Methods of Nonlinear Analysis D-12
Winter 2000
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Homework Assignment 7

Due Thursday, March 2, 2000.

1. Phase Equation for Spatially Ramped Control Parameters

Consider the modified Swift-Hohenberg model

$$\partial_t \psi = R(X)\psi - (\partial_x^2 + 1)^2 \psi - \psi^3 + B(X)(\partial_x \psi)^2 \psi$$

where $X = \epsilon x$ indicates a slow dependence of the two control parameters R and B on space. Use the WKB-type ansatz to derive a phase equation for this system of the form

$$\partial_T \Phi = D(q, R, B)\partial_X^2 \Phi + r(q, R, B)\partial_X R + b(q, R, B)\partial_X B.$$

2. Phase Equations for Standing Waves

Consider spatially slow modulations of standing waves and derive coupled phase equations for them. In analogy to the case of steady patterns and homogeneously oscillatory states start with an ansatz of the form

$$v = v(\epsilon^{-1}\Phi, \epsilon^{-1}\Psi)$$

where Φ and Ψ denote the spatial and temporal phases of the standing waves, respectively. Thus, under spatial translations the phases transform as

$$\Phi \to \Phi + \Delta\Phi \qquad \Psi \to \Psi,$$

while under temporal translations they transform as

$$\Phi \to \Phi, \qquad \Psi \to \Psi + \Delta\Psi.$$

Note that the standing waves we are interested in are also symmetric under spatial reflections. To get a better feeling for these two different phases consider standing waves solutions, i.e. spatially and temporally periodic solutions, of the coupled complex Ginzburg-Landau equations

$$\begin{aligned} \partial_t A - s\partial_x A &= d\partial_x^2 A + A + c|A|^2 A + g|B|^2 A \\ \partial_t B + s\partial_x B &= d\partial_x^2 A + B + c|B|^2 B + g|A|^2 B \end{aligned}$$

with $|A| = |B|$ and equal wavenumbers in both components. Use symmetry and scaling arguments to derive strongly nonlinear coupled equations for the phases. Expand then the phases around the simple solution

$$\Phi_0 = qX, \qquad \Psi_0 = \omega T$$

to obtain weakly nonlinear equations going to quadratic order in the deviation of the phases from (Φ_0, Ψ_0).

Perform a linear stability analysis of the state (Φ_0, Ψ_0) and discuss the two different ways in which this state can become linearly unstable.