Methods of Nonlinear Analysis D-12
Winter 2000
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Homework Assignment 2

Due Thursday, January 20, 2000.

1. Turing Patterns in a Reaction-Diffusion System

Consider the reaction-diffusion system

$$\begin{aligned}\partial_t u &= \nabla^2 u + u(1-u^2) - v\\ \partial_t v &= D\nabla^2 v - av(1-v^2) + bu.\end{aligned}$$

a) Find the threshold and critical wavenumber for the Turing instability for $D > 1$ and $a > 0$. Consider b as your bifurcation parameter.
b) Use slow spatial and temporal scales to derive the (two-dimensional) Newell-Whitehead-Segel equation for this Turing bifurcation. The reaction-diffusion system does not contain any quadratic terms. Do you need any terms at quadratic order in your amplitude expansion?
c) Confirm that the gradient terms in the NWS-equation could have been obtained directly from the linear stability calculation performed in part (a) (i.e. without any nonlinear calculation).

2. Stability of Stripe Patterns in Newell-Whitehead-Segel Equation

a) Determine the linear stability of stripe (roll) solutions for the Newell-Whitehead-Segel equation,

$$\partial_T A = (\partial_X - \frac{i}{2}\partial_Y^2)^2\, A + \mu A - |A|^2 A.$$

Determine first the long-wave stability limits (i.e. in the limit of very small Floquet parameters (K, L))[1]. Then show that no additional instability arises (i.e. no instability that appears first at a finite wavevector) by investigating the extrema in the dispersion relation, i.e. in the dependence of the growth rate $\sigma(K, L)$on the Floquet parameters.
b) Identify both instabilities that arise (Eckhaus and zig-zag instability) as phase instabilities. To do so determine the eigenvectors in the long-wave limit and rewrite them as phase modes, as was done in class for the one-dimensional case.

3. Stability of Normal Stripe Patterns in Ginzburg-Landau Equation for Systems with Axial Anisotropy

Similar to problem **2**, determine the linear stability of stripe solutions in the Ginzburg-Landau equation

$$\partial_T A = (\partial_X^2 + \partial_Y^2)\, A + \mu A - |A|^2 A,$$

which describes *normal* roll (stripes) in the anisotropic system. Determine first the long-wave stability limits (i.e. in the limit of very small Floquet parameters (K, L))[2]. Then show that no additional instability arises (i.e. no instability that appears first at a finite wavevector) by investigating the extrema in the dispersion relation, i.e. in the dependence of the growth rate $\sigma(K, L)$on the Floquet parameters. Do you obtain an Eckhaus and a zig-zag instability? Do you obtain an additional instability?

[1] Hint: make use of the isotropy of the physical system from which the NWS-equation has been derived to simplify the stability calculation.
[2] Hint: make use of the isotropy of the Ginzburg-Landau equation itself to simplify the stability calculation.