FRONT PATTERNS IN REACTION-DIFFUSION SYSTEMS

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Papers Read:

1) Hagberg and Meron, “Pattern Fomation in Non-Gradient Reaction-Diffusion Systems: The Effects of Front Bifurcation” Nonlinearity **7**, 1994, 805-835.

2)Haim, Li, Ouyang, McCormick, Swinney, Hagberg and Meron “Breathing Spots in a Reaction-Diffusion System” Physical Review Letters **77** July 1, 1996, 190-193.

PATTERNS IN REACTION-DIFFUSION SYSTEMS FALL INTO THREE

CATEGORIES: EXCITABLE, BISTABLE, AND HOPF-TURING.

FRONT BIFURCATIONS LEAD TO SYMMETRY BREAKING

IN BISTABLE SYSTEMS AND CAUSE EITHER PERSISTENT PATTERNS

OR TRAVELING PATTERNS. HOPF-TURING SYSTEMS FAR FROM ONSET

ALSO HAVE DYNAMICS SIMILAR TO BISTABLE SYSTEMS IN WHICH THE

TRAVELING PATTERNS, USUALLY REFERED TO AS BREATHING SPOTS, ARE

FORMED.

PRESENTED BELOW ARE THE ANALYSIS AND PATTERNS

OF BISTABLE AND FAR FROM ONSET HOPF-TURING SYSTEMS.

FITZHUGH - NAGUMO MODEL (FHN)

$$u_t = u - u^3 - v + u_{xx}$$

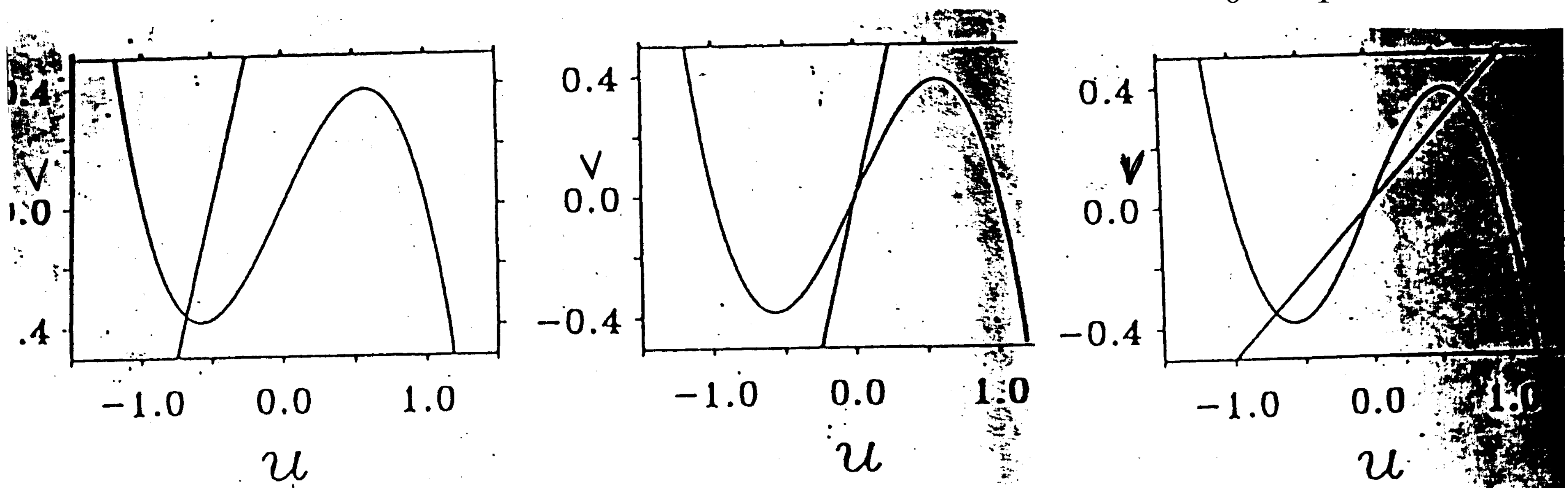
$$v_t = \varepsilon(u - a_1 v - a_0) + \delta\, v_{xx}$$

$\varepsilon = T_u / T_v$, the ratio of the time scales of each field.

$\delta = D_v / D_u$, the ratio of the diffusion constants of each field (typically $\delta = 0$)

THREE STATIONARY STATES OF FHN

NULLCLINES: $v = u - u^3$; $v = (u - a_0)/a_1$



(i) Excitable
nullclines intersect once outside wells

(ii) Hopf - Turing
nullclines intersect once inside wells

(iii) Bistable
nullclines intersect three times

FRONT BIFURCATION - BISTABLE CASE
SEARCH FOR CRITICAL $\varepsilon = \varepsilon_c$.

1) $\varepsilon = 0$: Variational Case:

Propagation speed: $c = 1/(\int_{-\infty}^{\infty} u'(v)^2)\,(F(u_-) - F(u_+))$

where $\Phi(u,v) = -u^2/2 + u^4/4 + vu$

Propagation of Front:

i) $v = 0, c = 0 \Rightarrow$ stationary

ii) $v < 0, c > 0 \Rightarrow$ front travels right

iii) $v > 0, c < 0 \Rightarrow$ front travels left

2) $\varepsilon >> 1$

   i) Two stable states
   ii) One front connecting these states
   iii) Front propagation direction is same as variational case.

3) $0 < \varepsilon << 1$:

   i) Two stable fronts coexist
   ii) One unstable front

GOAL: Determine ε where the solutions go from one front to two fronts.

Stationary Solution of FHN (a_0 , $\delta = 0$)

$$u_s(x) = -u_+ \tanh(\eta x) \quad \text{and} \quad v_s = u_s(x)/a_1$$

where $u_+ = (1 - 1/a_1)^{1/2}$, $\eta = (u_+)/(2)^{1/2}$

Propagating solution:

Let $\chi = x - ct$ and assume an expansion in terms of c.

$$u_p = u_s + cu_1 + c^2 u_2 + \ldots$$

$$v_p = v_s + cv_1 + c^2 v_2 + \ldots$$

$$\varepsilon = \varepsilon_c + c\varepsilon_1 + c^2 \varepsilon_2$$

$$L = d^2/d^2x + 1 + 1/a_1 - 3u_s^2$$

O(c):

$$L\, u_1 = (1/(\varepsilon_c a_1^2) - 1)u_s'$$

$$v1 = u_1/a_1 + u_s'/(\varepsilon_c a_1^2)$$

Because $L\, u_s' = 0$ and $L^\dagger\, u_s' = 0$ leads to solvability:

$$\varepsilon_c = 1/a_1^2$$

$$u_1 = 0 \qquad\qquad v_1 = u_s'$$

$O(c^2)$:

$$\mathcal{L}\, u_2 = -\varepsilon_1 a_1^2 u_s' + a_1 u_s''$$

$$v_2 = u_2/a_1 + a_1 u_s''$$

To satisfy solvability: $\varepsilon_1 = 0$

$u_2 = ½\, a_1 \chi u_s'$ $\qquad$ $v2 = ½\, a_1 u_s' + a_1 u_s''$

$O(c^3)$:

$$L\, u_3 = a_1^2 u_s''' - \varepsilon_2 a_1^2 u_s'$$

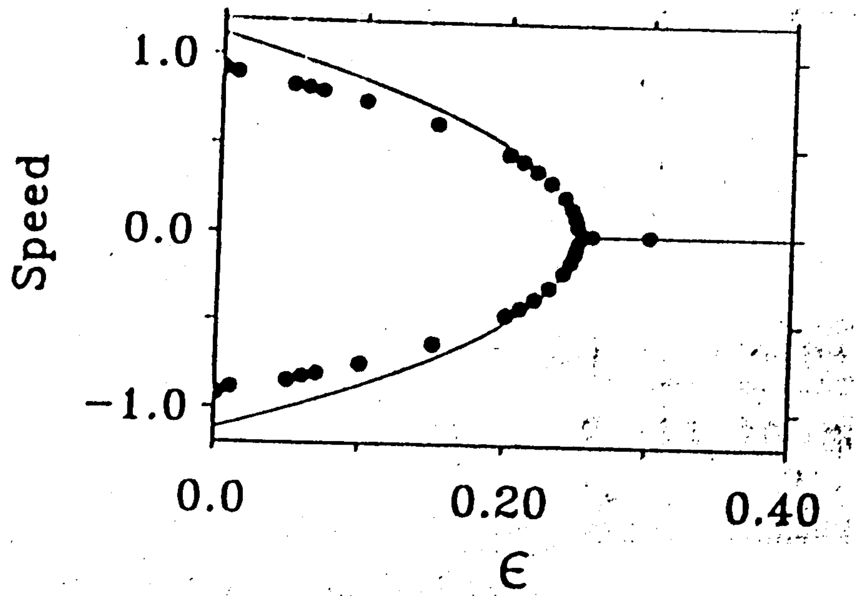
Solvability leads to : $\varepsilon_2 = -5/2(1 - 1/a_1)$

$$\text{So, } u_p(x) = u_s + \tfrac{1}{2} c^2 a_1 \chi u_s'$$

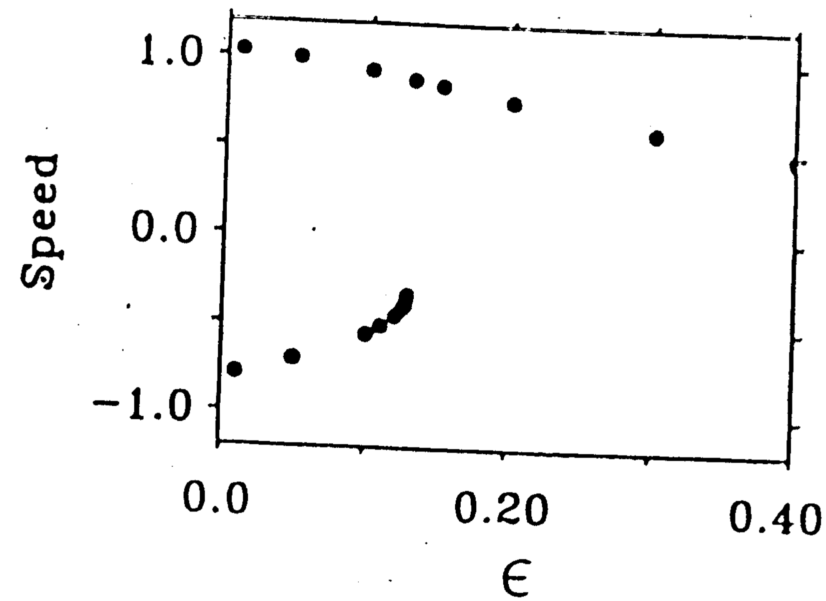
$$v_p(x) = u_s/a_1 + c u_s' + c^2(\tfrac{1}{2} \chi\, u_s' + a_1 u_s'')$$
$$\varepsilon = 1/a_1^2 - \tfrac{1}{2}(1 - 1/a_1)c^2$$

Bifurcation Diagram

$a_0 = 0$



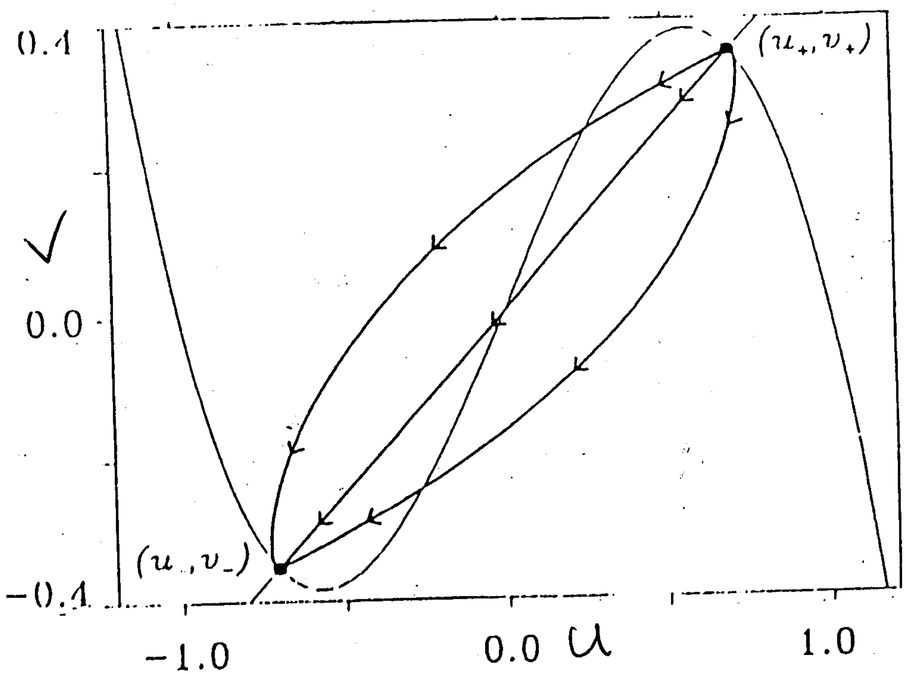
$a_0 \neq 0$

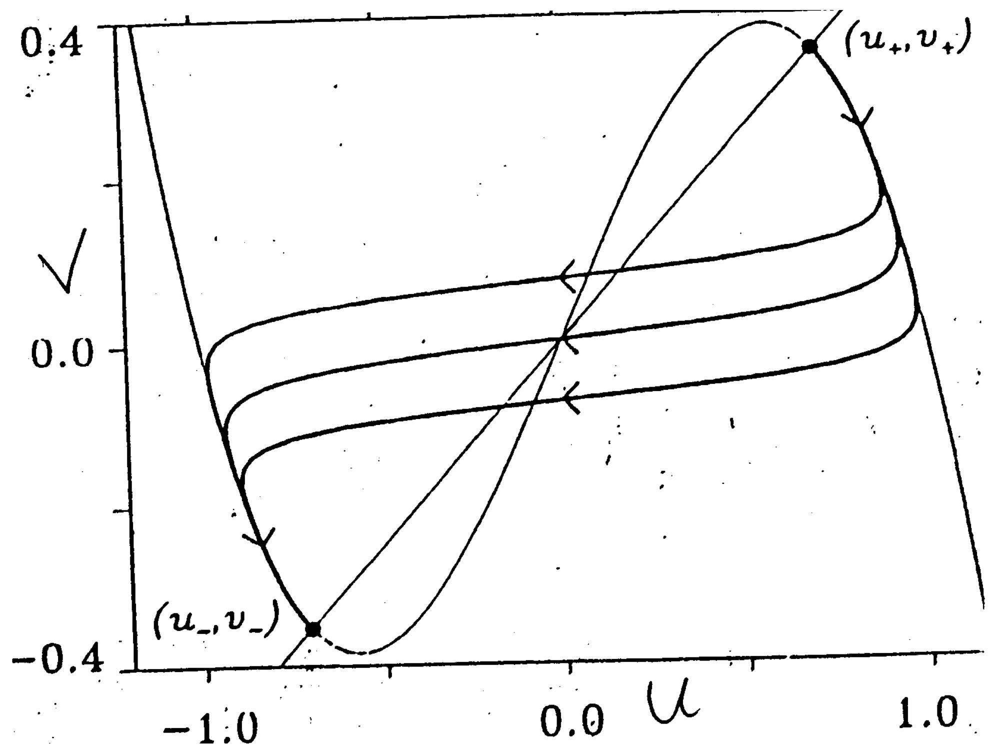


Pitchfork $\Rightarrow$ Saddle Node

Phase Portrait

$\delta = 0$ $\delta \neq 0$ and $\varepsilon/\delta << 1$





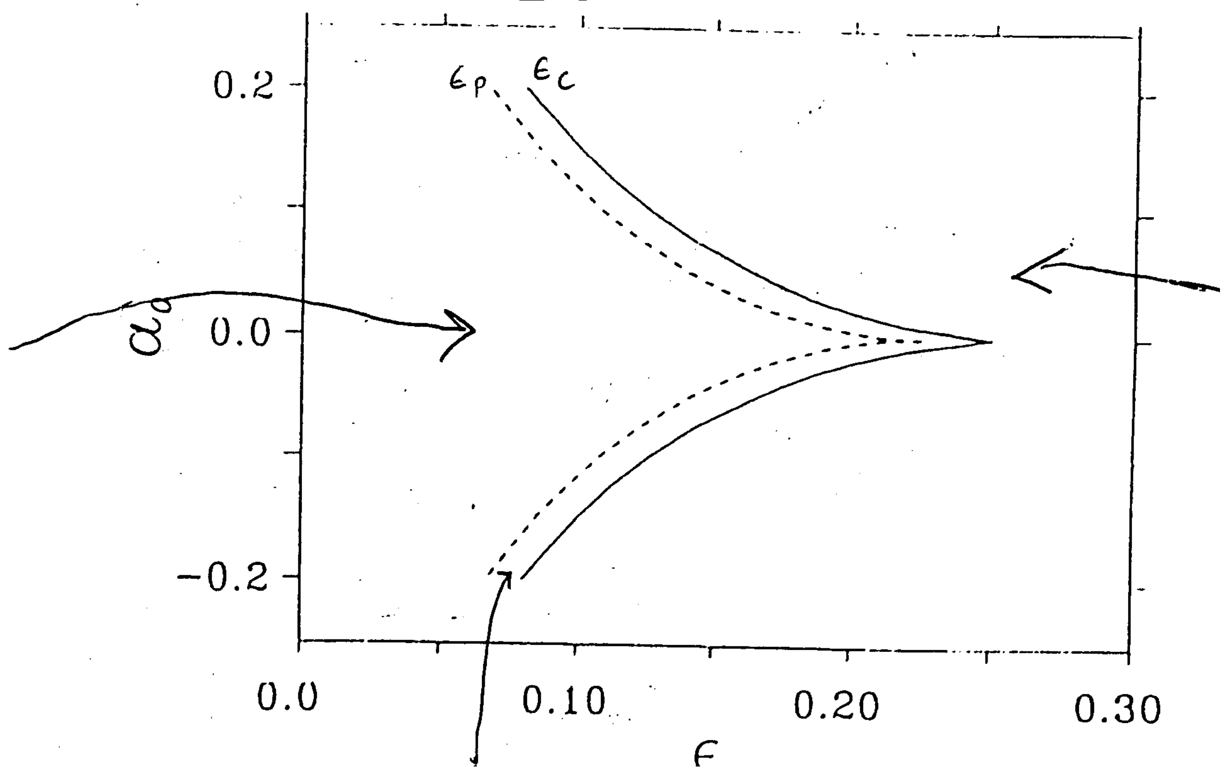
Trajectory through (0,0) shows stationary front

Other two trajectories break the odd symmetry and correspond to traveling fronts.

Dependency of ε on a_0:

Diagram of the transition from persistent patterns to traveling patterns

Stable Traveling Front



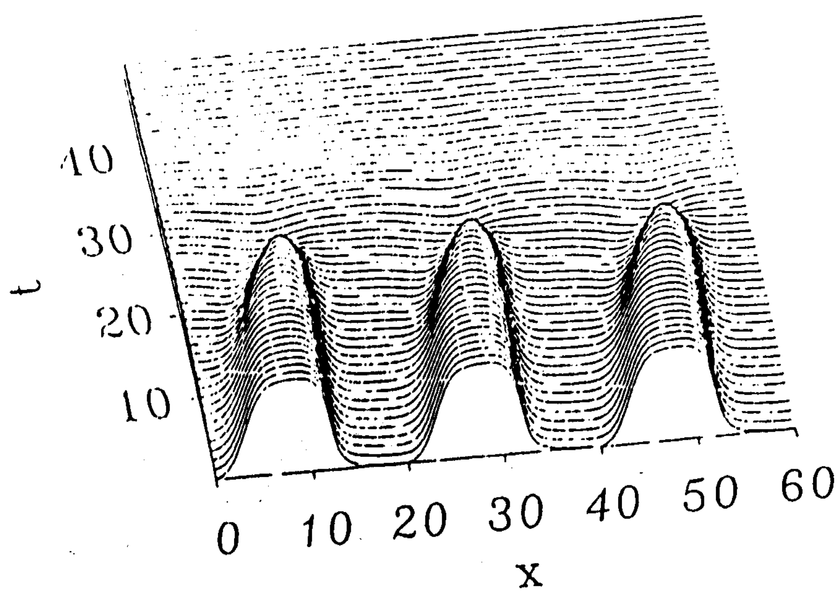
One Stable Front

3 Fronts, 2 stable

Front Patterns for Different ε Regimes, $\delta = 0$
Same Initial Pattern

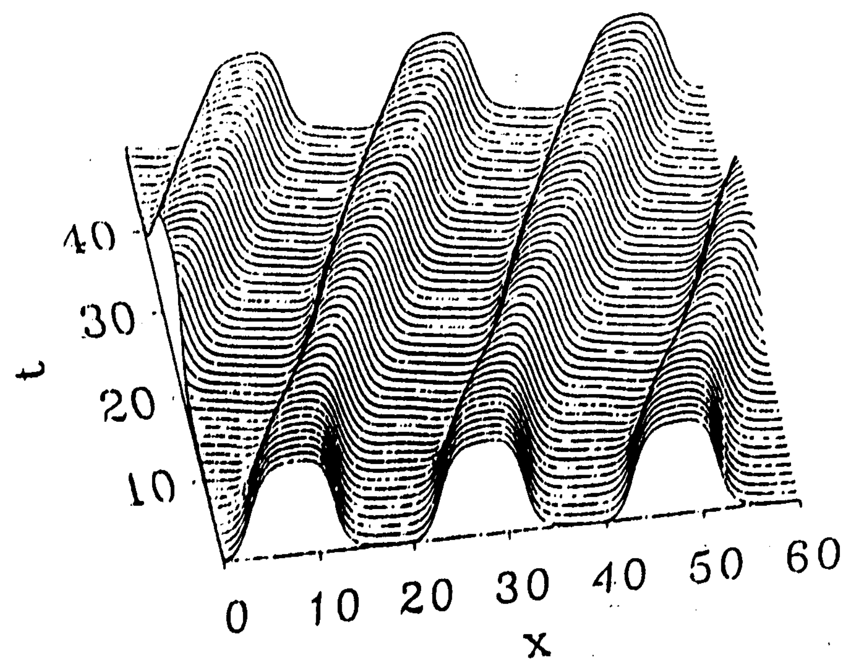
$\varepsilon > \varepsilon_c$

minimizes free energy functional



$\varepsilon < \varepsilon_p$

Traveling Solution



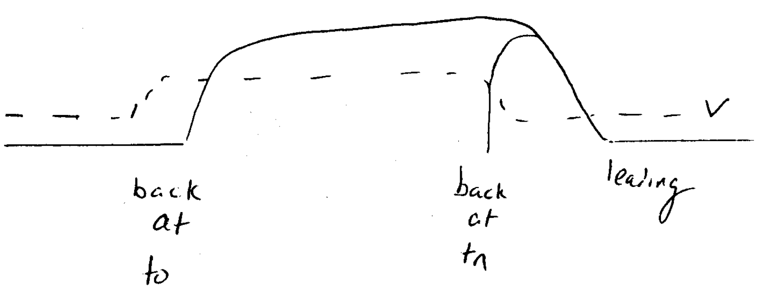
Two Traveling Fronts in a Domain

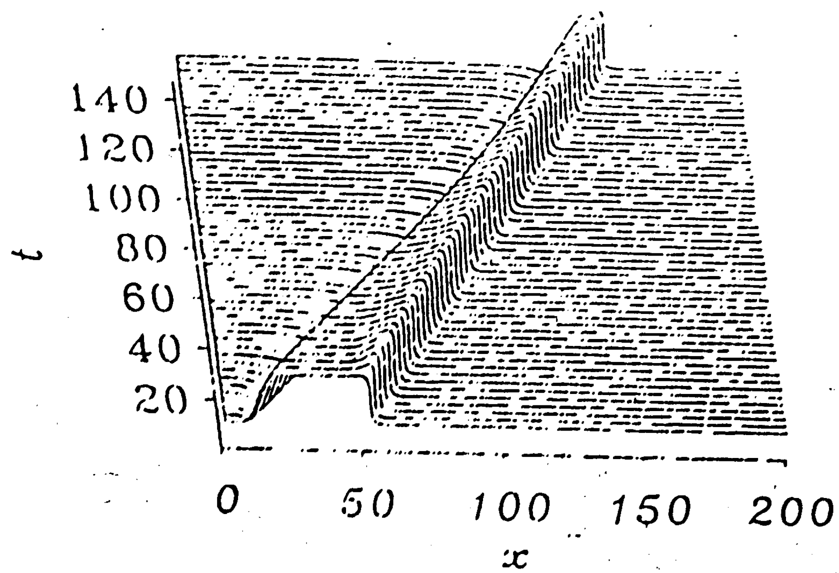
1) Speed of Back Front $\geq$ Speed of Leading Front:

Fronts Merge

Distance between fronts is :

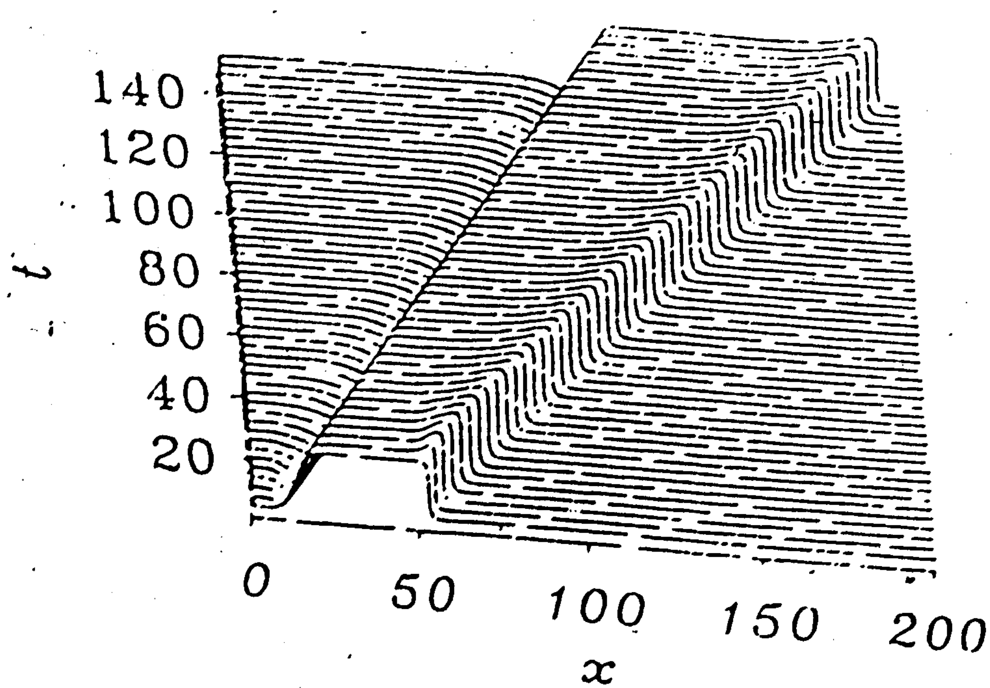
$$\lambda = (1/\varepsilon k)\ \ln((v_+ - v_-)/(v_+ - v_b^0)),\quad \text{where } k = (a_1 + ½)/c$$





2) Speed of Back Front ≤ Speed of Leading Front

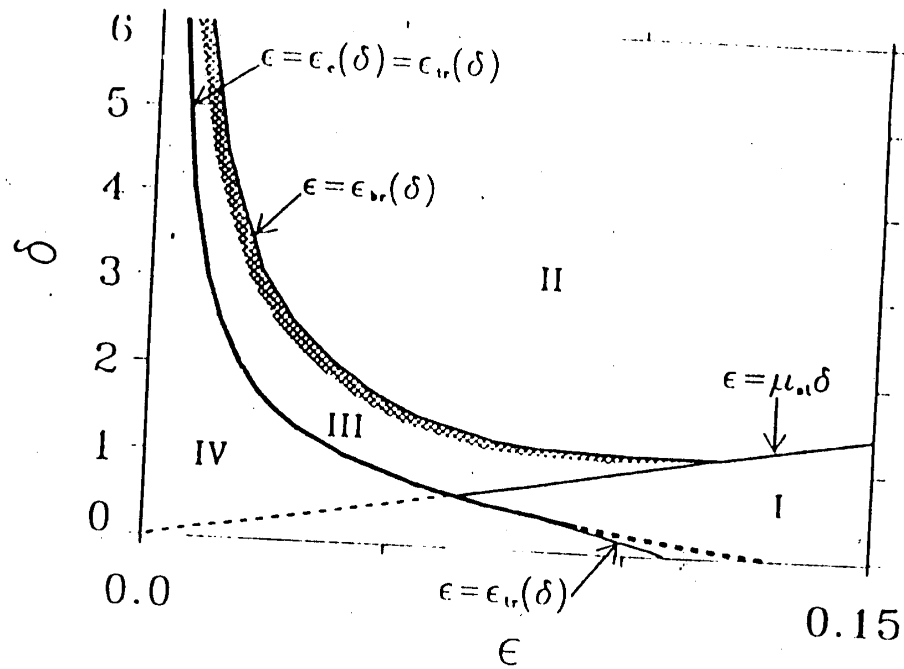
Front Expands



Front Bifurcation Regions in $\varepsilon - \delta$ Space

So far $0 \leq \delta \ll 1$ have been considered. Now examine what happens for $\delta > 1$.

Note $\varepsilon_c(\delta) = 9/\{(2a_1+1)\delta\}$ $\quad \varepsilon/\delta \ll 1$



Region I $\Rightarrow$ Transient single solution domain

Region II $\Rightarrow$ Stable stationary domains

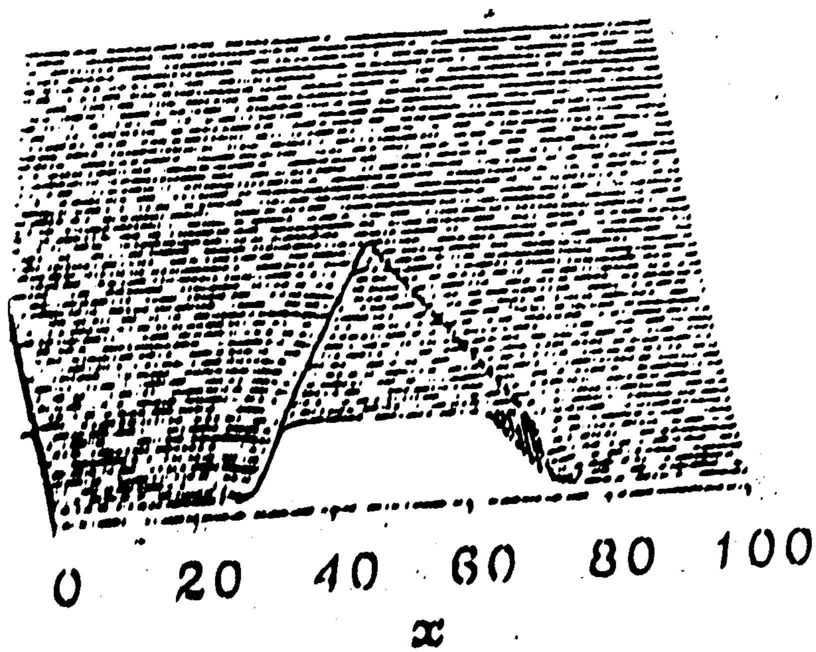
Region III shaded $\Rightarrow$ Steady breathing spots

Region III unshaded $\Rightarrow$ Dying breathing spots

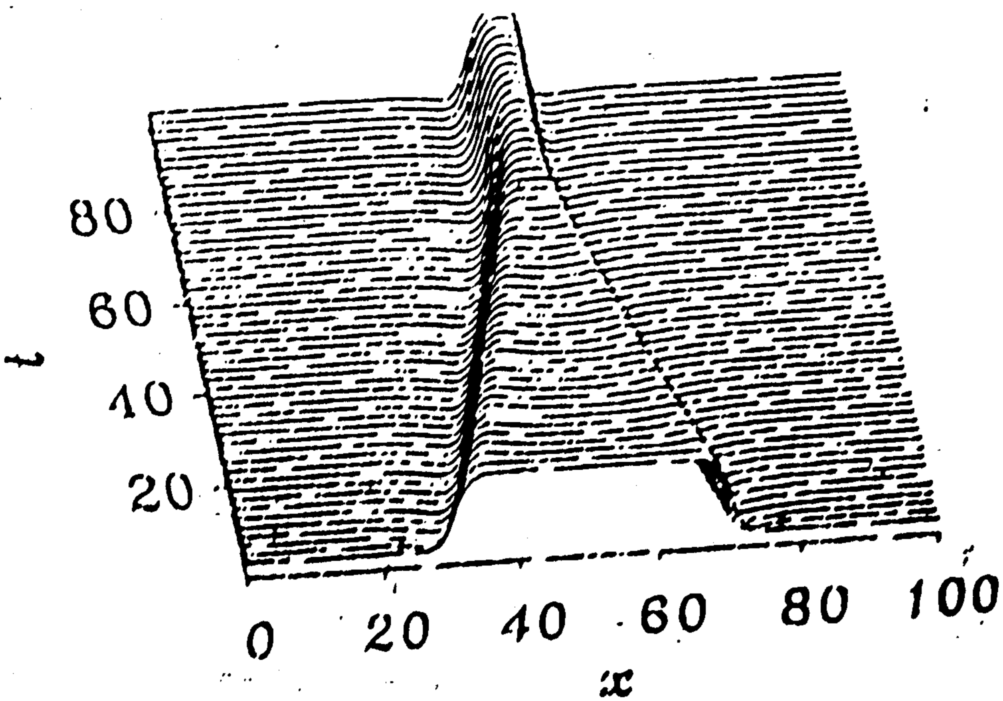
Region IV $\Rightarrow$ Traveling domains

Front Propagation Across Stationary Boundary
δ >> ε and define $\mu = \varepsilon/\delta$

Solution in Region I:
Front Dies



Solution in Region II:



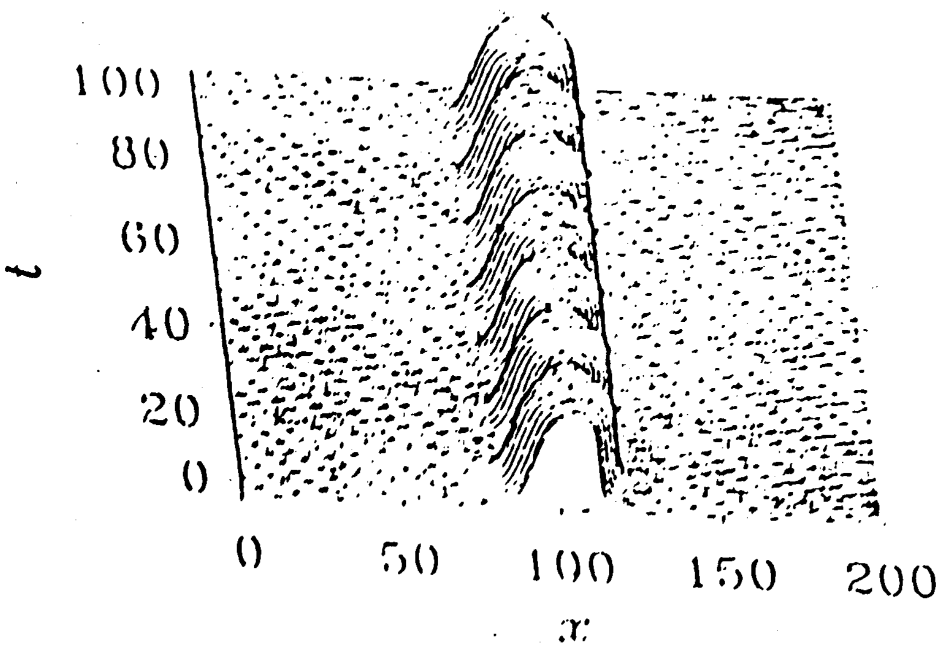
Solutions in Region III: Breathing Spots

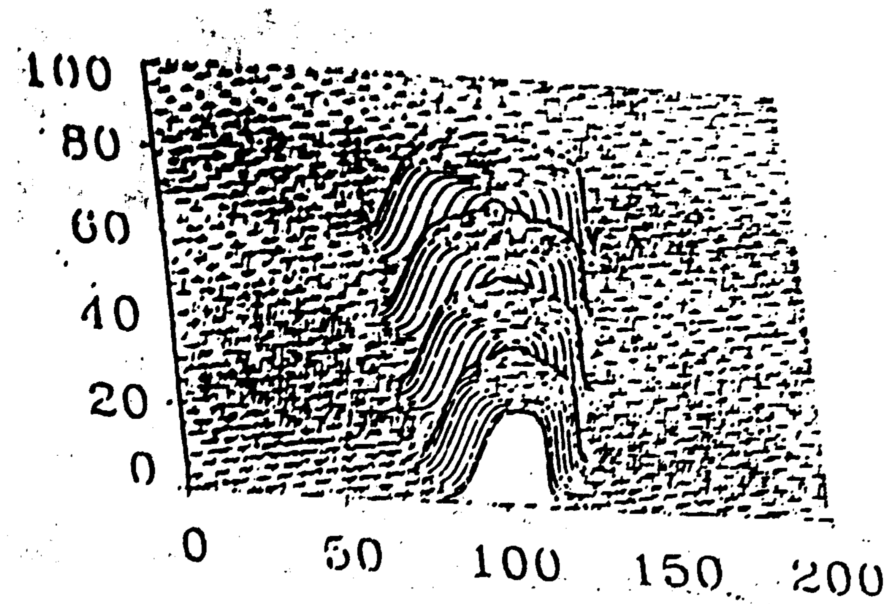
1) Across $\varepsilon_{br}(\delta)$,stationary solutions lose stability for Hopf-bifurcations

2) $a_0 > a_{0c} = -1+2(a_1 + ½)^{3/2} (2\varepsilon\delta)^{1/2}/3$
for breathing spots to occur

Region III shaded

Region III unshaded

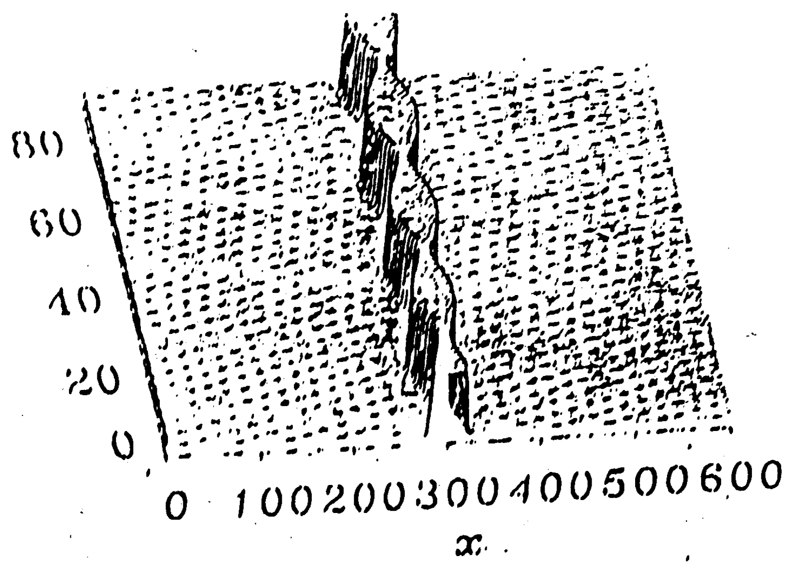


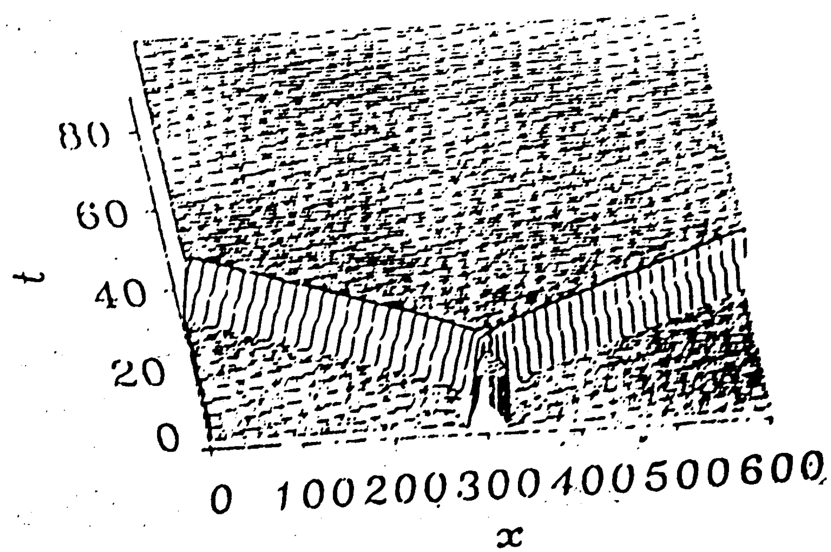


Two Fronts Moving Toward Each Other

1) $\varepsilon > \varepsilon_c$ Fronts Merge because only one stable solution exists for each front in this regime

2) $\varepsilon < \varepsilon_c$ Two stable states exist in this regime. Solution follows one state until fronts collide, then follow other state leading to the reflection.





Two Fronts Traveling in the Same Direction

1) $\varepsilon > \varepsilon_c$

(i) Right hand front changes its propagation direction

(ii) oscillations

2) $\varepsilon < \varepsilon_c$

Traveling Domain