Methods of Nonlinear Analysis D-12
Winter 2000
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Homework Assignment 1

Due Thursday, January 13, 2000.

Rayleigh-Bénard Convection

Introduction. In order to do at least one example of a weakly nonlinear analysis of an (almost) realistic system consider Rayleigh-Bénard convection. By using simplified boundary conditions, this calculation can be done completely analyticalyl. In the limit of low Prandtl number (viscosity large compared to heat conductivity) the calculation is quite doable. One of the many suitable references for this calculation is M.C. Cross & P.C. Hohenberg, Rev. Mod. Phys. 65 (1993) 851, appendix A3. It is probably helpful to read and make use of the relevant section in that paper.
Consider a liquid layer confined between two flat plates separated by a distance d that is heated from below, i.e. the temperature at the bottom plate is T_b while the top temperature is T_t. The liquid has an expansion coefficient α, i.e. its density is given by

$$\rho = \rho_0(1 - \alpha(T - T_b)). \tag{1}$$

The liquid is then described by the Navier-Stokes equation for the momentum

$$\rho\partial_t \mathbf{v} + \rho(\mathbf{v}\cdot\nabla)\mathbf{v} = \rho\mathbf{f} - \nabla p + \mu\Delta\mathbf{v}$$

with

$$\mathbf{f} = -g\mathbf{e}_z,$$

the continuity equation

$$\partial_t\rho + \nabla\cdot(\rho\mathbf{v}) = 0,$$

and the heat equation

$$\partial_t T + (\mathbf{v}\cdot\nabla)T = \kappa\Delta T.$$

The boundary conditions for the temperature are

$$T(z=0) = T_b \qquad T(z=d) = T_t.$$

For the fluid velocity the proper boundary conditions are

$$\mathbf{v}(z=0) = \mathbf{v}(z=d) = 0,$$

since the top and bottom plates are impermeable and due to the viscosity the fluid “sticks” to the plates (no-slip boundary condition). For these boundary conditions the calculation requires numerical work. To obtain an analytical solution, it has therefore been quite common to assume that the top and bottom plates do not apply any stress on the fluid (free-slip boundary conditions),

$$\mathbf{v}_z(z=0,d) = 0 = \partial_z\mathbf{v}_{(x,y)}(z=0,d).$$

Here $\mathbf{v} = (\mathbf{v}_x, \mathbf{v}_y, \mathbf{v}_z)$. Convection is driven by buoyancy. It is common to assume that the density change is small, which can formally be thought of as assuming $\alpha \ll 1$. Inserting the fluid expansion (1) into the forcing one obtains

$$\begin{aligned} \rho\mathbf{f} - \nabla p &= -\rho_0 g \mathbf{e}_z + \alpha g \rho_0 (T - T_b)\, \mathbf{e}_z - \nabla p \\ &= -\nabla(p + \rho_0 g z) + \alpha g \rho_0 (T - T_b)\, \mathbf{e}_z. \end{aligned}$$

Thus, the contribution of gravity from the constant-density term can be absorbed into the pressure term. Since the pressure will eventually be eliminated the leading-order term in α in the forcing is $\alpha g \rho_0 (T - T_b)\mathbf{e}_z$. In all other terms the effect of the expansion is usually ignored (e.g. in $\rho \partial_t \mathbf{v} = \rho_0 \partial_t \mathbf{v} - \alpha \rho_0 (T - T_b) \partial_t \mathbf{v}$ the contribution of $O(\alpha)$ is small compared to the first term). This approximation is called *Boussinesq approximation.*

Assignment.

1. Calculate the basic stationary solution $T_0, \mathbf{v}_0 = 0$ exactly.
2. Introduce dimensionless variables $T = T_0 + \theta \Delta T / R$, $x = d\xi$, $z = d\zeta$, $t = \frac{d^2}{\kappa}\tau$, $p = \rho_0 g d \Pi$, and $\mathbf{v} = \frac{\kappa}{d}\mathbf{V}$ and rewrite the equations in terms of these variables. You will find it useful to introduce the dimensionless Rayleigh number $R = \alpha g (T_b - T_t) d^3 / \nu\kappa$ and the dimensionless Prandtl number $Pr = \nu/\kappa$, where the kinematic viscosity is given by $\nu = \mu/\rho_0$. In the following restrict yourself to solutions that do not depend on y . Take the curl of the momentum equation twice and obtain for the momentum equation and the heat equation in the limit of large Prandtl number, $Pr \to \infty$,

$$\begin{aligned} 0 &= \partial_\xi^2 \theta + \Delta^2 W & (2) \\ \partial_\tau \theta + U \partial_\xi \theta + W \partial_\zeta \theta &= \Delta\theta + R W & (3) \end{aligned}$$

   where $\mathbf{V} = (U, W)$ is the dimensionless velocity.
3. Perform a linear stability analysis of the basic stationary solution and determine the critical Rayleigh number R_c and the critical wave number q_c .
4. Show that the linear operator L_0 obtained by linearizing (2,3) around the basic stationary solution is self-adjoint if the scalar product

$$< (U_1, W_1), (U_2, W_2) >= \int d\zeta \int d\xi \;\, (\partial_\xi \theta_1^* \, \partial_\xi \theta_2 + R_c W_1^* W_2)$$

   is used. Note that this makes using the Fredholm alternative much easier.
5. Derive the amplitude equation for convection rolls with wave number q_c , i.e. perform a weakly nonlinear analysis of (2,3).