Bounded Localized Solutions of the 1-D CGL in a backward bifurcation without saturation.

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Winter 2000
D12-2 Project

The 1-D Complex Ginzburg-Landau Equation has been the focus of much interest due to its surprising versatility. Recent papers show that what one might intuitively expect concerning the CGL eqn. undergoing a backward bifurcation without stabilizing higher order terms is not true. One may naively assume that since the bifurcation is sub-critical, then bounded solutions should not exist.

We will show that this is not the case and that indeed, bounded, localized solutions do exist and they take on certain forms depending on the parameters. First shown by Bretherton and Spiegel, the reason is due to non-linear dispersion as the complex coefficients get large and dominate. The phase gradient mechanism (PGM) can also be used to understand the dynamics. We then present experiments confirming the analysis.

Modeling a binary fluid mixture in thermohaline convection, Bretherton and Spiegel first investigated the equation:

$$\partial_t A = A + (1 + ic_1)\partial_x^2 A + i|A|^2 A$$

Simulations performed gave these results:
a) $c_1 = 100$. b) $c_1 = 1$. c) $c_1 = -10$.

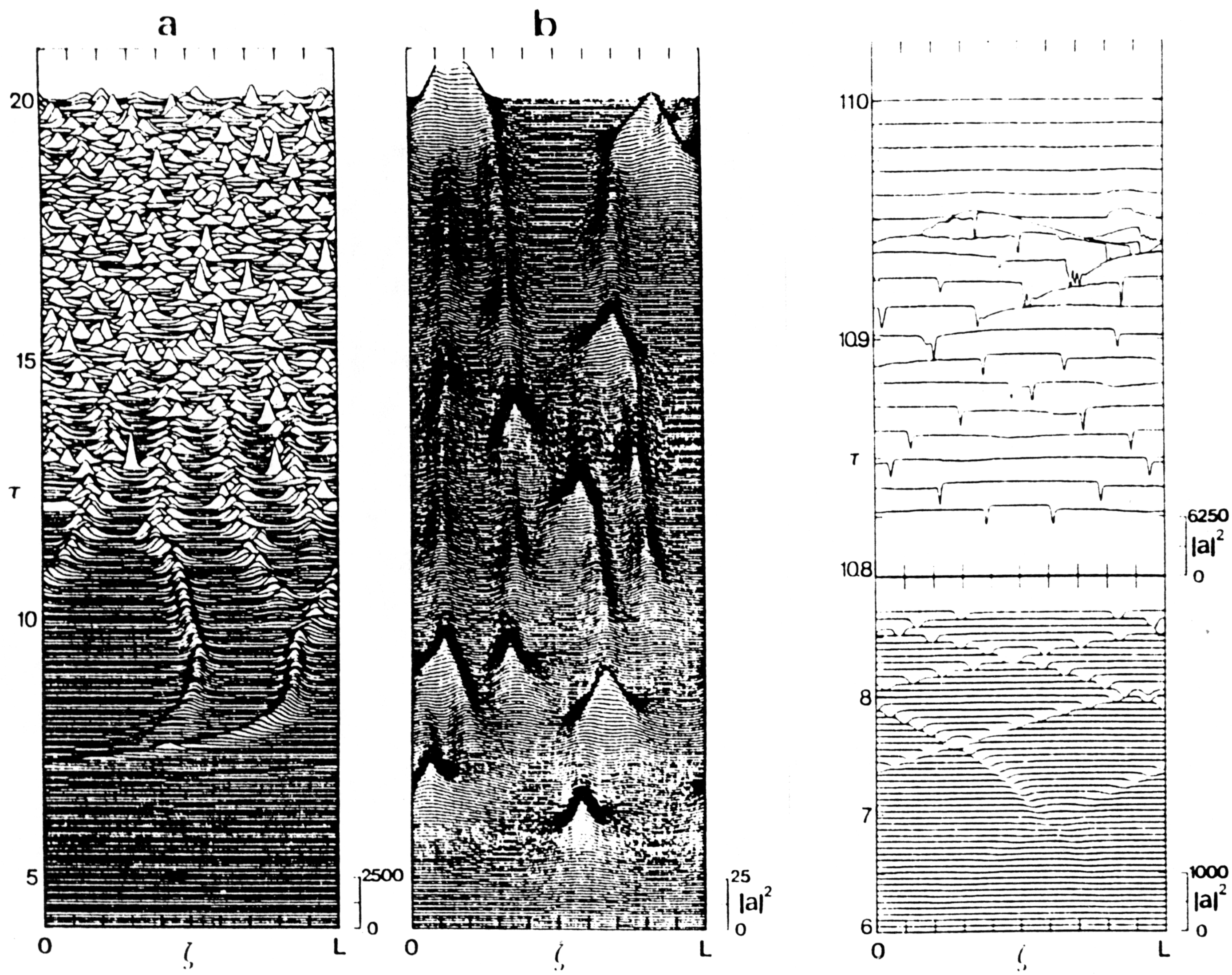


Fig. 3. The amplitude $|a|^2$ for $L = 20$ and (a) $\mu = 100$, (b) $\mu = 1$, after it evolves from white noise of mean-square amplitude $|a|^2 = 0.01$ at $\tau = 0$.

CGL eqn:

$$\partial_t A = A + (1 + ic_1)\partial_x^2 A + (\alpha + ic_2)|A|^2 A$$

$\alpha = -1.$

Equation is invariant under a simultaneous change in sign of c_1 and c_2, so let $c_2 \geq 0$.

Simulations: Most of the bounded solutions show spatio-temporal chaos.

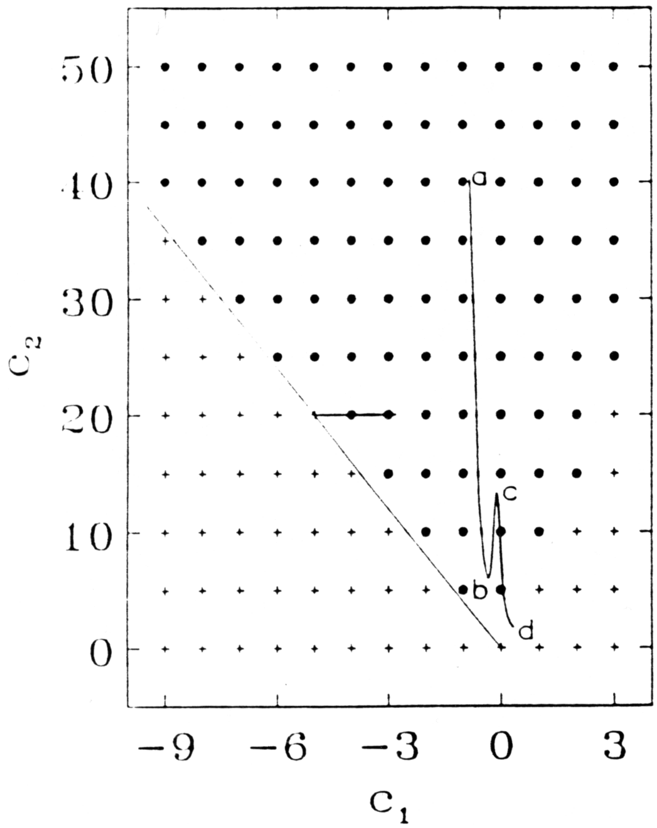


FIG. 1. Classification of the long-time behavior of solutions of Eq. (1) for $\alpha = -1$ in the c_1-c_2 plane. The system length is $L = 2\pi/0.3$ with periodic boundary conditions. The circles correspond to bounded and the crosses to unbounded solutions. The line $c_2 = -4c_1$ separates the two ranges on the left. Along the horizontal line at $c_2 = 20$ we found stable stationary pulses (with wavelength $\lambda = L = 2\pi/2.8$). The curve near $c_1 = 0$ refers to a water-alcohol mixture with varying separation ratio Ψ. At the indicated positions one has a, $\Psi = -7.5\times 10^{-5}$; b, -3.5×10^{-4}; c -0.005; and d, -0.5.

Simulations: Near the stability boundary there seems to be a range of bounded solutions showing quasi-stationary and quasi-periodic pulses.

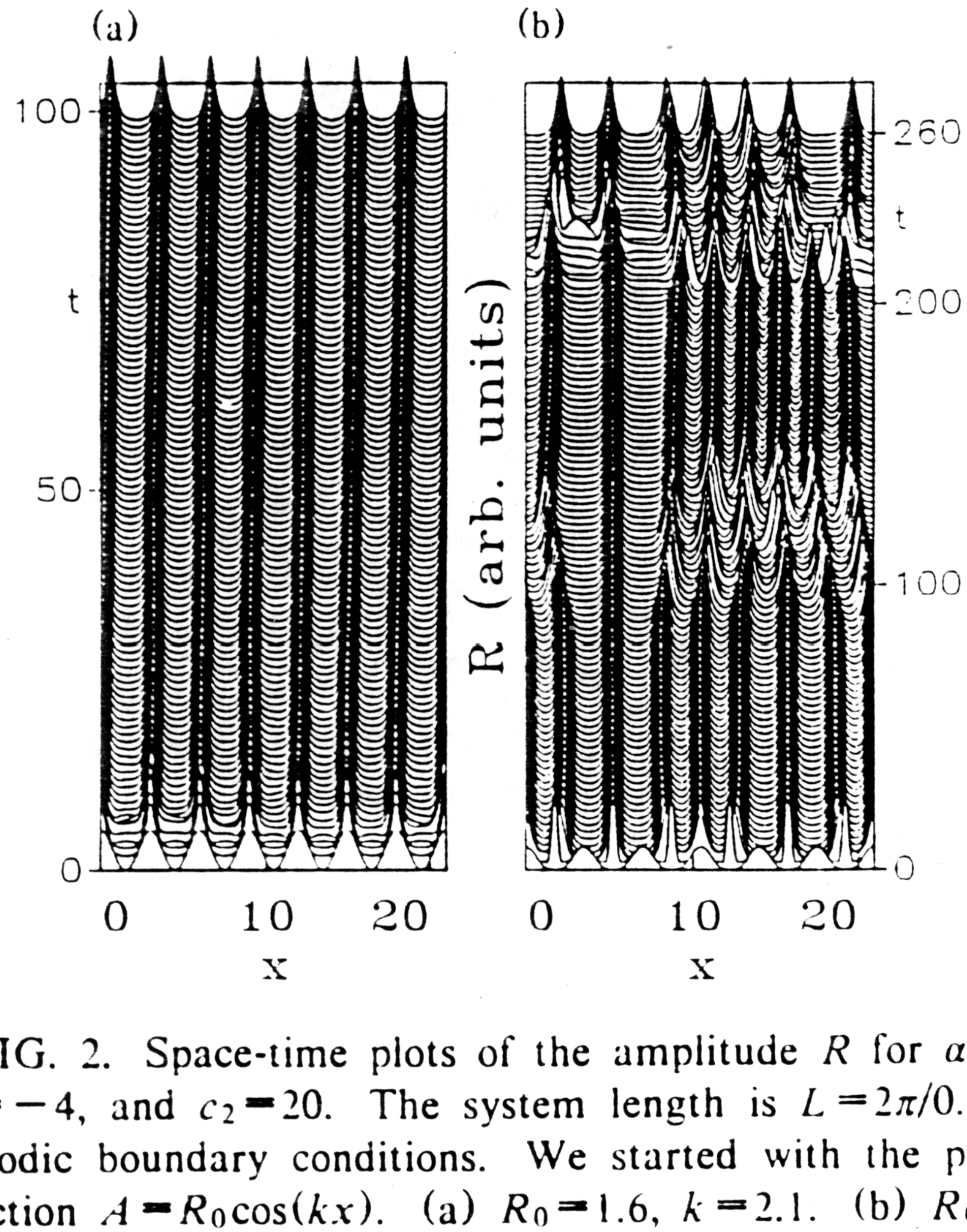


FIG. 2. Space-time plots of the amplitude R for $\alpha=-1$, $c_1=-4$, and $c_2=20$. The system length is $L=2\pi/0.3$ with periodic boundary conditions. We started with the periodic function $A=R_0\cos(kx)$. (a) $R_0=1.6$, $k=2.1$. (b) $R_0=0.8$, $k=1.8$.

Analysis:

Let $-c_1, c_2 \to \infty$.

$$A(x,t) = R(x,t)\, e^{i\phi(x,t)},$$

$\beta \equiv \dfrac{-c_1}{c_2}$, ratio of the parameters, and $\varepsilon \equiv 1/c_1$.

We now plug into the CGL eqn to get,

$$\partial_t R = \partial_x^2 R - R(\partial_x \phi)^2 - \frac{1}{\varepsilon}\partial_x R \partial_x \phi - \frac{R}{\varepsilon}\partial_x^2 \phi$$

$$R\partial_t \phi = \frac{1}{\varepsilon}\partial_x^2 R - R(\partial_x \phi)^2 + 2\partial_x R \partial_x \phi + R\partial_x^2 \phi + \frac{\beta}{\varepsilon} R^3$$

We can rearrange the above two equations to affect certain cancellations to get :

$$\varepsilon^2 \partial_t R + \varepsilon R \partial_t \phi = (1+\varepsilon^2)(\partial_x^2 R - R(\partial_x \phi)^2) + (\varepsilon^2 + (\beta + \varepsilon^2)R^2)R$$

$$-1/2\varepsilon \partial_t R^2 + \varepsilon^2 R^2 \partial_t \phi = (1+\varepsilon^2)\partial_x (R^2 \partial_x \phi) - \varepsilon\left[1 + (1-\beta)R^2)\right]R^2$$

We then expand through a nonlinear analysis with ε as the small parameter and assume that the phase, ϕ, to be very large.

$$R = R_0 + \varepsilon^2 R_1 + \cdots,$$

$$\phi = \frac{1}{\varepsilon}\phi_0 + \varepsilon\phi_1 + \cdots.$$

$$\mathrm{O}(\varepsilon^{-2}): \quad R_0 \phi'^2 = 0,$$

$$\mathrm{O}(\varepsilon^{-1}): \quad \partial_x (R_0^2 \phi_0') = 0.$$

$$\Rightarrow \phi_0 = \phi_0(t).$$

i.e. Phase is independent of space to leading order.

(Primes denote spatial derivatives.)

Define $\gamma(t) \equiv \partial_t \phi_0(t)$.

$$\mathrm{O}(\varepsilon^0): \quad \partial_x^2 R_0 - \gamma R_0 + \beta R_0^3 = 0,$$

$$\mathrm{O}(\varepsilon^1): \ -\partial_x (R_0^2 \phi_1') = \tfrac{1}{2}\partial_t R_0^2 - (1+\gamma) R_0^2 + (1-\beta) R_0^4.$$

For β, γ > 0, we seek periodic solutions so that the Duffing type eqn at $O(\varepsilon^0)$ leads to:

$$R_0(x,t) = \left[\frac{2\gamma(t)}{[2-m(t)]\beta}\right]^{\frac{1}{2}} \mathrm{dn}\left[\left[\frac{2\gamma(t)}{[2-m(t)]\beta}\right]^{\frac{1}{2}} x \,|\, m(t)\right]$$

$\mathrm{dn}(u\,|\,m)$ is a Jacobian elliptic function with period

2K(m). Where $(1-\mathrm{m})^{\frac{1}{2}} < \mathrm{dn}(u\,|\,m) < 1$, and K(m) is a complete elliptic integral of the first kind, $0 < \mathrm{m} < 1$.

When $\mathrm{m} \to 1$, $\mathrm{K} \to \infty$, and $\mathrm{dn}(\mathrm{u},1) \to \mathrm{sech}(\mathrm{u})$.

$\therefore R_0(x,t) = \sqrt{\dfrac{2\gamma(t)}{\beta}}\,\mathrm{sech}(\gamma^{1/2}x)$, pulse solution.

When $\mathrm{m} \to 0$ small harmonic oscillations occur.

$\lambda = 2\mathrm{K}(\mathrm{m})\left[\dfrac{2-m(t)}{\gamma(t)}\right]^{\frac{1}{2}}$, wavelength of $R_0(x,t)$.

λ should be time independent so once $\gamma(t)$ is known, m(t) is fixed.

Plug R_0 into the O(ε) eqn and require that ϕ_1' be periodic with wavelength, λ, so that the r.h.s. integrated over one wavelength is 0.

$\Rightarrow D\partial_t \gamma = 2E(m)\gamma(1-\gamma/\gamma_0).$

Where E(m) is a complete elliptic integral of 2nd kind.

$D = D(K(m), E(m), m) > 0 \; \forall$ m.

Since E(m) > 0, if $\gamma = \gamma_0$, (i.e. freq $= \gamma_0/\varepsilon$) then amplitude is stable when $D > 0$.

$$\gamma_0 = 3\beta\left[(\beta-4) - 2(\beta-1)\frac{1-m}{2-m}\frac{K(m)}{E(m)}\right]^{-1}$$

$$\gamma_0 > 0 \text{ for } \beta > \beta_d(m) \equiv 1 + 3\left[1 - 2\frac{1-m}{2-m}\frac{K(m)}{E(m)}\right]^{-1}$$

and diverges at $\beta_d(m)$.

If m = 1, then $\beta_d = 4$. If m < 1, then $\beta_d > 4$.

Note amplitude blows up at β_d. For given positive value λ, a solution exists for $\beta > 4^+$.

Numerical simulations comparing results of the analytics to the full CGL eqn.

λ and c_2 are held fixed.

Amplitude blows up for $\beta < \beta_d \approx 4$.

$\beta_h = 7.1$ for $c_2 = 20$ and $\beta_h = 7.8$ for $c_2 = 100$.

Pulses become unstable against oscillations for $\beta > \beta_h$.

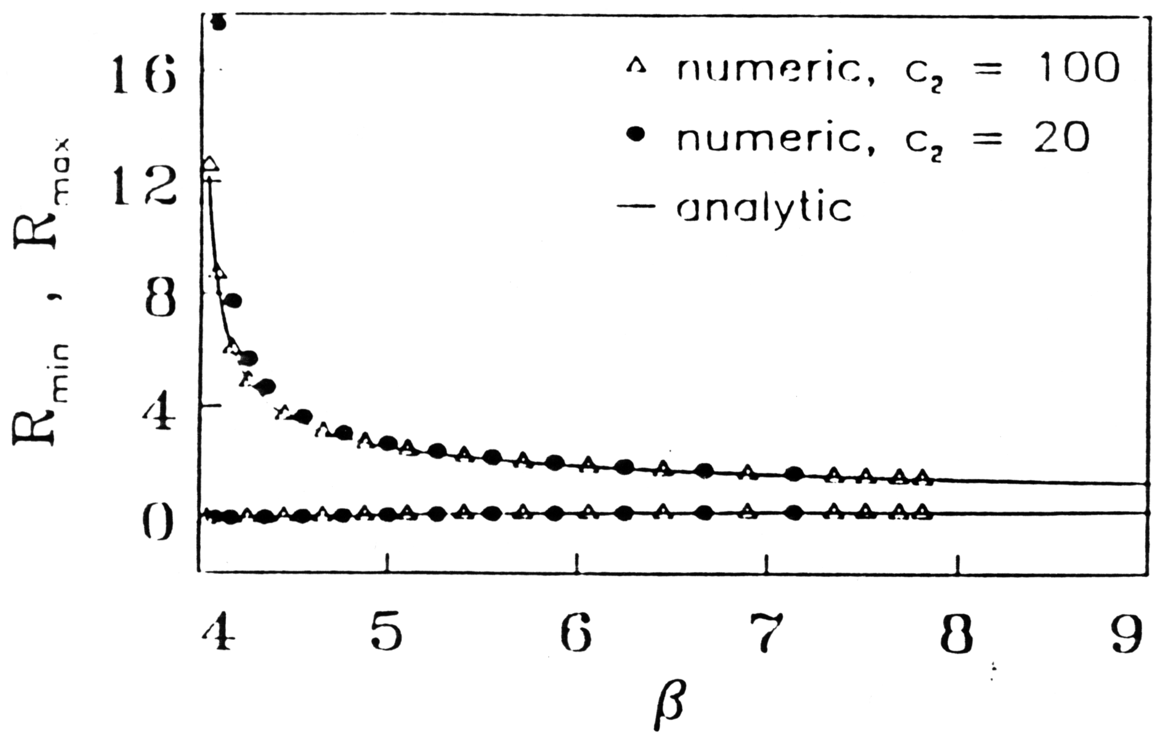
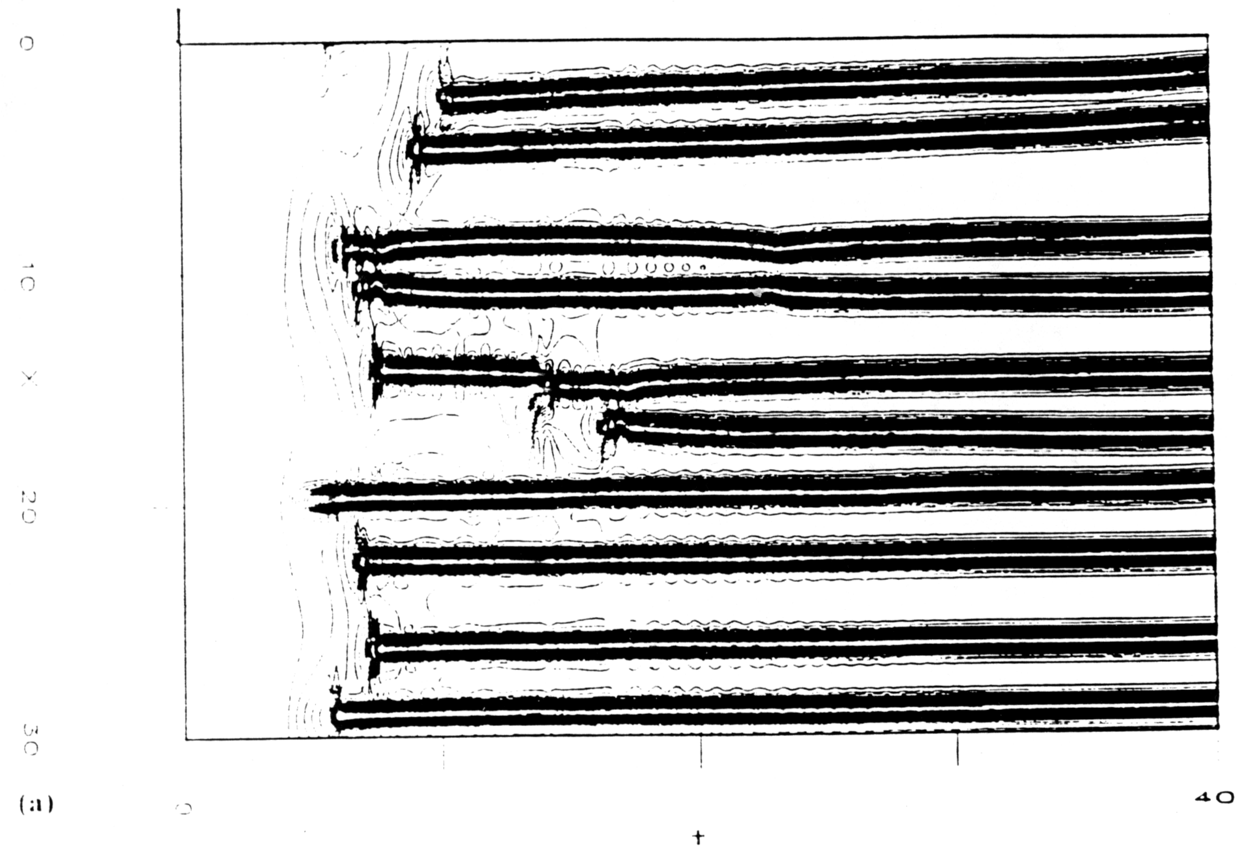


FIG. 3. Minimum R_{min} and maximum R_{max} of stable, stationary pulses as a function of $\beta = -c_2/c_1$ for $L = \lambda = 2\pi/2.8$. The solid line is calculated from Eq. (6). The circles refer to numerical simulations with $c_2 = 20$ and the triangles to $c_2 = 100$.

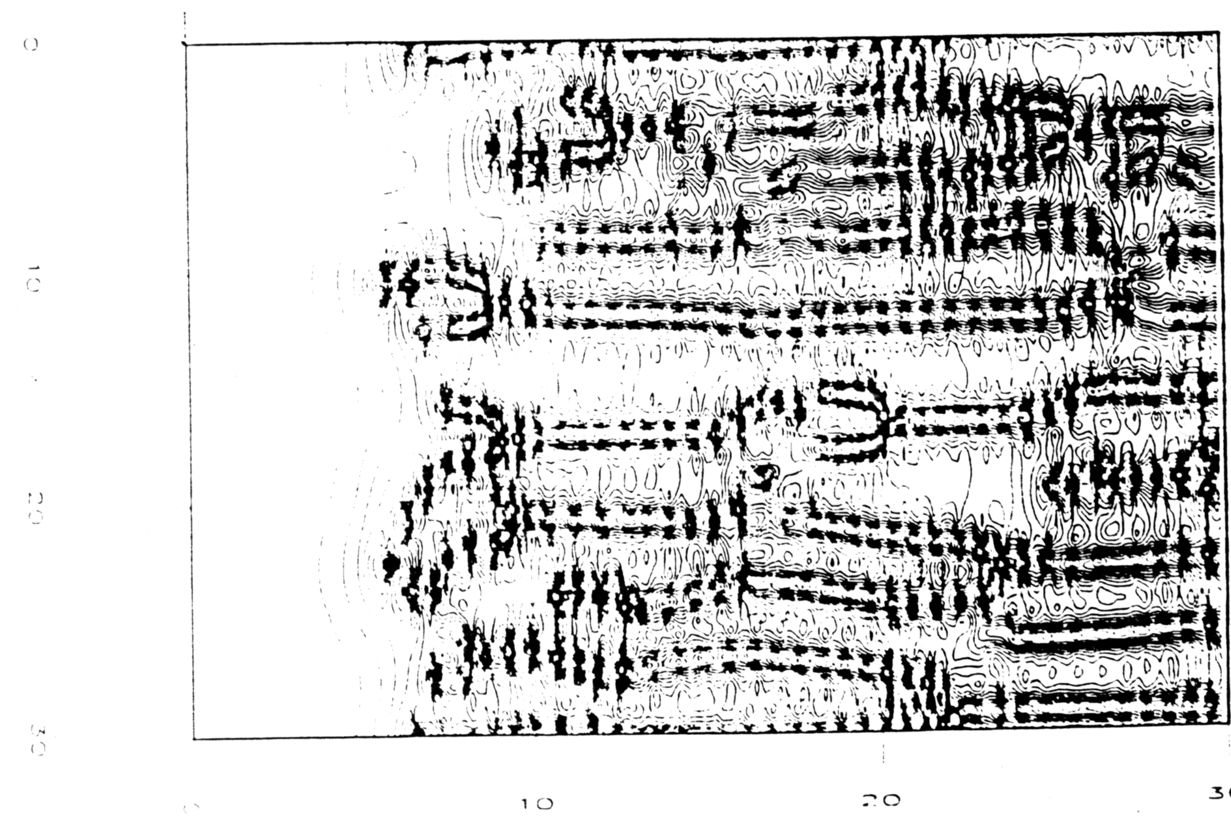
Figures. $c_2 = 15$ throughout.

a) Stationary pulses: $\beta = 5, c_1 = -3$.

b) Oscillatory Instability: $\beta = 10, c_1 = -1.5$.



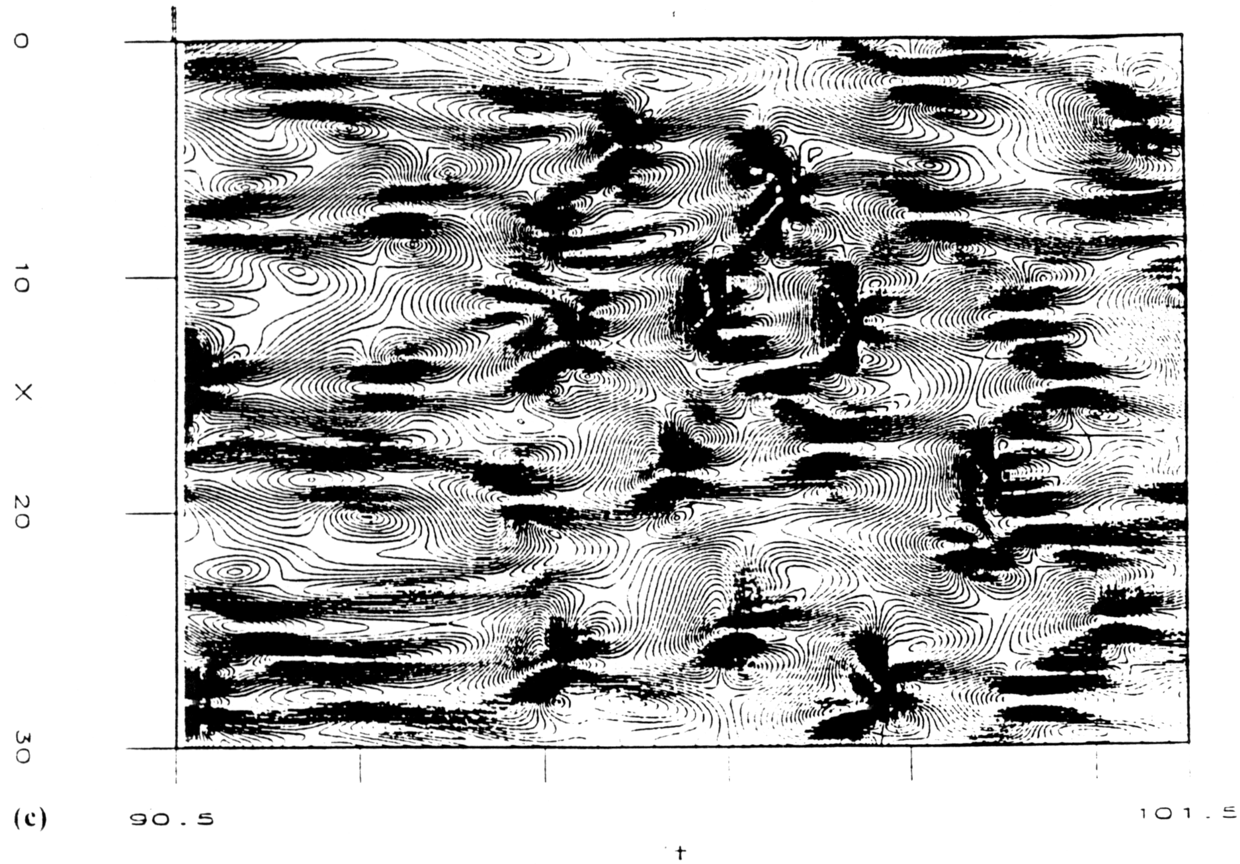
(a)



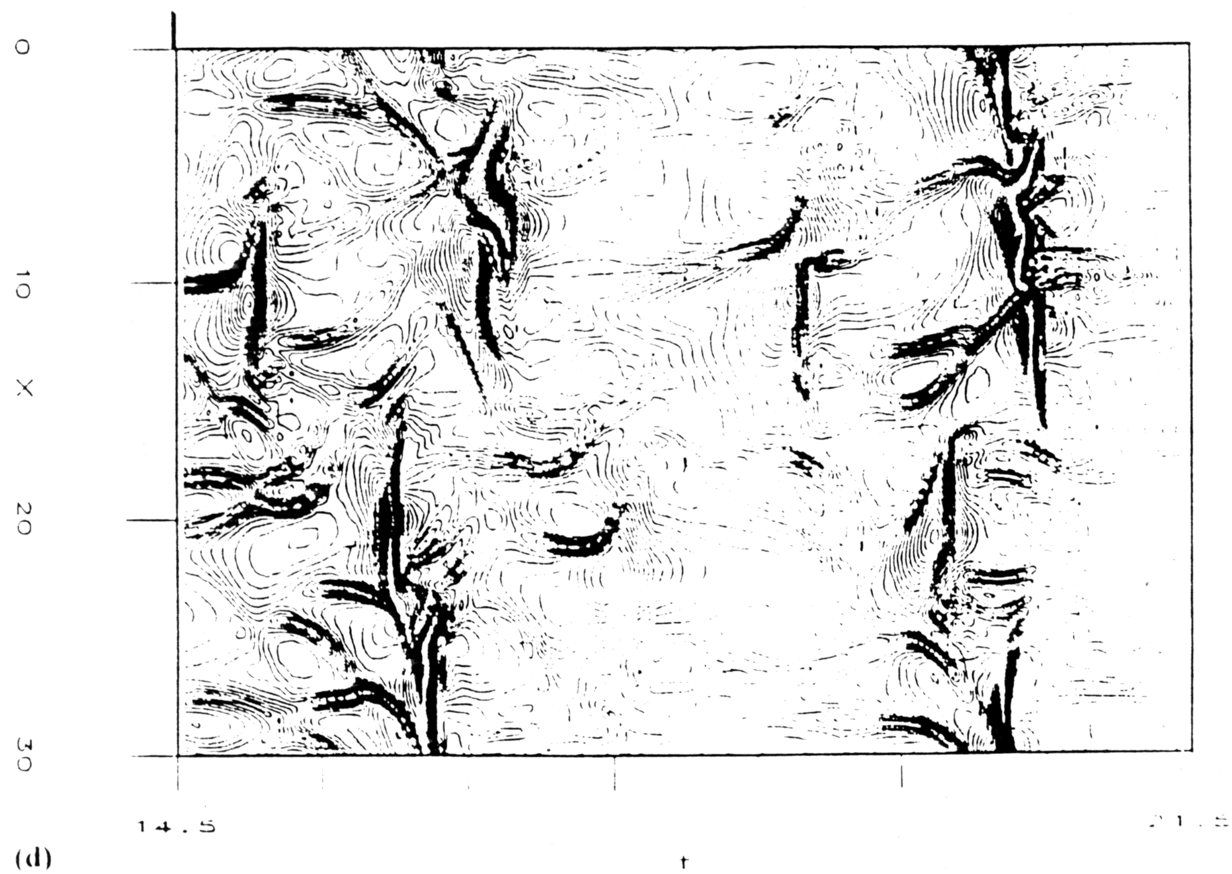
(b)

c) Spatio-Temporal Chaos: $\beta = \infty, c_1 = 0.$

d) De-focusing Regime: $\beta = -7.5, c_1 = 2.$



(c)



(d)

Stabilization Mechanism.

Let $A = \mathrm{R}e^{i\phi}$, $\mathrm{k} = \partial_x \phi$ (Phase Gradient).

$$\partial_t R = R + R^3 - k^2 R - 2c_1 k \partial_x R - c_1 R \partial_x k + \partial_x^2 R,$$

$$\partial_t k = -c_2 \partial_x R^2 - 2c_1 k \partial_x k + c_1 \partial_x \left[\frac{\partial_x^2 R}{R}\right] + \partial_x \left[\frac{\partial_x \left(R^2 k\right)}{R^2}\right].$$

Set $c_1 = 0$ and disregard the last term in both eqns.

Assume $c_2 >> 1$.

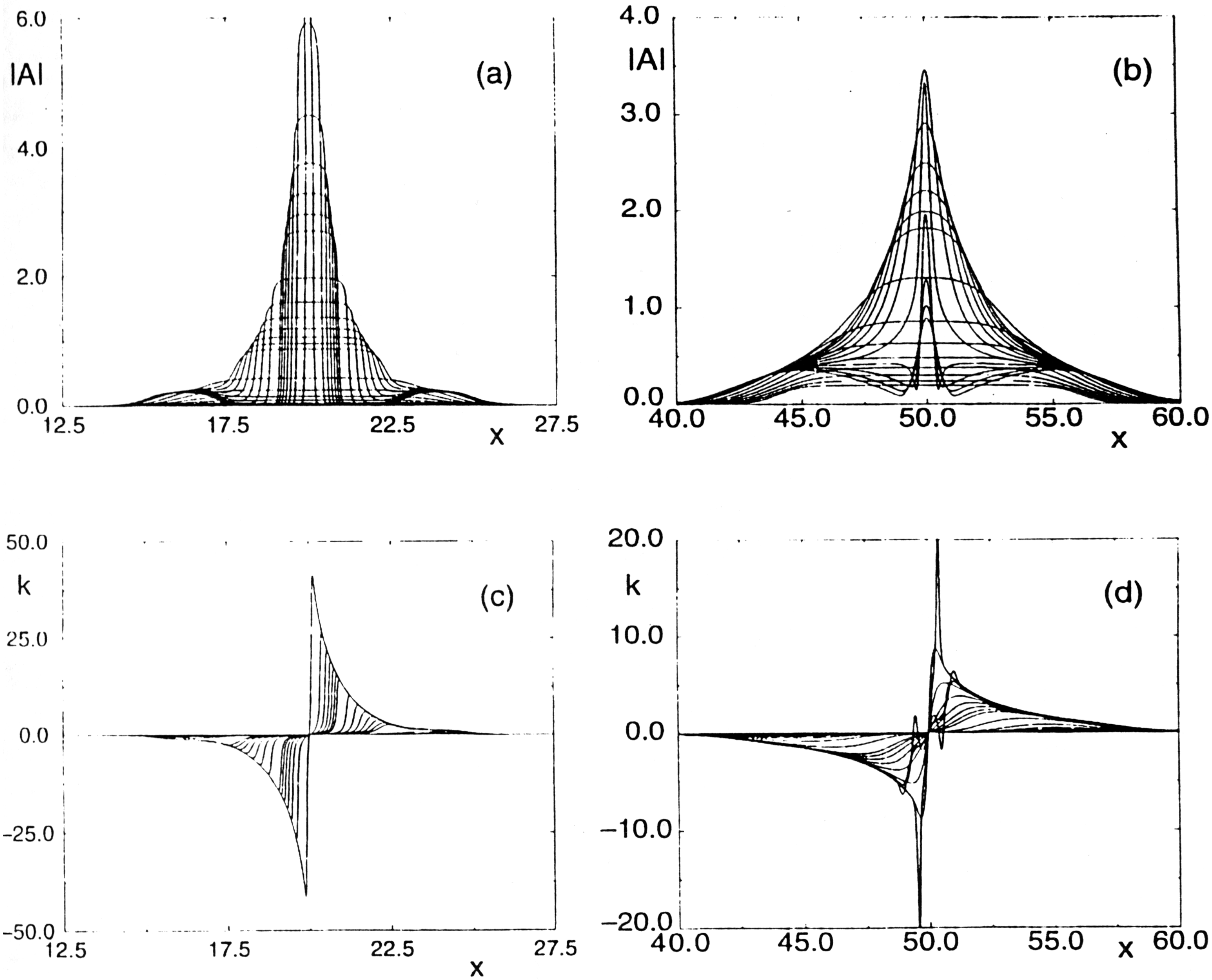
We then get for the simplified model:

$$\partial_t R^2 = 2R^2 \left(1 + R^2 - k^2\right),$$

$$\partial_t k = -c_2 \partial_x R^2.$$

Consider an I.C. of a small amplitude plateau, which goes to 0 on both sides. Let k=0 initially.
R is approximately constant in the plateau region. As $\partial_t R^2$ increases, $\partial_x R^2$ gives rise to Phase gradients, k, that lead to a saturation of the amplitude in this region. This Phase Gradient Mechanism (PGM) makes the two plateau regions move toward the center. Then the sharp gradients cause the propagation speed to steadily increase and the pulse narrows. Then suppression occurs. The simplified model is only valid up to this suppression point. In the full model, the amplitude settles down and phase slips then occur. Instead, we get freezing of the phase gradient as $\partial_t k \rightarrow 0$ and $\partial_t R \rightarrow R(1-k^2)$.
Note we neglected smoothing term and other higher order terms.

Comparison of simplified model with full CGL eqn. Plots of R (top) and k (bottom).



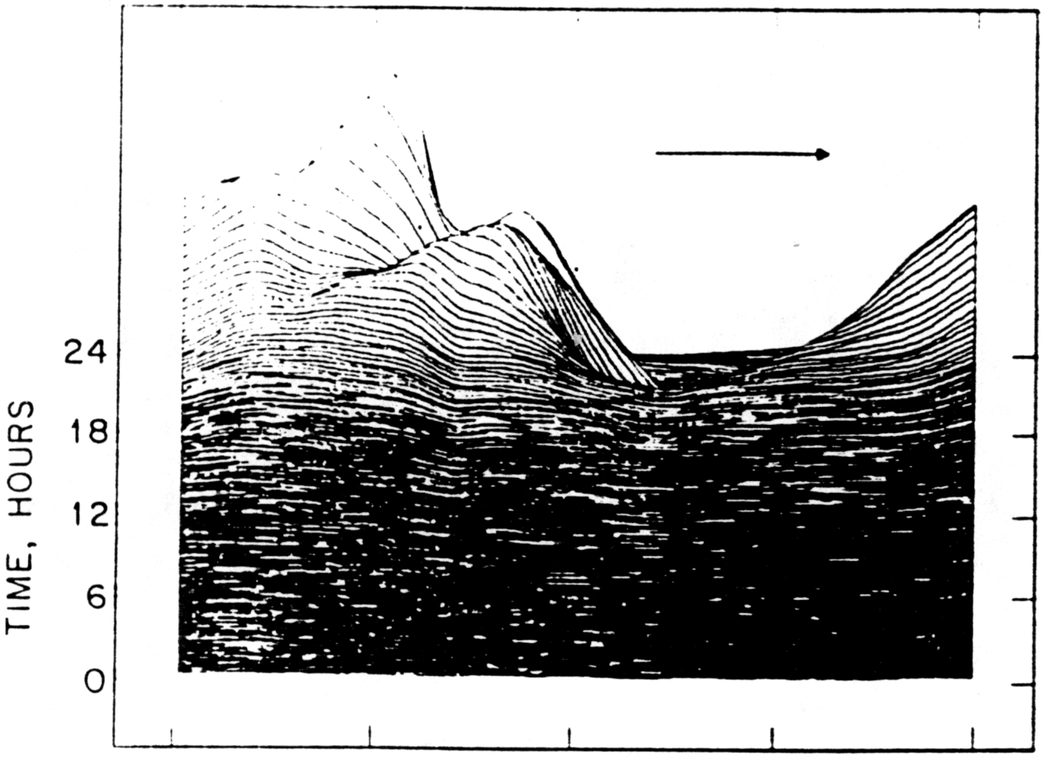
ig. 3. Simulations of the model equations (10) and (11) ((a) and (c)) and the CGLE (3) ((b) and (d)) for $b = 0$, $c = 15$; we show the modulus $|A|$ and the local phase gradient $k := \partial_x \phi$ for some successive time steps. The initial conditions were $A = 0.01(\tanh(x-16) - \tanh(x-24))$ (a) and (c)) and $A = 0.1(\tanh(x-44) - \tanh(x-56)))$ ((b) and (d)).

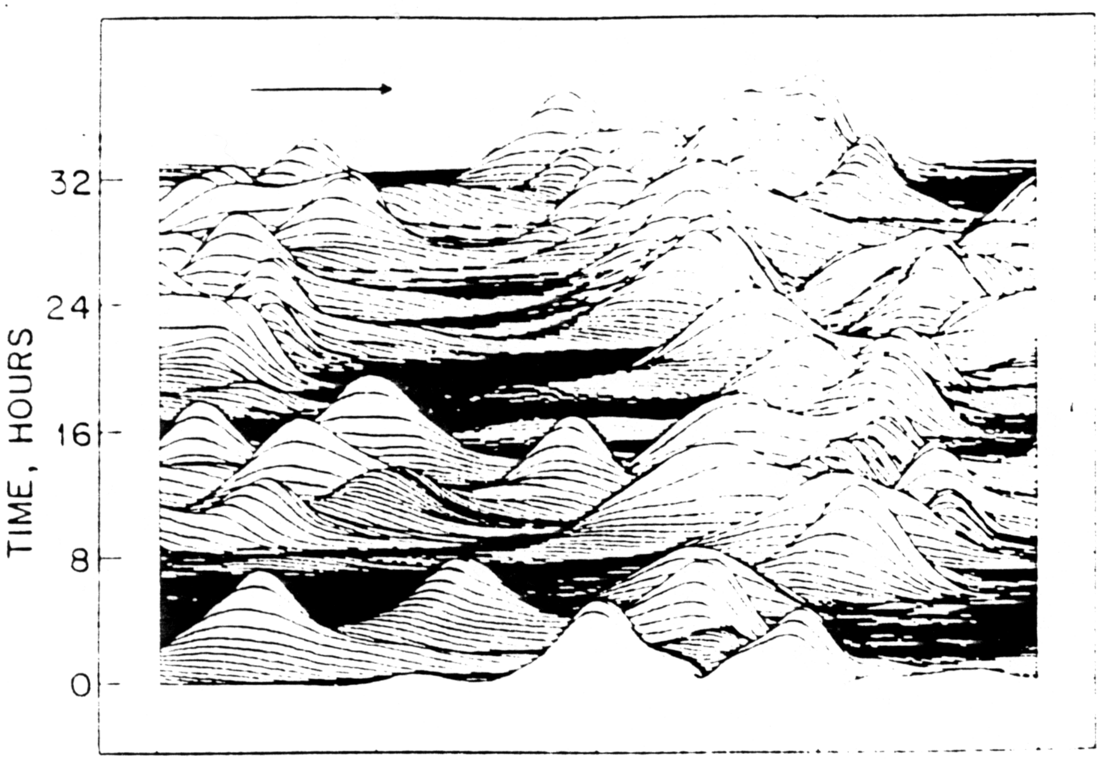
Experiments:
Binary mixture of alcohol and water heated from below in an annular cell.
Let ε be the linear onset and Ψ be the separation ratio. ε=0 characterizes no growth.
For ε>0, traveling wave solutions appear and for Ψ >-.05 we get erratic pulsing which becomes more frequent as ε increases. Also the left and right traveling waves are uncorrelated so this suggests that the coupling term is negligible.
As ε increases, the density of the pulses increase in time and they become more localized in space.
The nonlinear term c_2 acts to damp the waves in regions of strong spatial gradients. This makes the edges steepen and contract and the peaks then grow linearly without saturation until they collapse.

Space-time plots:

a) ε=1.8x10-4, ψ=-0.021.

b) ε=5.6x10-4, ψ=-0.021.





Conclusions:

Even for a sub-critical bifurcation the CGL eqn. allows for stable localized structures that do not blow up even though higher order stabilizing terms were omitted. These structures can take various forms depending on the parameters c_1 and c_2.

Spatially periodic soliton type solutions exist and can be characterized analytically. An Oscillatory instability kicks in for $\beta > \beta_h$. Then as β gets large, we get localized pulses exhibiting spatio-temporal chaos. This corresponds with the relevant experimental regime. Then if $c_1 > 0$, a qualitatively different solution is found that seems related to hole-like solutions represented elsewhere.

References:

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