Dynamics of Defects within the Two-Dimensional Ginzburg - Landau Equation and the NLSE

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Properties of defects

Defects are locations in an amplitude field where the complex amplitude goes to zero and the phase is undefined.

The phase of each defect changes by 2π when encircling a defect in the clockwise direction.

Phase change	Topological Charge
2π	+1
-2π	-1
$2\pi n$	+n

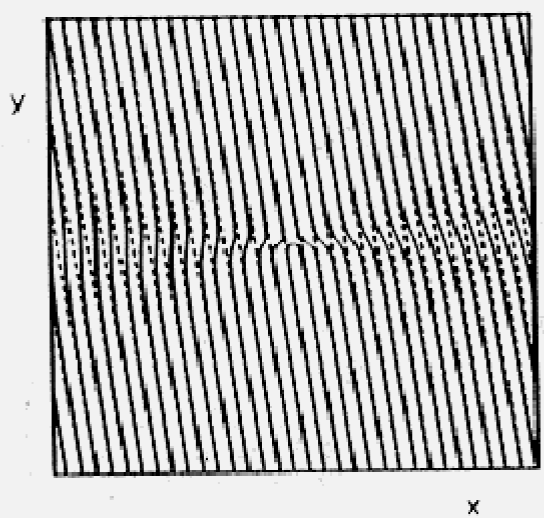


Fig. 6. The static dislocation in the oblique-roll case for parameters typical for EHC ($p_c/q_c = 0.15$, $\xi_1 = 0.30l$, $\xi_2 = 0.05l$, $a = 0.20$, $\epsilon = 0.1$).

The above simulation of a defect in a roll pattern formed in a anisotropic liquid crystal demonstrates that due to the increase in phase traveling around the defect, there is an additional roll on one side of

Core Radius of Vortices

In a paper by JC Neu, defects are studied in the nonlinear Schrodinger equation and the nonlinear heat equation,

$$\Delta\psi + (1 - |\psi|^2)\psi = - i\, \psi_t$$
$$\Delta\psi + (1 - |\psi|^2)\psi = \psi_t$$

The time independent case of each of these reduces to the following,

$$\Delta\psi + (1 - |\psi|^2)\psi = 0 \quad (1)$$

The vortex solution to this of the form,

$$\psi = U(r) \exp[\, i(n\theta + \theta_0)]$$

Substituting this into (1), we get,

$$U_{rr} + U_r/r - n^2U/r^2 + (1 - |U|^2)\, U = 0$$

$$U(0) = 0 \text{ and } U(\infty) = 1$$

The numerical solution of this is illustrated below showing that the "core radius" of the vortex increases as n increases. Asymptotically, $U(r) \sim 1 - n^2/2r^2$.

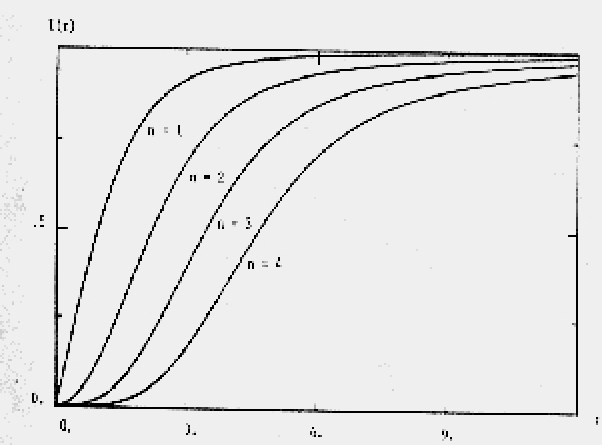


Fig. 1

In the NLHE, vortices with n = +1 or –1 are stable, and those with |n| > 1 are unstable.

Motion of vortices in the NLHE

J.C. Neu / Vortices in complex scalar fields

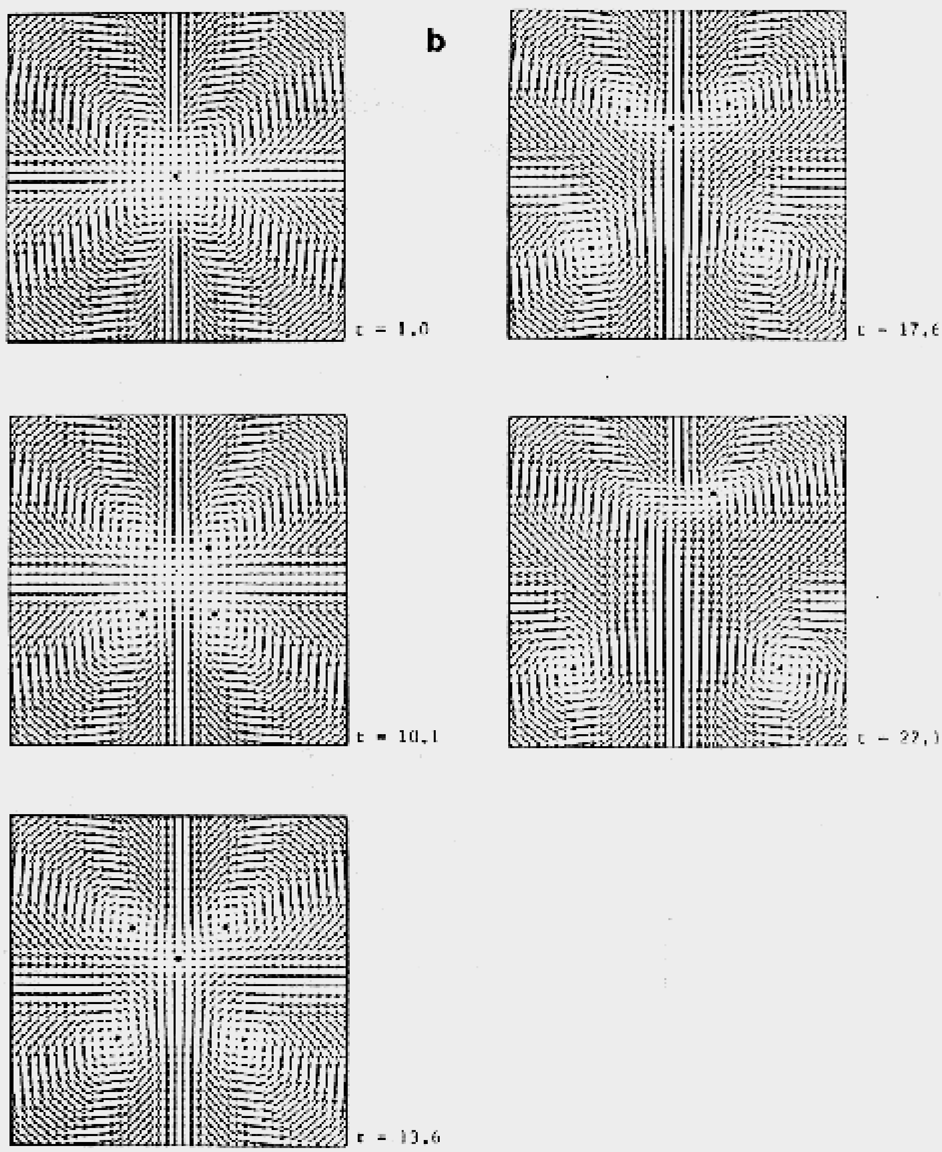


Fig. 3. Continued.

An n= 3 vortex divides into four n=+1 vortices and a n=-1 vortex. Eventually, two of the oppositely charged vortices combine leaving three n=1 vortices. This demonstrates that, for the nonlinear heat equation, vortices with winding numbers, |n| > 1, are unstable against division into vortices with a charge of |n|=1.

Motion of dislocations in roll patterns can cause changes in the underlying pattern

A. **Climb**, or motion along the roll axis changes the spacing of the rolls. Note that for Q<0, P = 0, the dislocation moves upward, and the pattern behind the dislocation is brought closer to band center.

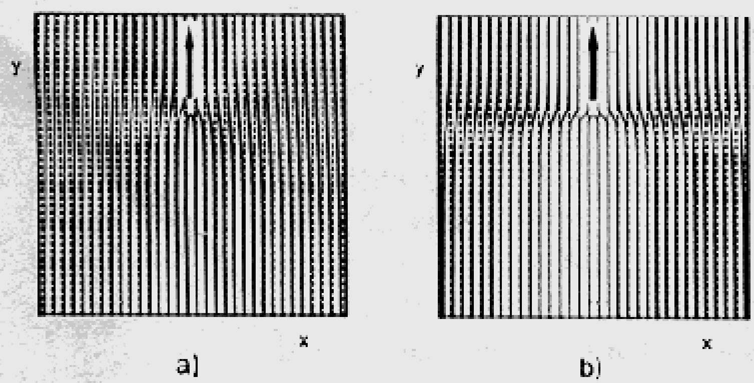


Fig. 3. A climbing motion in the normal-roll case for the same parameters as in fig. 2 except for $Q \neq 0$. (a) $Q = -0.0133$, $V = 0.004$, and in physical units $\Delta q_x = 0.0042\varepsilon^{1/2} l^{-1}$, $\Delta q_y = 0$, $v_x = 0$, $v_y = 0.0158\varepsilon^{1/2} l/T_0$. (b) $Q = -0.0505$, $V = 0.0198$, $\Delta q_x = -0.0160\varepsilon^{1/2} l^{-1}$, $\Delta q_y = 0$, with the same values for v_x, v_y

B. **Glide**, or motion perpendicular to the roll axis changes the orientation of the rolls.

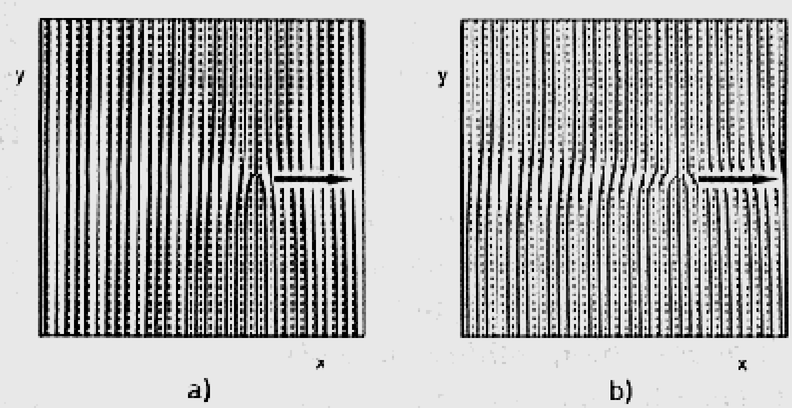


Fig. 4. A gliding motion in the normal-roll case for the same parameters as before except for $P \neq 0$. (a) $P = -0.0133$, $V = 0.0058$, $\Delta q_x = 0$, $\Delta q_y = 0.0042\varepsilon^{1/2} l^{-1}$, $v_x = 0.0158\varepsilon^{1/2} l/T_0$, $v_y = 0$; (b) P, Q, v_x, v_y as in (a), $\Delta q_x = 0$, $\Delta q_y = 0.021\varepsilon^{1/2} l^{-1}$.

Determination of the dependence of the dislocation velocity on the perturbation, Q, from the critical wavenumber

Carrying out weakly nonlinear analysis of an anisotropic system, near the threshold, the spatially varying quantities take the form,

$$u_j = \varepsilon^{1/2} [\, c_j A(X', Y', T')\, e^{i(qx + py)} + cc] f_j(z,T)$$

Where c_j can be i or 1, $q_c = (q , p)$ is the critical wavevector, and X, Y and T are slowly varying variables scaled as:

$$X = \epsilon^{1/2}\, x$$

$$Y = \epsilon^{1/2}\, y$$

$$T = \epsilon t$$

An expansion of the physical equations to order $\varepsilon^{3/2}$, yields a solvability condition in the form of the following amplitude equation.

$$T_0\, \partial_T A = [\xi_1{}^2 \partial^2{}_X + \xi_2{}^2 \partial^2{}_Y + 2 \xi_1 \xi_2\, a\, \partial_X \partial_Y + 1 - |A|^2] A$$

Where a = 0 for normal rolls and $0 \subset a < 1$ for oblique rolls.

Rotating this amplitude equation and rescaling yields a simpler form,

$$\partial_T A = (\Delta + 1 - |A|^2)\, A$$

Stationary Solutions

Substituting $A = e^{i(QX + PY)}$, into the amplitude equation shows that there are stationary solutions of the form,

$$A = (1 - Q^2 - P^2)^{1/2} e^{i(QX + PY)},$$

Stability analysis of perturbations,

$$A = B e^{iQx} e^{iPx} [1 + a_+ e^{\sigma t} e^{iKX} e^{iLX} + a_- e^{\sigma t} e^{-iKX} e^{-iLX}]$$

Shows that solutions are stable for $Q^2 + P^2 < 1/3$, which is a direct generalization of the Eckhaus Stability limit.

Stationary dislocations

Stationary dislocations take the form,

$A = F(r)\, e^{i\phi}$

$\phi = \tan^{-1}(Y/X)$

Where $r = (X^2 + Y^2)^{1/2}$,

Plotting F(r) numerically, yields

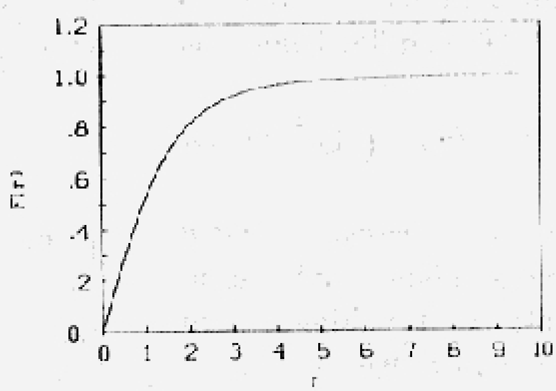


Fig. 1. The amplitude $f(r)$ of a static dislocation as a function of the distance from the dislocation core $|r|$ in reduced units.

Showing that the amplitude increases linearly near the core of the dislocation and for large r is approximated by,

$F(r) \sim 1 - 1/(2r^2)$

Moving Dislocations

When Q is not equal to 0, the dislocations will no longer be stationary. Considering nonstationary solutions,

$$A = B(X,Y,T)\, e^{\,i(QX + PY)} \qquad (1)$$

Where $|B|^2$ g $1 - Q^2 - P^2$, as r g ϖ and the phase changes by two π going around the dislocation for large r. (ie, from the far field, the slowly moving dislocation appears stationary.)

Because the amplitude equation is rotationally invariant, we can without loss of generality choose a coordinate system in which velocity V, of the dislocations is in the Y - direction.

Substituting (1) into the amplitude equation for A yields,

$$\partial_T B = [\Delta + 2iQ\partial_X + 2iP\partial_Y + 1 - Q^2 - P^2 - |B|^2]B$$

Shifting this into a moving frame yields,

$$-V\partial_Y B = [\Delta + 2iQ\partial_X + 2iP\partial_Y + 1 - Q^2 - P^2 - |B|^2]B$$

Projection onto the Zero Eigenmodes

Because the system is translation invariant, the translation modes, $\partial_X B$ and $\partial_Y B$ are zero eigenvectors of the system.

Projecting onto the zero eigenmodes and adding the complex conjugates yields,

$$
\begin{aligned}
-V \langle\partial_X B^*, \partial_Y B\rangle - V \langle\partial_X B, \partial_Y B^*\rangle = {} & \langle\partial_X B^*, \partial^2_X B\rangle + \langle\partial_X B, \partial^2_X B^*\rangle \\
& + \langle\partial_X B^*, \partial^2_Y B\rangle + \langle\partial_X B, \partial^2_Y B^*\rangle \\
& + iQ\langle\partial_X B^*, \partial_X B\rangle - iQ \langle\partial_X B, \partial_X B^*\rangle \\
& + iP\langle\partial_X B^*, \partial_Y B\rangle - iP \langle\partial_X B, \partial_Y B^*\rangle \\
& + (1 - Q^2 - P^2)[\langle\partial_X B^*, B\rangle + \langle\partial_X B, B^*\rangle] \\
& + \langle\partial_X B^*, |B|^2 B\rangle + \langle\partial_X B, |B|^2 B^*\rangle]
\end{aligned}
$$

$$
\begin{aligned}
-V \langle\partial_Y B^*, \partial_Y B\rangle - V \langle\partial_Y B, \partial_Y B^*\rangle = {} & \langle\partial_Y B^*, \partial^2_X B\rangle + \langle\partial_Y B, \partial^2_X B^*\rangle \\
& + \langle\partial_Y B^*, \partial^2_Y B\rangle + \langle\partial_Y B, \partial^2_Y B^*\rangle \\
& + iQ\langle\partial_Y B^*, \partial_X B\rangle - iQ \langle\partial_Y B, \partial_X B^*\rangle \\
& + iP\langle\partial_Y B^*, \partial_Y B\rangle - iP \langle\partial_Y B, \partial_Y B^*\rangle \\
& + (1 - Q^2 - P^2)[\langle\partial_Y B^*, B\rangle + \langle\partial_Y B, B^*\rangle] \\
& + \langle\partial_Y B^*, |B|^2 B\rangle + \langle\partial_Y B, |B|^2 B^*\rangle]
\end{aligned}
$$

To simplify these expressions, notice using one or more iteration of integration by parts, and the fact that the solution is uniform at infinity.

$$\langle\partial_X B^*, \partial^2_X B\rangle = - \langle\partial_X B, \partial^2_X B^*\rangle$$

$$\langle\partial_Y B^*, \partial^2_Y B\rangle = - \langle\partial_Y B, \partial^2_Y B^*\rangle$$

$$\langle\partial_X B^*, B\rangle = - \langle\partial_X B, B^*\rangle$$

$$\langle\partial_Y B^*, B\rangle = - \langle\partial_Y B, B^*\rangle$$

$$\langle\partial_Y B^*, |B|^2 B\rangle = - \langle\partial_Y B, |B|^2 B^*\rangle$$

The two projection equations simplify to the following,

$$V[< \partial_X B^* \partial_Y B> + cc] = -2iP[< \partial_X B^* \partial_Y B> - cc] \quad (1)$$

$$V <|\partial_Y B|^2> = iQ [< \partial_X B^* \partial_Y B> - cc] \quad (2)$$

Notice that for the case in which P=0, the amplitude equation has the symmetry, B(-X) =B*(X), so equation (1) is satisfied.

Which shows that the velocity of a dislocation is in the direction perpendicular to Q, the perturbation in the critical wavevector.

An Equation for the Velocity of a Dislocation

Using integration by parts on the right hand side of the second solvability condition yields,

$$V \langle|\partial_Y B|^2\rangle - cc = iQ\,[- \langle\partial_X \partial_Y B^*, B\rangle + \langle\partial_X \partial_Y B, B^*\rangle$$

$$+ \int_{-\infty}^{\infty} B\,\partial_X B^*\Big|_{Y=-\infty}^{Y=\infty} dX + ?_{-\infty}^{B} B\,\partial_Y B^*\Big|_{X=-\infty}^{X=\infty} dY]$$

$$V \langle|\partial_Y B|^2\rangle - cc = iQ\,[\int_{-\infty}^{\infty} B\,\partial_X B^*\Big|_{Y=-\infty}^{Y=\infty} dX + ?_{-\infty}^{B} B\,\partial_Y B^*\Big|_{X=-\infty}^{X=\infty} dY] \quad (3)$$

Using the limit that the dislocation goes to the uniform solution as r goes to infinity,

$$B \sim (1-Q^2)^{1/2}\, e^{i\theta(X,Y)} \qquad \text{for large } r$$

Inserting this into (3) it becomes obvious that the right hand side of three is a closed contour integral of $-i(1-Q^2)\partial_s\theta$ where ∂_s is the tangential derivative. Because the contour integral of $\partial_s\theta$ is equal to 2π, equation (3) reduces to

$$V(Q) = -\mu\, 2\pi Q(1 - Q^2)$$

Where,

$$\mu^{-1} = \langle|\partial_y B|^2\rangle$$

Note, for small perturbations $Q < 0$, where the pattern is widened compared to the critical wavelength, the dislocation climbs in the positive Y direction.

The frequency behind the dislocation is increased moving the system toward bandcenter.

An Expression for V as Q g 0

Because m diverges for large r , we cannot simply evaluate μ, for Q = R = 0, and we need a better expression for B.

Inserting the expression,

$$B \sim (1-Q^2)^{1/2} e^{i\theta(X,Y)}$$

Valid for large r into the amplitude equation yields the phase equation,

$$V\partial_Y\theta + \Delta\theta = 0$$

Because there is a discontinuity in theta as we go around the defect, it is necessary to insert the derivative of a delta function on the left hand side of this equation. Assuming the discontinuity line is in the y direction, then solving using a fourier transform yields:

$$\partial_Y\theta = |V|\ X\ K_1(|V|\ r\ /\ 2)\ \exp(-VY/2)/\ (2r)$$

In the limiting cases $Vr/2 \ll 1$ and $Vr/2 \gg 1$, where the Bessel function can be simplified, we obtain,

$$\partial_Y \theta = \tan^{-1}(Y/X) \exp(-VY/2)/ (2r) \qquad Vr/2 \ll 1$$

$$\partial_Y \theta = (V \pi)^{1/2} X \exp(-V [Y+r]/2) / (2r^{3/2}) \qquad Vr/2 \gg 1$$

For disturbances in front of the dislocation $Y>0$, the decay is exponential and behind the disturbance, $Y<0$, $r=|Y|$, the decay is algebraic.

Using these two limits, the velocity equation becomes,

$$V \ln(V_0/V) = -2Q \quad VR \gg 1$$

$$V \ln(r/\xi_0) = -2Q \quad VR \ll 1$$

With $\xi_0=1.13$ and $V_0 = 3.29$.

Asymptotic Vortex Dynamics for the Non-linear Schrodinger Equation

Nonlinear Schrodinger Equation

$\Delta\psi + (1 - |\psi|^2)\psi = - i\, \psi_t$

We derive trajectories for vortices in the nonlinear Schrodinger equation by matching a core expansion in a vicinity O(1) from the vortices with a far field expansion $O(1/\epsilon)$ from the vortices.

Assumptions:

1. The vortices are far apart.
2. The vortices move slowly.

A vortex trajectory is associated with $\mathbf{X} = \mathbf{Q}(T,\varepsilon)$.

Introduce slow time and space scales

$\mathbf{X} = \epsilon\, \mathbf{x}$

$T = \epsilon^2 T$

Core expansion:

$\psi = \psi(\mathbf{r},T,e)$

$\mathbf{r} = \mathbf{x} - \mathbf{Q}(T,\varepsilon)/\varepsilon$

Far field expansion:

$\Psi = \Psi(\mathbf{X},T,\varepsilon)$

Leading Order Core and Far-Field Solutions

Because the vortices are separated by distances (1/ε) and are moving slowly, the leading order core expansion is expected to be a vortex state:

$$\psi = U(r)\exp[\,i(n\theta + \theta_0(T,\varepsilon))]$$

Far from the vortex, the far field approaches the uniform solution which has a modulus of one. So the far field solutions has a first order solution:

$$\Psi^0 = U(\mathbf{X},T,\varepsilon)\exp[i\Theta^0(X,T,\varepsilon)]$$

Where $\Theta^0(X,T,\varepsilon)$ is a slowly varying phase.

Far Field Expansion

Scaling the nonlinear Schrodinger equation, to the slow time and space variables yields,

$$\varepsilon^2\Delta\psi + (1 - |\psi|^2)\psi = - i\, \varepsilon^2\psi_T$$

Substituting the far field expansion,

$$\Psi^0 = U(\mathbf{X},T,\varepsilon) \exp[i\Theta^0(\mathbf{X},T,\varepsilon)]$$

Into the far field NLSE, and separating the real and imaginary components yields,

$$(U^2-1)U = -\varepsilon^2 U(\Theta_T + |\nabla\Theta|^2) + \varepsilon^2\Delta U \quad (1)$$

$$U\Delta\Theta = -U_T - 2\nabla U\cdot\nabla\Theta \quad (2)$$

Equation 1 shows that $U = 1 + O(\varepsilon^2)$, which substituted into the above expressions yields,

$$U = 1 - \varepsilon^2 U(\Theta_T + |\nabla\Theta|^2)/2 + O(\varepsilon^4)$$

$$\Delta\Theta = O(\varepsilon^2)$$

So, the far field expansion to first order is

$$\Psi^0 = \exp[i\Theta^0(\mathbf{X},T,\varepsilon)]$$

Where the slowly varying phase satisfies,

$$\Delta\Theta^0 = 0$$

Core Expansion for Large r

$\psi = U(r) \exp[\, i(n\theta + \theta_0(T,\varepsilon))]$

For large r, we know that $U(r) = 1 + O(1/r^2)$

So,

$\psi = \exp[\, i(n\theta + \theta_0(T,\varepsilon))] + O(1/r^2)$

Because, the phase of the core expansion, for large r must match that of the outer expansion, we have the condition

$(n\theta + \theta_0(T,\varepsilon)) = \Theta^0(X,T,\varepsilon)$

Must hold as εr goes to 0 , for large r.

Simplified Far-Field Solution

Because Θ, satisfies Laplaces equation, the solution for the first order far field phase subject to the boundary conditions takes the form,

$$\Theta^0 = n\theta(R) + H(X,T,\varepsilon)$$

Where H is a harmonic function of X near X=Q.

Expanding the harmonic function,

$$\Theta^0 \sim n\theta(R) + q_0 + K\cdot R + O(R^2)$$

Then expanding the exponential function yields in the far-field expansion yields,

$$\Psi^{\,0} = \exp[\, i(n\theta + \theta_0)] \,[\, 1 + iK\cdot R + O(R^2)]$$

Where $q_0 = q_0(T,e) = H(Q,T,e)$ and $K = \nabla H(Q,T,e)$

Core Expansion

Substituting ,

$\psi \sim \psi(r,T,\varepsilon)$

$\mathbf{r} = \mathbf{x} - \mathbf{Q}(T,\varepsilon)/\varepsilon$

$\Delta\psi + (1 - |\psi|^2)\psi = - i\, \varepsilon^2 \psi_T$

$\Delta\psi + (1 - |\psi|^2)\psi = -i(\varepsilon^2\psi_T \text{-} \varepsilon Q' \cdot \Delta y)$

Substituting the expansion,

$\psi \sim \psi^0 + \varepsilon\, \psi^1$

Yields first and second order expansions,

$\Delta\psi^0 + (1 - |\psi^0|^2)\psi^0 = 0$

$L\psi^1 = iQ' \cdot \Delta\psi^0$

Where $Lu = \Delta u + (1 - 2|\psi^0|^2)u - (\psi^0)^2 u^*$

Matching the Core and Far-Field Solutions

Imposing the matching condition between the core and far field expansions,

$$\psi^0(\mathbf{r},T,\varepsilon) + \varepsilon\psi^1(\mathbf{r},T,\varepsilon) - \Psi^0(\varepsilon\mathbf{r},T,\varepsilon) = \varepsilon\psi^1(\mathbf{r},T,\varepsilon)$$

$$- i\varepsilon\mathbf{K}\cdot\mathbf{r}\ \exp[i((n\theta + \theta^0)] + O(\varepsilon^2 r^2 + 1/r^2)\ (3)$$

Or,

$$\psi^1(r,T,\varepsilon) - i\mathbf{K}\cdot\mathbf{r}\ \exp[i((n\theta + \theta^0)] = O(r)$$

Where $r = O(\varepsilon^p)$ $-1< p <-1/3$ is necessary to ensure that the error terms in (3) are smaller than the difference in listed in 3.

Equation ,

$$L\psi^1 = iQ' \cdot \nabla\psi^0$$

As well as the matching condition can be used to determine the velocity of the trajectories Q'.

Because L is self adjoint,

$$\int_D \mathrm{Re}(\ u\ (Lv)^* - v\ (Lu)^*)\ dx\ = \int_{\partial D} \mathrm{Re}(\ u\ (\partial_n v)^* - v\ (\partial_n u)^*)\ dx$$

If we let, $u = e \cdot \nabla\psi^0$ and $v = \psi^1$, where ψ^0 is the leading order solution, a vortex state, ψ^1 is determined by the matching condition, and D is a circular region with a radius for which the matching condition is valid.

After a considerable amount of effort, this simplifies to,

$Q' = 2\ K + O(1)$

Where $\mathbf{K} = \nabla H(\mathbf{Q},T,e)$

Velocity Equation for Dislocations in the NLSE

Q' = 2 **K** + O(1)

Where K =K = ∇H(Q,T,e) .

Where H was previously referred to in the phase equation for the near and far field phases,

$$\Theta^0 = n\theta(R) + H(X,T,\varepsilon) \qquad (1)$$

If the system has more that one vortex, a boundary condition as above must hold for each of the vortices, so the above condition generalized to the following:

$$\Theta^0 = \sum_{i=1}^{N} n_i\theta(X - Q_i) + C(T) \qquad (2)$$

Comparing (1) and (2), we expect H to have the form,

$$H = \sum_{j \neq i} n_j\theta(X - Q_j)$$

Where $\theta(R)$ from equation (1) is equal to $\theta(X - Q_i)$.

ODE Governing Dislocation Trajectories in the NLSE

Then it follows that, because where **X** = (x,y)

$$\nabla\theta(\mathbf{X}) = \nabla\tan^{-1}(y/x) = (-y/x^2 , 1/x) / (1 + y^2/x^2)$$

$$\nabla\theta(\mathbf{X}) = (-y , x)/ (x^2 + y^2) = (-y , x) /|\mathbf{X}|^2$$

K simplifies to,

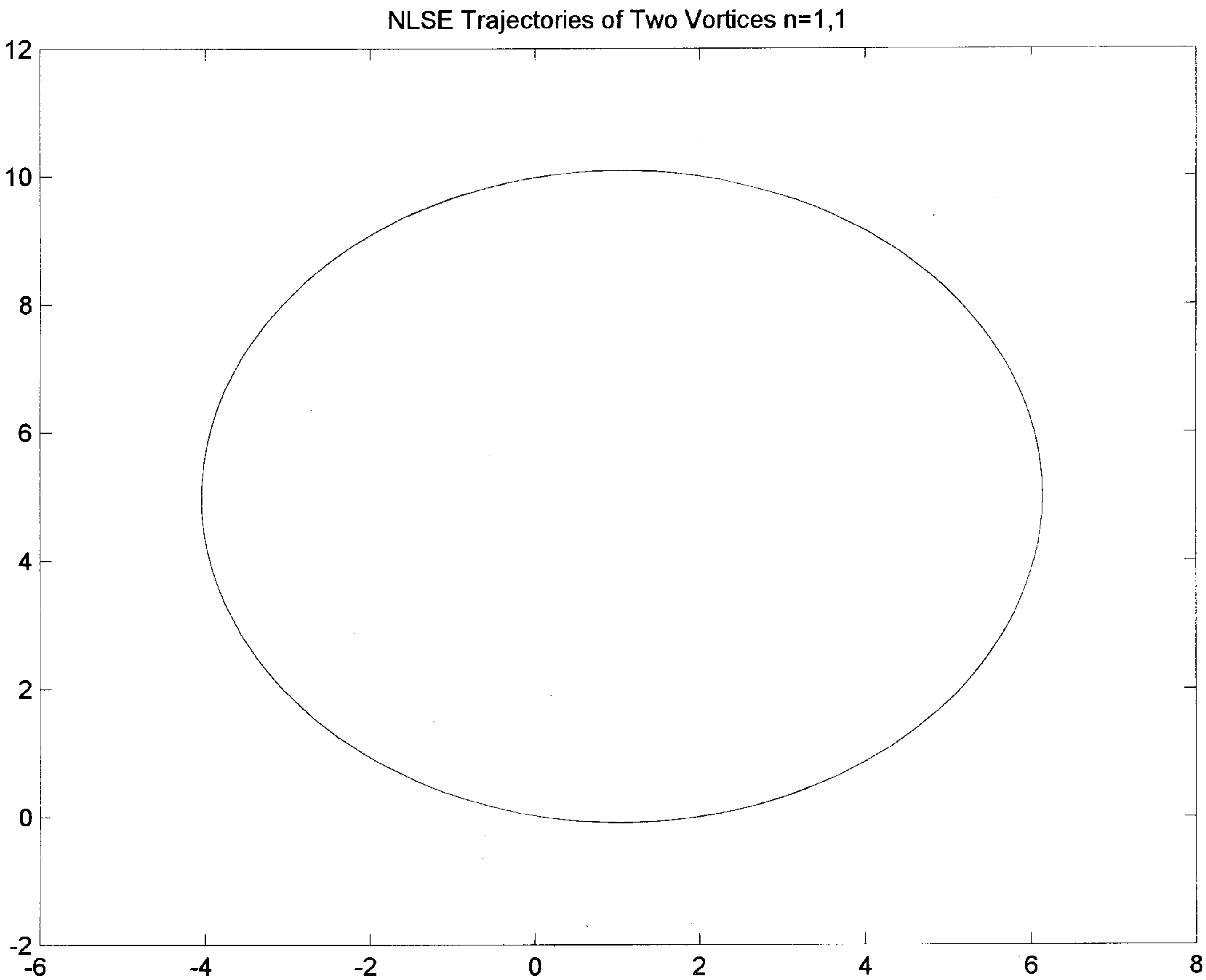
$$K = \nabla H(Q_i,T,\varepsilon) = \sum_{j\,\mathrm{R}i} n_j\, J(Q_i - Q_j)/|Q_i-Q_j|^2$$

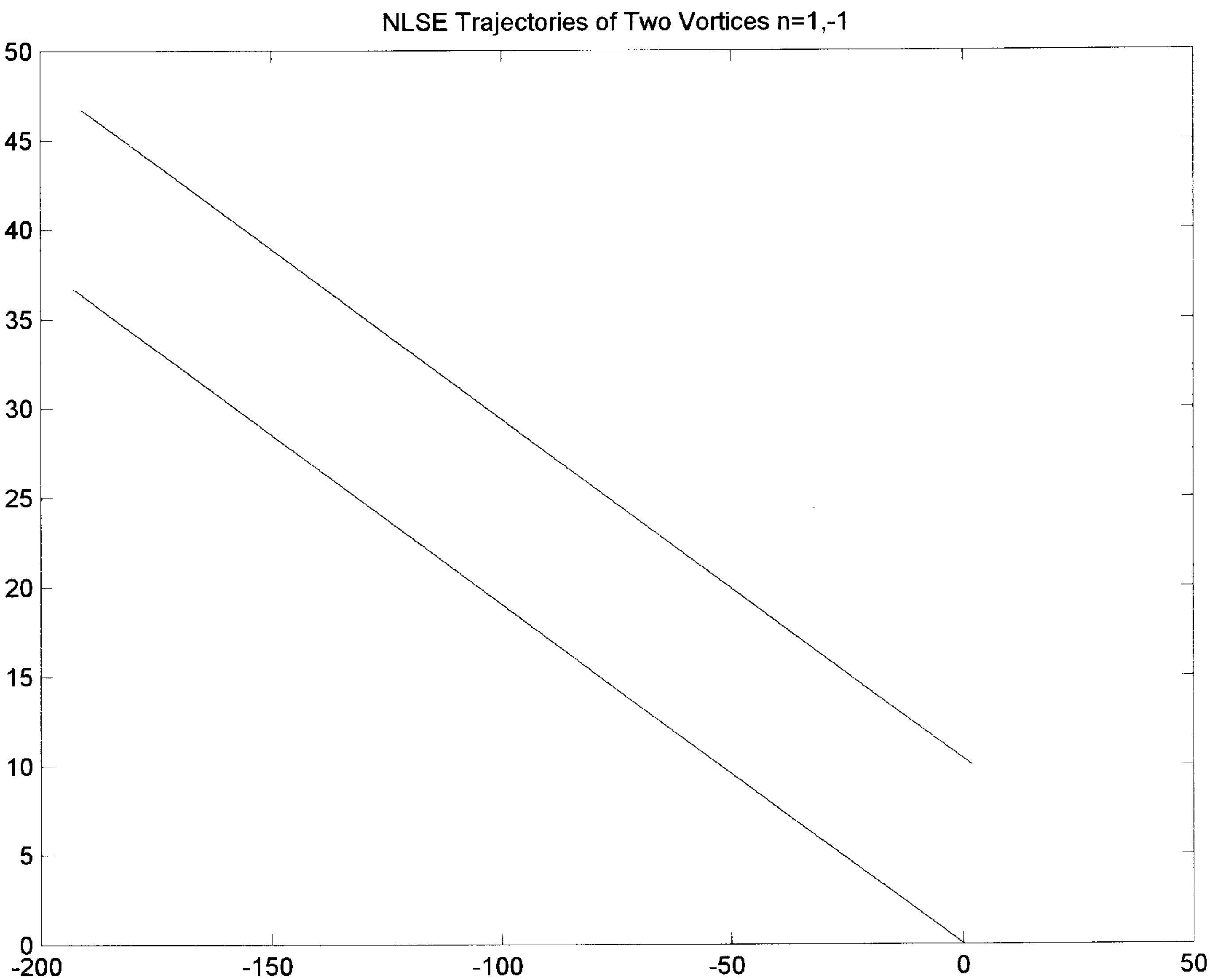
Where

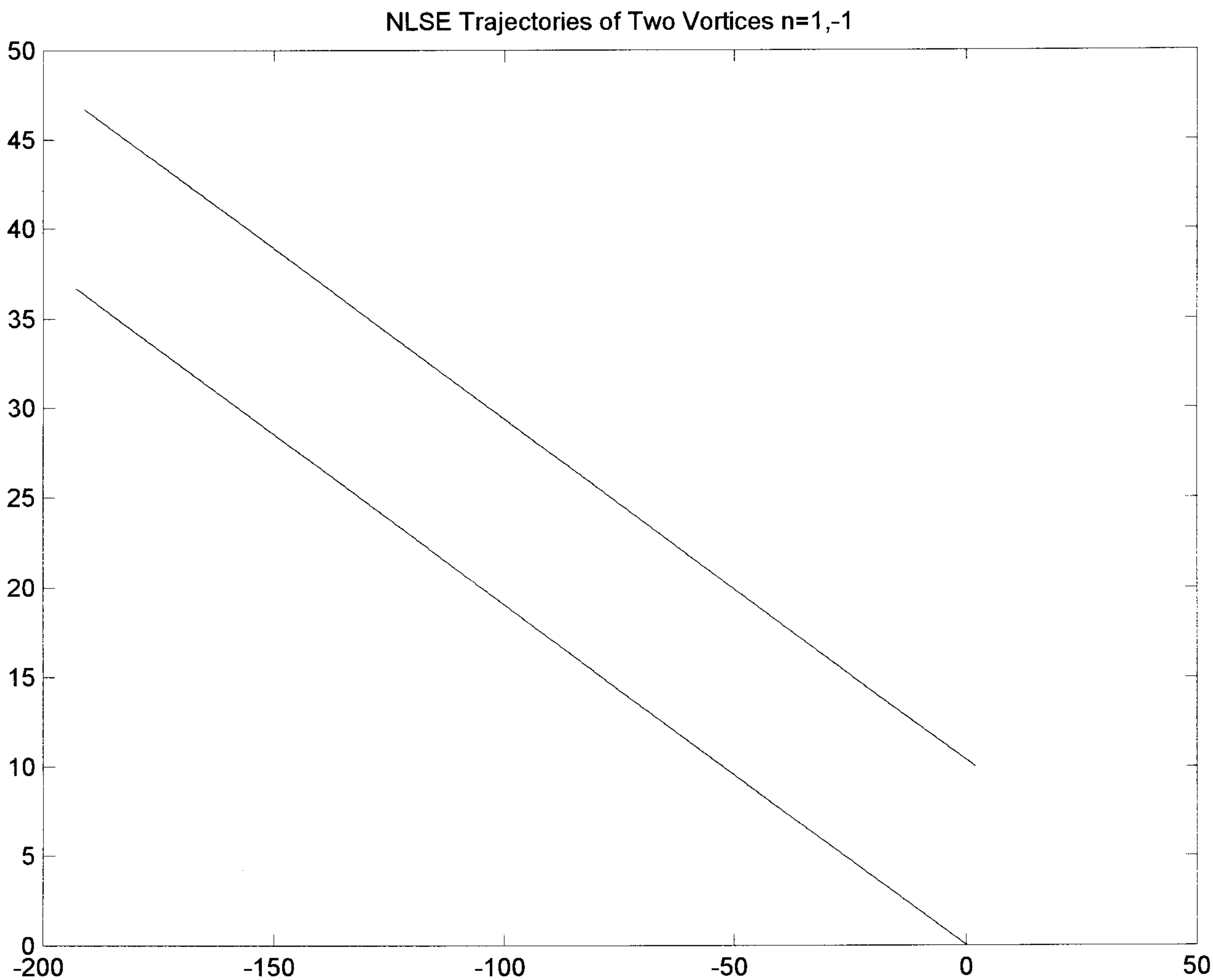
$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

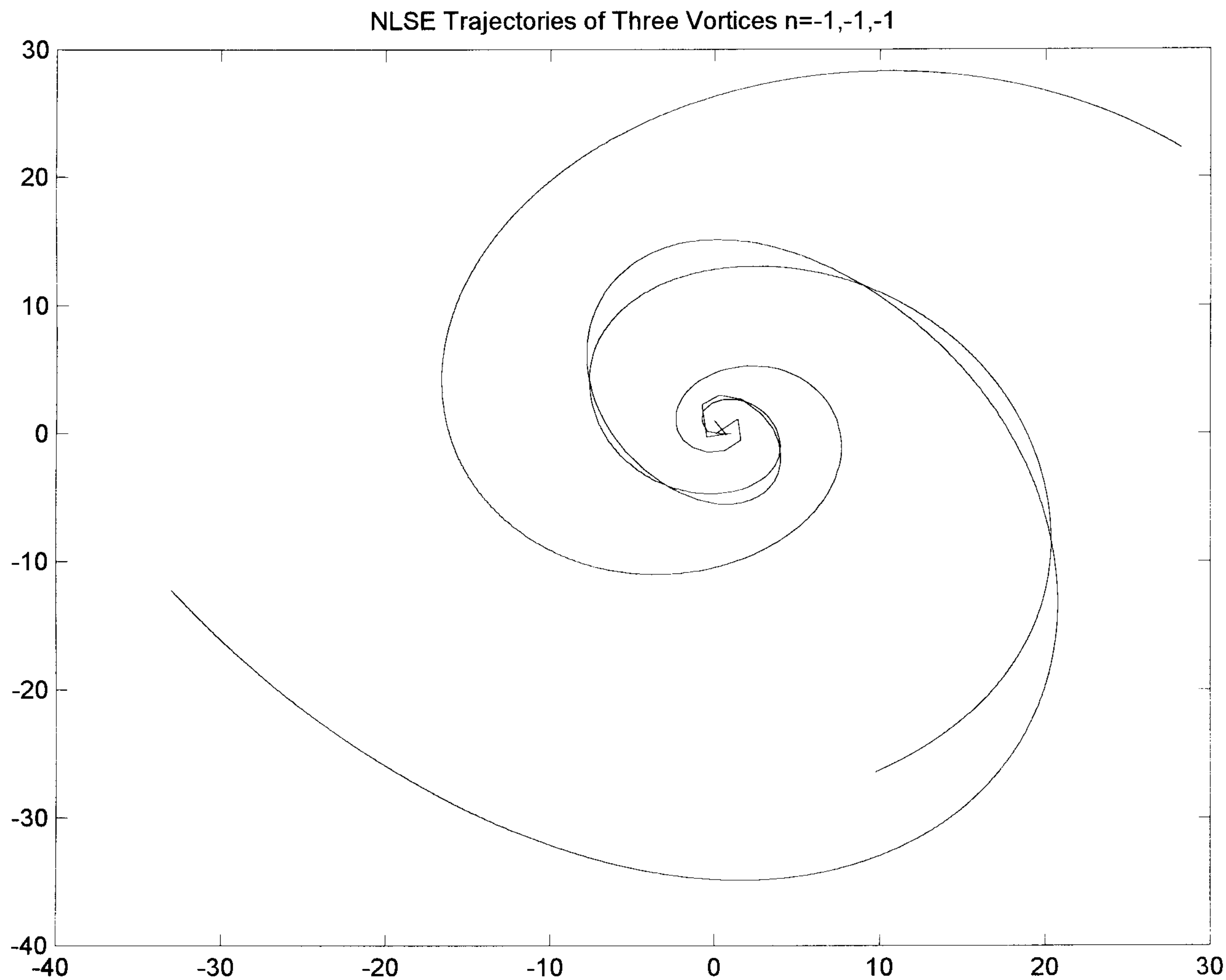
So, the differential equation governing the trajectories of the vortices becomes,

$$\mathbf{Q_i = 2 \sum_{j\,\mathrm{R}i} n_j\, J(Q_i - Q_j)/|Q_i-Q_j|^2}$$


NLSE Trajectories of Two Vortices n=1,1
12
10
8
6
4
2
0
-2
-6
-4
-2
0
2
4
6
8


NLSE Trajectories of Two Vortices n=1,-1
50
45
40
35
30
25
20
15
10
5
0
-200
-150
-100
-50
0
50


NLSE Trajectories of Two Vortices n=1,-1
50
45
40
35
30
25
20
15
10
5
0
-200
-150
-100
-50
0
50


NLSE Trajectories of Three Vortices n=-1,-1,-1
30
20
10
0
-10
-20
-30
-40
-40
-30
-20
-10
0
10
20
30

Conclusions:

1. The velocity of the defects in a system governed by the Ginzburg-Landau system, can be approximated for a system near band - center.

2. This velocity is perpendicular to the perturbation in the underlying wavevector.

3. As the GLE defects move, they perturb the underlying system such that it moves toward band center.

4. Assuming defects in the Nonlinear Schrodinger Equation, are far apart and move slowly, asymptotic methods can be used to derive ODE's governing their motion.

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