbit have conjugate isotropy subgroups. Hence by (1.2) we may assume that $(z_1, z_2) = (a, b)$ where $a, b \geq 0$ are real. By applying κ we may assume that $a \geq b \geq 0$. The computation of isotropy subgroups and fixed-point subspaces is now routine, but as an illustration we give details.

Clearly $(0, 0)$ is fixed by the whole of $\mathbf{O}(2) \times \mathbf{S}^1$ and is the only point so fixed. If $a > b > 0$ then (a, b) cannot be fixed by any group element not in $\mathbf{SO}(2) \times \mathbf{S}^1$ (that is, involving κ) since $|z_1|$ and $|z_2|$ must be preserved. Now

$$(\varphi,\theta)(a,b) = (e^{i(\theta-\varphi)}a, e^{i(\theta+\varphi)}b)$$

so (φ,θ) fixes (a,b) if and only if $\theta \pm \varphi \equiv 0 \pmod{2\pi}$. Hence $(\varphi,\theta) = (0,0)$ or (π,π).

Similarly if $a > 0$, the point $(a,0)$ can be fixed only by elements (φ,θ) of $\mathbf{SO}(2) \times \mathbf{S}^1$, and now we need $\theta - \varphi \equiv 0 \pmod{2\pi}$, that is, $\varphi = \theta$.

Finally (a,a) is fixed by κ. Hence its isotropy subgroup is generated by κ, together with a subgroup of $\mathbf{SO}(2) \times \mathbf{S}^1$. It can be fixed by $(\varphi,\theta) \in \mathbf{SO}(2) \times \mathbf{S}^1$ only if $\theta \pm \varphi \equiv 0 \pmod{2\pi}$, which as earlier leads to $\{(0,0),(\pi,\pi)\}$. Together with κ these generate $\mathbf{Z}_2 \oplus \mathbf{Z}_2^c$ as claimed. □

(d) Nonstandard Actions of $\mathbf{O}(2)$

We now indicate how to modify the preceding calculations for a (nontrivial) nonstandard action of $\mathbf{O}(2)$. Consider the representation ρ_l for which

$$\varphi \cdot z = e^{il\varphi}z \qquad (\varphi \in \mathbf{SO}(2))$$
$$\kappa \cdot z = \bar{z},$$

where $l > 1$ is an integer. We can obtain this representation by composing the standard representation ρ_0 with the group homomorphism

$$\alpha\colon \mathbf{O}(2) \to \mathbf{O}(2)$$
$$\varphi \mapsto l\varphi$$
$$\kappa \mapsto \kappa$$

whose kernel is

$$\mathbf{Z}_l = \{2m\pi/l\colon m = 0,\ldots,l-1\} \subset \mathbf{O}(2).$$

Then the (nonstandard) action $\rho_l(\gamma)$ of $\gamma \in \mathbf{O}(2)$ on $\mathbb{C}$ is the same as the standard action of $\alpha(\gamma)$ on $\mathbb{C}$. Thus the representation ρ_l behaves just as the standard representation ρ_0 does except that $\mathbf{Z}_l$ acts trivially.

The same is true of the corresponding action of $\mathbf{O}(2) \times \mathbf{S}^1$ on $\mathbb{C}^2$, in which $\mathbf{O}(2)$ acts diagonally by $(w_1,w_2) \mapsto (e^{il\varphi}w_1, e^{il\varphi}w_2)$. As in subsection (a) we can choose coordinates (z_1,z_2) so that this action takes the form

$$\theta(z_1,z_2) = (e^{i\theta}z_1, e^{i\theta}z_2)$$
$$\varphi(z_1,z_2) = (e^{-il\varphi}z_1, e^{il\varphi}z_2)$$
$$\kappa(z_1,z_2) = (z_2,z_1).$$

Again this is just like the standard action, except that $\mathbf{Z}_l$ acts trivially. The orbit data are the same as in Table 1.2, except that $\mathbf{Z}_l$ must be added to every isotropy subgroup. Thus the rotating wave solutions have an additional cyclic symmetry $\mathbf{Z}_l$ of order l, and the standing waves have dihedral group symmetry $\mathbf{D}_l = \langle \kappa, \mathbf{Z}_l \rangle$ (plus the original kernel $\mathbf{Z}_2^c$).

The invariants and equivariants for nonstandard actions of $\mathbf{O}(2)$ are the same as those for the standard action (since the kernel $\mathbf{Z}_l$ acts trivially and hence does not change the invariance or equivariance conditions). The preceding considerations apply generally to nonstandard representations of $\mathbf{O}(2)$ or $\mathbf{SO}(2)$, and similar ideas apply to nonstandard representations of $\mathbf{D}_n$ and $\mathbf{Z}_n$.

§2. Invariant Theory for $\mathbf{O}(2) \times \mathbf{S}^1$

In order to determine the direction of branching (super- or subcritical) and the stability of the branches of periodic solutions, we must compute the $\mathbf{O}(2) \times \mathbf{S}^1$ invariants and equivariants for the preceding action. (As just noted, these are identical for all nontrivial actions of $\mathbf{O}(2)$.) The results are as follows:

Proposition 2.1.
(a) *Every $\mathbf{O}(2) \times \mathbf{S}^1$-invariant germ f has the form*

$$f(z_1,z_2) = P(N,\Delta)$$

where $N = |z_1|^2 + |z_2|^2$, $\Delta = \delta^2$, and $\delta = |z_2|^2 - |z_1|^2$.
(b) *Every $\mathbf{O}(2) \times \mathbf{S}^1$-equivariant germ g has the form*

$$g(z_1,z_2) = (p+iq)\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + (r+is)\delta\begin{bmatrix} z_1 \\ -z_2 \end{bmatrix}, \tag{2.1}$$

where p, q, r, and s are $\mathbf{O}(2) \times \mathbf{S}^1$-invariant germs.

Proof. As usual Schwarz's theorem lets us assume that f and g are polynomial. First we consider the invariance of f under $\mathbf{SO}(2) \times \mathbf{S}^1$, which implies that

$$f(e^{i(\theta-\varphi)}z_1, e^{i(\theta+\varphi)}z_2) = f(z_1,z_2).$$

Define $\psi_1 = (\psi/2, \psi/2)$, $\psi_2 = (-\psi/2, \psi/2) \in \mathbf{SO}(2) \times \mathbf{S}^1$. Then

$$f(z_1, e^{i\psi_2}z_2) = f(z_1,z_2) = f(e^{i\psi_1}z_1, z_2). \tag{2.2}$$

Thus $f = h(u,v)$ where $u = |z_1|^2$, $v = |z_2|^2$. By κ-invariance $h(u,v) = h(v,u)$, whence $h(u,v) = k(u+v, uv)$. Then $N = u + v$ and $\Delta = (v-u)^2 = N^2 - 4uv$ provide an alternative Hilbert basis. Note that $\delta = v - u$.

Now suppose that $g(z_1,z_2) = (g_1(z_1,z_2), g_2(z_1,z_2))$ is equivariant under $\mathbf{O}(2) \times \mathbf{S}^1$. Then (again using ψ_1 and ψ_2)

$$\begin{aligned} &\text{(a)}\quad g_1(z_1,z_2) = e^{-i\psi_1}g_1(e^{i\psi_1}z_1, z_2)\\ &\text{(b)}\quad g_1(z_1,z_2) = g_1(z_1, e^{i\psi_2}z_2)\\ &\text{(c)}\quad g_2(z_1,z_2) = g_1(z_2,z_1). \end{aligned} \tag{2.3}$$

Identity (2.3(b)) implies that $g_1 = h(z_1, |z_2|^2)$, and (2.3(a)) implies that $h(z_1,|z_2|^2) = k(|z_1|^2, |z_2|^2)z_1$.

Recall that $u = |z_1|^2$, $v = |z_2|^2$. Using the coordinate change $(u, v) \mapsto (u + v, u - v)$ we can write

$$k(u, v) = l(u + v, v - u)$$

and decompose l into an even and an odd function in the second coordinate. Thus

$$k(u, v) = l_0(u + v, (v - u)^2) + l_1(u + v, (v - u)^2)(v - u).$$

Summarizing the preceding results we have

$$g_1(z_1, z_2) = P(N, \Delta)z_1 + R(N, \Delta)\delta z_1.$$

Then (2.3(c)) just specifies g_2 in terms of g_1:

$$g_2(z_1, z_2) = P(N, \Delta)z_2 - R(N, \Delta)\delta z_2.$$

Finally, note that P and R are complex-valued invariant functions. Thus we complete the proof of (2.1) by setting $P = p + iq$ and $R = r + is$. □

§3. The Branching Equations

Suppose that we have a system of ODEs on $\mathbb{R}^4$,

$$\dot{x} + X(x, \lambda) = 0 \tag{3.1}$$

where X is smooth and $\mathbf{O}(2)$-invariant. Suppose also that

$$(dX)_{0,0} = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

and that the eigenvalues of $(dX)_{0,\lambda}$ cross the imaginary axis with nonzero speed. Then XVI, §4 implies that there is a Liapunov–Schmidt reduction to a mapping $g(z_1, z_2, \lambda, \tau)$ of $\mathbb{C}^2 \times \mathbb{R} \times \mathbb{R} \to \mathbb{C}^2$ commuting with $\mathbf{O}(2) \times \mathbf{S}^1$, whose zeros are in one-to-one correspondence with periodic solutions of (3.1) of period near 2π. Here τ is the period-perturbing parameter.

By Proposition 2.1, g has the form

$$g = (p + iq)\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + (r + is)\delta\begin{bmatrix} z_1 \\ -z_2 \end{bmatrix} \tag{3.2}$$

where p, q, r, s are functions of N, Δ, λ, and τ. The Liapunov–Schmidt reduction shows that

$$\begin{aligned} &\text{(a)} \quad p(0) = 0, \qquad p_\tau(0) = 0, \\ &\text{(b)} \quad q(0) = 0, \qquad q_\tau(0) = -1, \\ &\text{(c)} \quad p_\lambda(0) \neq 0 \qquad \text{(eigenvalue crossing condition).} \end{aligned} \tag{3.3}$$

To find the zeros of g, it suffices to look at representative points on $\mathbf{O}(2) \times \mathbf{S}^1$-orbits. That is, when solving $g = 0$ we may assume that $z_1 = a$ and $z_2 = b$ are real. It is now easy to check that the equation $g = 0$ reduces to the entries of Table 3.1.

Table 3.1. Branching Equations for $\mathbf{O}(2) \times \mathbf{S}^1$ Hopf Bifurcation

Orbit Representative		Isotropy Subgroup	Branching Equations
(a) $(0, 0)$		$\mathbf{O}(2) \times \mathbf{S}^1$	—
(b) $(a, 0)$,	$a > 0$	$\widetilde{\mathbf{SO}}(2)$	$p - a^2 r = 0$ $q - a^2 s = 0$
(c) (a, a),	$a > 0$	$\mathbf{Z}_2 \oplus \mathbf{Z}_2^c$	$p = 0$ $q = 0$
(d) (a, b),	$a > b > 0$	$\mathbf{Z}_2^c$	$p = q = r = s = 0$

Table 3.2. Branches of Periodic Solutions for $\mathbf{O}(2)$ Hopf Bifurcation

Name	Isotropy Subgroup	Branching Equation
Trivial solution	$\mathbf{O}(2) \times \mathbf{S}^1$	$z = 0$
Rotating wave	$\widetilde{\mathbf{SO}}(2)$	$\lambda = \dfrac{(-p_N(0) + r(0))}{p_\lambda(0)} a^2 + \cdots$
Standing wave	$\mathbf{Z}_2 \oplus \mathbf{Z}_2^c$	$\lambda = \dfrac{-2p_N(0)}{p_\lambda(0)} a^2 + \cdots$

By (3.3(b)) we can always use the equation involving q to solve for τ. This is a specific instance of general arguments in the proof by Liapunov–Schmidt reduction. Generically $r(0)$ and $s(0)$ are nonzero; thus generically there are no solutions with isotropy subgroup $\mathbf{Z}_2^c$.

It is now easy to solve for the leading terms of the branching equations for rotating waves ($\widetilde{\mathbf{SO}}(2)$) and standing waves ($\mathbf{Z}_2 \oplus \mathbf{Z}_2^c$). These are given in Table 3.2. Assuming that $p_\lambda(0) < 0$, or equivalently that the trivial solution is stable subcritically, it is now possible to establish the directions of branching for rotating and standing waves shown in Figure 1.2 earlier.

§4. Amplitude Equations, $\mathbf{D}_4$ Symmetry, and Stability

Return now to the system of ODEs (3.1) and assume that it is in Birkhoff normal form. As we saw in XVI, §6, this means that X commutes with $\mathbf{O}(2) \times \mathbf{S}^1$ rather than just $\mathbf{O}(2)$. By Proposition 2.1,

$$X = (p + iq)\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + (r + is)\delta\begin{bmatrix} z_1 \\ -z_2 \end{bmatrix} \tag{4.1}$$

where p, q, r, s are functions of N, Δ, λ, and $p(0) = 0$, $q(0) = 1$. By Theorem XVI, 10.1, the Liapunov–Schmidt reduced function g of (3.2) for this X has the explicit form

$$g(z_1, z_2, \lambda, \tau) = X(z_1, z_2, \lambda) - (1+\tau)i\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}. \tag{4.2}$$

Thus the branching equations for X are precisely those given in Tables 3.1 and 3.2, since the τ-dependence enters in a simple way in those equations.

(a) Amplitude Equations

One of the remarkable facts about the Birkhoff normal form (4.1) is that it lets us separate the 4×4 system of ODEs into amplitude and phase equations, which in turn permit a simple analysis of the stability of the rotating and standing waves. Write

$$\begin{aligned} z_1 &= xe^{i\psi_1} \\ z_2 &= ye^{i\psi_2}. \end{aligned} \tag{4.3}$$

Then (4.1) implies that the system (3.1) becomes

$$\begin{aligned} \dot{z}_1 + (p + iq + (r + is)\delta)z_1 &= 0 \\ \dot{z}_2 + (p + iq - (r + is)\delta)z_2 &= 0. \end{aligned} \tag{4.4}$$

In the amplitude/phase variables (4.3) these become

$$\begin{aligned} &\text{(a)} \quad \dot{x} + (p + r\delta)x = 0 \\ &\phantom{\text{(a)}} \quad \dot{y} + (p - r\delta)y = 0 \\ &\text{(b)} \quad \dot{\psi}_1 + (q + s\delta) = 0 \\ &\phantom{\text{(b)}} \quad \dot{\psi}_2 + (q - s\delta) = 0 \end{aligned} \tag{4.5}$$

where p, q, r, s are functions of $N = x^2 + y^2$, $\Delta = (y^2 - x^2)^2$, and λ; and $\delta = y^2 - x^2$. (This calculation may be performed by differentiating the identity $x^2 = z_1\bar{z}_1$ to obtain

$$x\dot{x} = \operatorname{Re}(z_1 \cdot \bar{z}_1) = -(p + r\delta)z_1\bar{z}_1.$$

Similarly one uses the identity $y^2 = z_2\bar{z}_2$.)

Nontrivial zeros of the amplitude equations correspond to invariant circles and invariant tori. In particular when $y = 0$ (the rotating waves branch) zeros correspond to circles; when $x = y$ (the standing waves branch) zeros correspond to invariant 2-tori in the original four-dimensional system. On such a torus we see from (4.3(b)) that $\dot{\psi}_1 = \dot{\psi}_2$ since $\delta = 0$. Therefore, trajectories on this 2-torus are all circles. Compare this observation with the group-theoretic Remark 1.2(b). In particular, this qualitative feature of the flow persists even

when the vector field is not in Birkhoff normal form. Finally, a zero $x > y > 0$ of the amplitude equations corresponds to an invariant 2-torus with linear flow ($\dot{\psi}_1$ and $\dot{\psi}_2$ constant). This linear flow is in general quasiperiodic. As remarked earlier, however, generically $r(0) \neq 0$, so there are (locally) no zeros of the amplitude equations with $x > y > 0$. So such tori do not occur in nondegenerate $\mathbf{O}(2)$-symmetric Hopf bifurcation. It is not surprising, however, that when considering degenerate cases such as $r(0) = 0$, such tori do occur. This observation of Erneux and Matkowsky [1984] will be discussed in more detail later.

Lemma 4.1. *A zero of the amplitude equations is asymptotically stable if and only if the corresponding steady-state, periodic trajectory, or invariant 2-torus is (orbitally) asymptotically stable in the four-dimensional system.*

Proof. A zero (x_0, y_0) of the amplitude equations (4.5(a)) is asymptotically stable if every trajectory $(x(t), y(t))$ with initial point sufficiently close to (x_0, y_0) stays near (x_0, y_0) for all $t > 0$, and $\lim_{t\to\infty}(x(t), y(t)) = (x_0, y_0)$. Let M be the connected component of the orbit of $\mathbf{O}(2) \times \mathbf{S}^1$ that contains (x_0, y_0): this consists of points (z_1, z_2) with absolute values x_0, y_0. By definition this means that any trajectory $(z_1(t), z_2(t))$ of the four-dimensional system, with $(z_1(0), z_2(0))$ sufficiently close to M, converges in norm to M. On M, $\dot{\psi}_1$ and $\dot{\psi}_2$ are constant, so the trajectory $(z_1(t), z_2(t))$ converges to a single trajectory on M. This is what is meant by orbital asymptotic stability. □

(b) $\mathbf{D}_4$-Symmetry

Let the dihedral group $\mathbf{D}_4$ act on $\mathbb{C}$ as symmetries of the square. That is, the action is generated by

$$\begin{aligned} &\text{(a)} \quad z \mapsto \bar{z} \\ &\text{(b)} \quad z \mapsto iz. \end{aligned} \tag{4.6}$$

By Examples XII, 4.1(c), the general $\mathbf{D}_4$-equivariant mapping has the form $pz + q\bar{z}^3$ where p and q are functions of $u = z\bar{z}$ and $v = \operatorname{Re}(z^4)$. If we let $z = x + iy$, then we have $u = x^2 + y^2 = N$ and $v = x^4 - 6x^2y^2 + y^4 = -(x^2 + y^2)^2 + 2(y^2 - x^2)^2 = 2\Delta - N^2$. Moreover, as mappings $\mathbb{R}^2 \to \mathbb{R}^2$, the mappings

$$z \mapsto z \text{ and } z \mapsto \bar{z}^3$$

correspond to

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x^3 - 3xy^2 \\ y^3 - 3x^2y \end{bmatrix} = -N\begin{bmatrix} x \\ y \end{bmatrix} - 2\delta\begin{bmatrix} x \\ -y \end{bmatrix}.$$

It follows that *the general form of the amplitude equations is exactly the same* *as the general form of the* $\mathbf{D}_4$*-equivariant mappings from* $\mathbb{R}^2$ *into* $\mathbb{R}^2$. The $\mathbf{D}_4$ symmetry may be thought of as what remains of the original $\mathbf{O}(2) \times \mathbf{S}^1$ symmetry after reduction to the amplitude equations.

This has two consequences: one mildly useful and one extremely important. Suppose we let

$$h(x, y, \lambda) = p(N, \Delta, \lambda)\begin{bmatrix} x \\ y \end{bmatrix} + r(N, \Delta, \lambda)\delta\begin{bmatrix} x \\ -y \end{bmatrix}. \tag{4.7}$$

Then the amplitude equations (4.5) have the form

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} + h(x, y, \lambda) = 0. \tag{4.8}$$

The mildly useful observation is that the $\mathbf{D}_4$ symmetries restrict the form of dh at solutions, so that the asymptotic stability of steady states can easily be computed. This we do later; however, these results can also be obtained by direct calculation without knowledge of the underlying $\mathbf{D}_4$ symmetry.

The important observation is that it is now possible to classify the *degenerate* $\mathbf{O}(2)$-equivariant Hopf bifurcations, by classifying the $\mathbf{D}_4$-equivariant germs h up to $\mathbf{D}_4$-equivalence. We do this in §6. Moreover, the solutions so obtained include not only the periodic solutions of rotating and standing waves, but also the invariant 2-tori with linear flow.

Remark. This situation has already occurred in our study of degenerate Hopf bifurcation in Chapter VIII. There we had a simple conjugate pair of purely imaginary eigenvalues, with no spatial symmetry. The Birkhoff normal form commutes only with $\mathbf{S}^1$ acting on $\mathbb{C}$ and has the form

$$\dot{z} + p(z\bar{z}, \lambda)z + q(z\bar{z}, \lambda)iz = 0.$$

This equation splits into amplitude and phase equations on setting $z = xe^{i\psi}$, giving

$$\dot{x} + p(x^2, \lambda)x = 0$$
$$\dot{\psi} + q(x^2, \lambda) = 0.$$

The form of the amplitude equation is just that of the general $\mathbf{Z}_2$-equivariant mapping. Indeed we studied degenerate Hopf bifurcation in Chapter VIII by using the classification of $\mathbf{Z}_2$-equivariant mappings under $\mathbf{Z}_2$-equivalence given in Chapter VI.

This kind of reduction (Γ-equivariant Hopf bifurcation $\to \Gamma \times \mathbf{S}^1$ Birkhoff normal form $\to \Sigma$-equivariant amplitude equations) seldom happens in such a nice way, but when it does, we have a method for studying the degenerate Γ-equivariant Hopf bifurcations. For an invariant-theoretic interpretation, see Exercise 4.1.

We end this section with a discussion of the asymptotic stability of rotating and standing waves. These results are summarized in Table 4.1.

Table 4.1. Stabilities of Rotating and Standing Waves in $\mathbf{O}(2)$ Hopf Bifurcation

Name	Orbit Representative	Isotropy Subgroup	Signs of Eigenvalues
Trivial solution	0	$\mathbf{D}_4$	$p(0, 0, \lambda)$ [twice]
Rotating wave	$y = 0$	$(x, y) \mapsto (x, -y)$	r
			$p_N - r + x^2(2p_\Delta - rN) - 2x^4 r_\Delta$
Standing wave	$x = y$	$(x, y) \mapsto (y, x)$	$-r$
			p_N

A rotating wave ($y = 0$) has isotropy subgroup in $\mathbf{D}_4$ generated by the reflection $(x, y) \mapsto (x, -y)$. At a rotating wave solution the Jacobian dh must commute with the matrix $\left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$. Thus dh is diagonal and the eigenvalues of dh are just $(p + \delta r)_x$ and $p - \delta r$. Since $\delta = -x^2$ and $p + \delta r = 0$ on this branch, we obtain the information in Table 4.1.

A standing wave ($x = y$) has isotropy subgroup generated by $(x, y) \mapsto (y, x)$. Thus dh commutes with the matrix $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ and so has the form $\left[\begin{smallmatrix} a & b \\ b & a \end{smallmatrix}\right]$, which has eigenvalues $a \pm b$. Now

$$a = [(p + \delta r)x]_x$$
$$b = (p + \delta r)_y x$$

and $p = \delta = 0$ along the standing waves branch. This information lets us compute the signs of eigenvalues of dh at the standing wave solution, also listed in Table 4.1.

Table 4.1 leads to the stability assignments given in Figure 1.2, since in nondegenerate $\mathbf{O}(2)$-equivariant Hopf bifurcation we assume that

$$p_\lambda(0) \neq 0, \qquad p_N(0) \neq 0, \qquad r(0) \neq 0, \qquad p_N(0) - r(0) \neq 0. \tag{4.9}$$

As we shall see, these are precisely the nondegeneracy conditions needed to classify the least degenerate $\mathbf{D}_4$-equivariant bifurcation problems.

Exercises

4.1. This exercise provides an alternative viewpoint on the introduction of amplitude/phase variables. Consider the system (3.1), $\dot{x} + X(x, \lambda) = 0$, where X is as in (4.1). Obtain expressions for dN/dt and $d\Delta/dt$ as functions of N and Δ, where N and Δ are the invariant generators of Proposition 2.1(a). Note that orbits of $\mathbf{O}(2) \times \mathbf{S}^1$ are parametrized by the values of invariants, so these expressions may be intepreted as the dynamics of *orbits*. Show that the resulting equations are equivalent to the amplitude equations (4.3(a)). Interpret the $\mathbf{D}_4$-equivariance of the amplitude equations in terms of the geometry of the image of the mapping $(z_1, z_2) \mapsto (N, \Delta)$.

4.2. More generally, suppose that $\dot{x} + X(x, \lambda)$ is a Γ-equivariant ODE, and let $\{I_1, \ldots, I_r\}$ be a system of invariant generators for Γ. Show that dI_j/dt is Γ-