The Role of Localized Objects in the Benjamin-Feir Stable Intermittent Regime of the Complex Ginzburg-Landau Equation

- **What is Spatio-Temporal Intermittency?**

  - A state in which stable plane-wave solutions of the CGL equation coexist with localized objects: holes, shocks...

- **Why does Intermittency arise?**

  - Why not? The existence of stable plane waves does not rule out amplitude defects or other NL solutions of the CGL equation

The Complex Ginzburg-Landau Equation has 2 Free Parameters, α and β

$$\partial_t A = A + (1 + i\alpha)\partial_x^2 A - (1 + i\beta)|A|^2 A \qquad (1)$$

- For $\alpha\beta >$ -1, there exists a band of wavenumbers for which plane waves are linearly stable. This is our regime of interest. (see phase diagram)

- Both numerical simulations of equation (1) as well as experiments in which a continuous bifurcation to travelling waves is observed show intermittent states in regimes where plane waves are stable. How can we classify the constituent objects of intermittency?

Holes: Bekki-Nozaki Holes and Homoclons

- We begin our classification of intermittency by noting that the stable plane waves are often separated by localized *depressions* of the amplitude. We call these "holes". We can say more. The holes asymptotically connect two plane waves of wavenumbers q_r and q_l, the wavenumbers of the right and left "wings" of the hole, *which may or may not be equal*. It turns out that this is an important distinction and, in fact, distinguishes between the two heretofore oberved types of holes. (see figures 1,2)

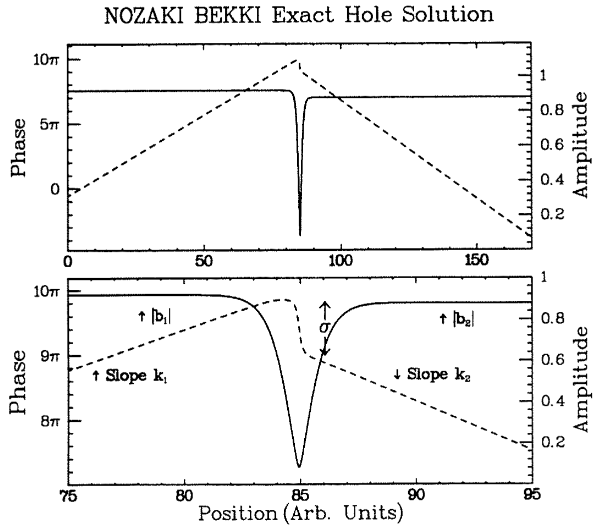

Figure 1: Profile of a Bekki-Nozaki hole. Amplitude is given by the solid line, phase by the dotted line. The lower figure is a close-up of the upper. Note how the hole connects two waves of different wave numbers (here k_l and k_r). Also note the phase "jump" σ which is continuous.

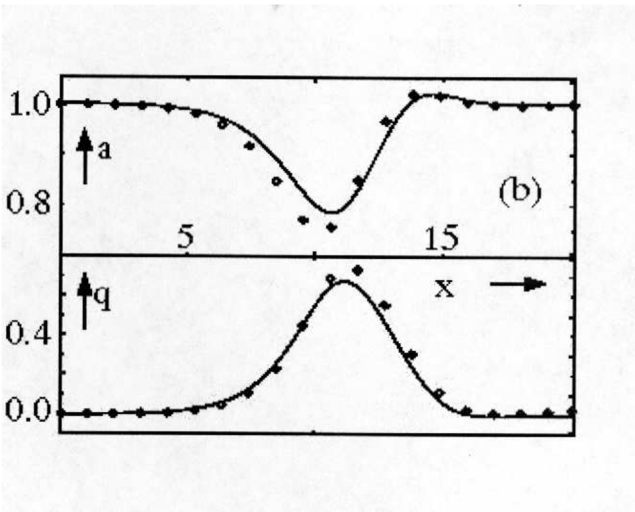

Figure 2: Profile of a Homoclon. Amplitude is given in the upper figure, wavenumber in the lower. Note how the Homoclon connects two waves of the *same* wavenumber. Here the peak in wavenumber corresponds to a jump in the phase.

Bekki-Nozaki Holes

- In 1984 Kazuhiro Nozaki and Naoaki Bekki discovered three new *exact* solutions to the CGL. [1](see central panel) One of these was a propagating "hole", asymptotically connecting two plane waves of different wavenumbers. In their words, "...both asymptotic periodic patterns must be stable to modulational perturbations...for the stability of the hole."[2] Thus we would expect to see the holes in exactly the regime of interest if at all.
- The hole solutions can be completely parameterized by the coefficients of the CGL and the wave numbers of the connecting plane waves. So we could write,

  $$\mathrm{H(x)} = \mathrm{H(x;}\ \alpha, \beta, q_r, ql).$$

  Thus their stability must also depend on these parameters. We ask the questions: How do Bekki-Nozaki holes arise? Are they stable? And, of course, what role, if any, do they play in spatio-temporal intermittency?

[1] Bekki, Nozaki "Exact Solutions of the Generalized Ginzburg-Landau Equation." Jour. Phys. Soc. Jap. (May 1984)
[2] Formations of Spatial Patterns and Holes in the Generalized Ginzburg-Landau Equation. Physics Letters (July 1985)

Creation of Bekki-Nozaki Holes
(A phase instability)

The plane wave solutions of the CGL are local attractors, and for infinitesimal perturbations of plane waves with wavenumbers in the Eckhaus *stable* band we expect an exponential relaxation. Thus the creation of holes or other localized objects requires a finite perturbation.[3] In particular, we can imagine a finite localized depression of the amplitude. What would happen? For a strong enough depression the local wavenumber will become Eckhaus *unstable*, causing a sharp drop in amplitude. This mechanism has been invoked by theorists and experimentalists.[4,5]

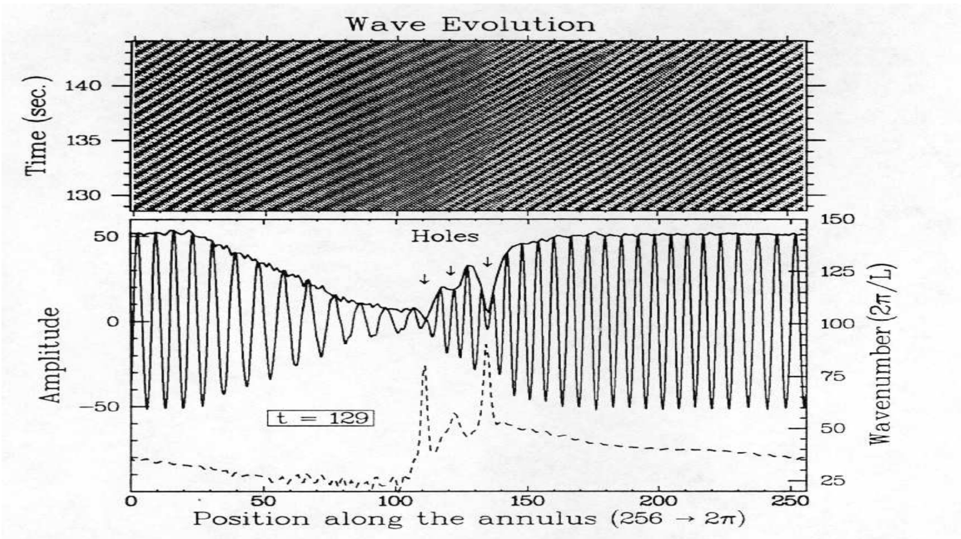

Figure 3: Experimental evidence of Bekki-Nozaki Holes. The experiment is Rayleigh-Bénard convection in an annulus. Note the large amplitude depression which generated the defects by making the local wavenumber Eckhaus unstable.

[3] This is a necessary, but not sufficient condition. As noted, the stability of the holes depends on the values of the 4 parameters, and given q_r, q_l, there is a range of values in α and β for which the holes are unstable. It turns out they are unstable in the regime of interest but exist for a finite time.

[4] L. Gil "Space and time intermittency behaviour of a one-dimensional complex Ginzburg-Landau equation." Nonlinearity (1991)

[5] Janiaud, Jucquois, Lega, Croquette "Experimental Evidence of Bekki-Nozaki Holes"

Stability of Bekki-Nozaki Holes

- The Bekki-Nozaki hole is an exact solution of the CGL and it is possible to linearize about this solution and study its stability.[6] The holes possess two distinct spectra of eigenvalues: a continuous one corresponding to the asymptotic plane wave behavior of the holes, and a discrete one giving the stability of the "core" or region of reduced amplitude. The holes are linearly "core" unstable in the region of intermittency (see figure 4).

[6] Chaté and Manneville "Stability of the Bekki-Nozaki hole solutions to the one-dimensional complex Ginzburg-Landau equation" Physics Letters A (1992)

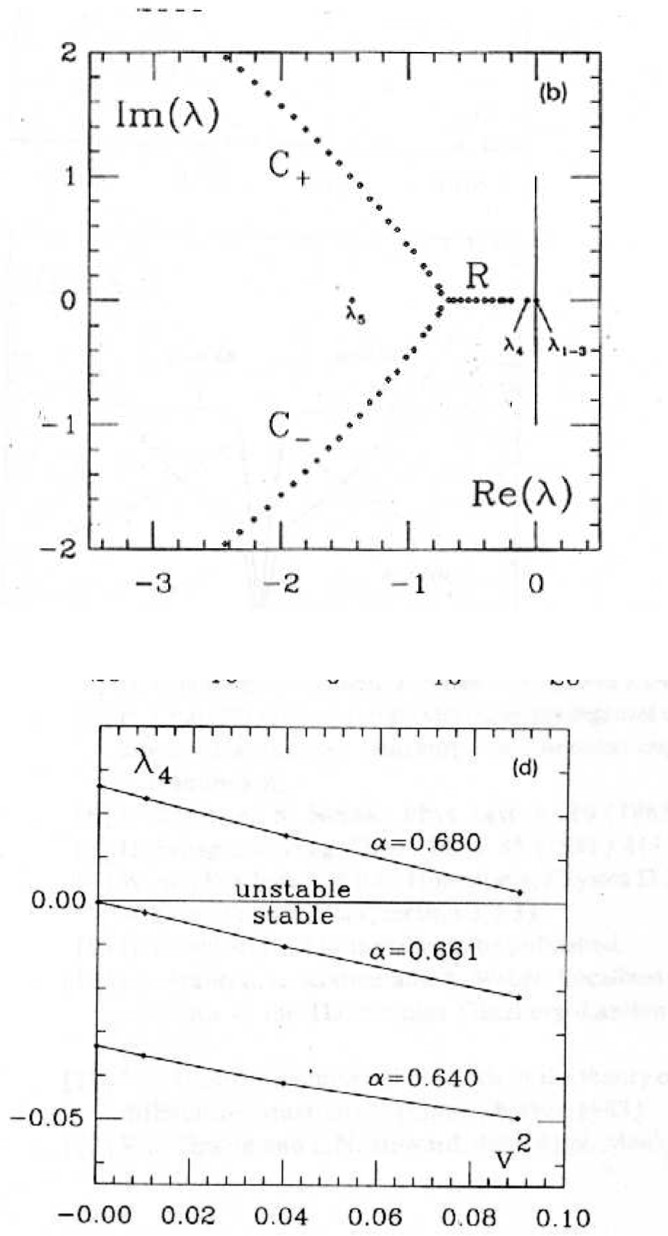

Figure 4: Upper: Spectrum of linearized operator for Bekki-Nozaki hole solution. There is a continuous spectrum (the "V") corresponding to the stability of the asymptotic "wings" as well as a number of discrete eigenvalues which reflect the stability of the core. λ_4 goes unstable first. Lower: Eigenvalue $*\lambda_4$ vs. square of velocity of hole for various values of α

More on Stability of BN Holes

Bekki-Nozaki holes *are* unstable, but nevertheless have been observed in experiment as mentioned. How can we be sure that what we see are really Bekki-Nozaki holes? A convincing experiment was conducted in order to derive the coefficients α and β of the CGL from properties of the observed holes.[7] Compare the derived, theoretical fit to the actual data in the figure below.

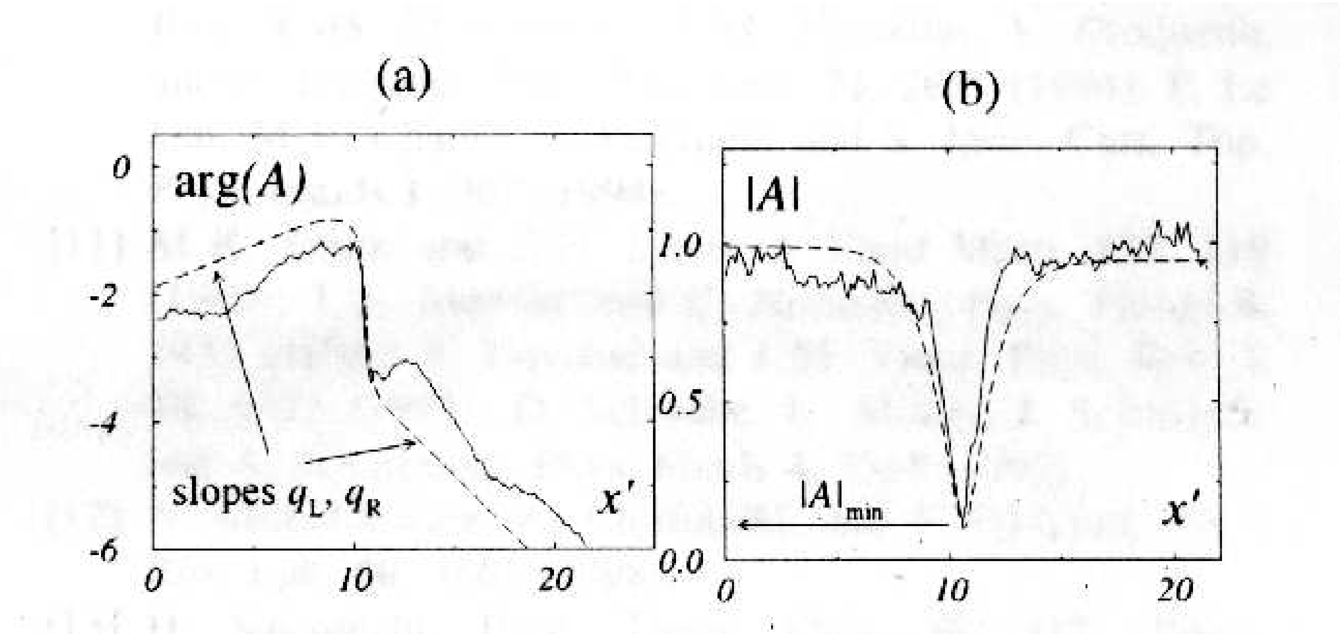

Figure 5: (a) Local wavenumber at hole. Dashed lines are the exact solution fits. (b) Amplitude. Experiment was laterally heated Rayleigh-Bénard convection

[7]Burguete, Chaté, Daviaud, Mukolobwiez. "Bekki-Nozaki Amplitude Holes in Hydrothermal Nonlinear Waves" Phys. Rev. Let. (1999)

More Stability

(Heteroclinic Orbits, Unstable Manifolds and Stable Hole-Shock Solutions)

- If we look at the space-time trajectory of hole solutions in a numerical simulation of (1), we see that they travel at a more or less constant speed (figure 6). Inserting such an ansatz, A(x,t) = $e^{-i\omega t}\hat{\mathrm{A}}$(x-vt) reduces (1) to three coupled ODE's.(see central panel) The fixed points of this system are just the stable plane wave solutions. The Bekki-Nozaki holes connect two such solutions asymptotically and therefore constitute a trajectory in phase space connecting two fixed points. They describe a heteroclinic orbit.

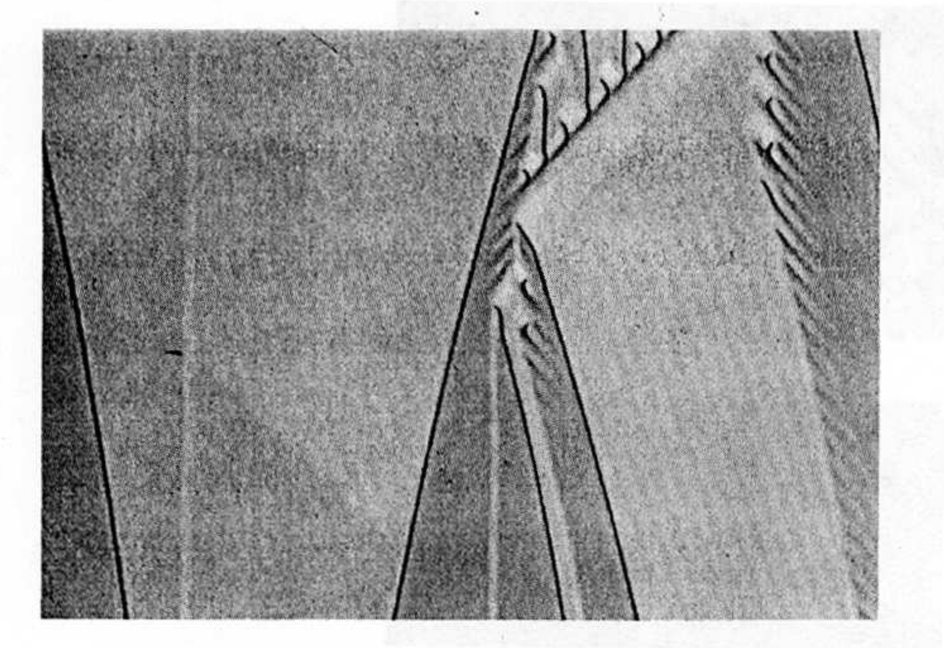

Figure 6: Space-time plot of amplitude, where lighter means larger amplitude. α = -0.25, β = -1.75 Note the approximately linear dark lines are holes moving at an approximately constant velocity. Chaté, Nonlinearity 7 (1994) 185-204

- Still, we have seen that this heteroclinic orbit is linearly unstable. The apparent contradiction can be explained by considering the heteroclinic orbit as an unstable fixed point in the CGL phase space. Waves are locally attracted to the fixed point and converge quickly along its stable manifold. They then escape slowly along the one-dimensional unstable manifold; slow enough so that the hole exists for finite time.

- One interesting case seen in experiment and numerical simulation is the Hole-Shock solution: a Bekki-Nozaki hole and a shock, which coexist and together constitute a stable object (figure 7). We know that

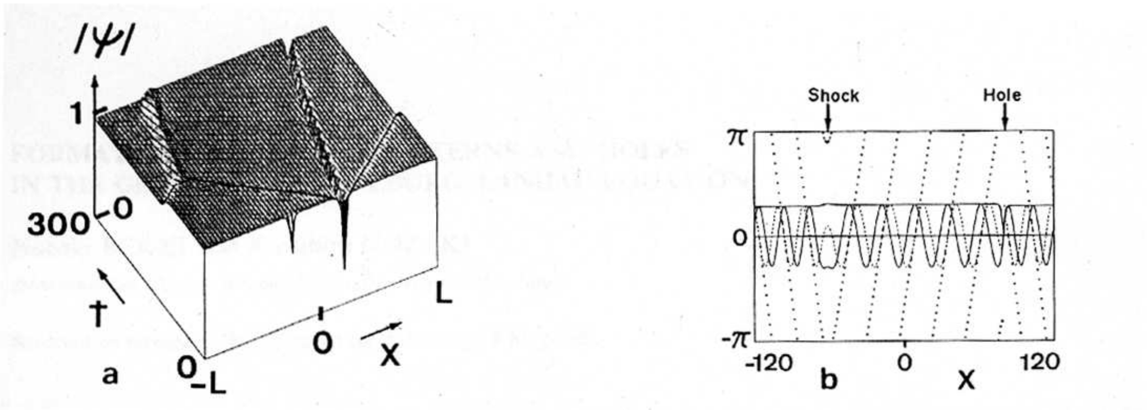

Figure 7: Left: space-time evolution of a Bekki-Nozaki hole. It emmits two shocks which collide to form a single shock. The hole-shock pair then propagates with the same velocity. Right: Profile of the stable hole-shock pair. Nozaki, Bekki, Jour. Phys. Soc. Japan Vol. 53 No.5 (1984)

amplitude modulations governed by the CGL can also be described by a phase-diffusion equation which, given periodic boundary conditions, means that total phase is conserved (for Eckhaus-stable wavenumbers). The BN hole, which connects two waves of different wave-numbers, generates a phase-jump at its core. In order to conserve the total phase, it “emmits” a shock or shocks. (figures 8,9)

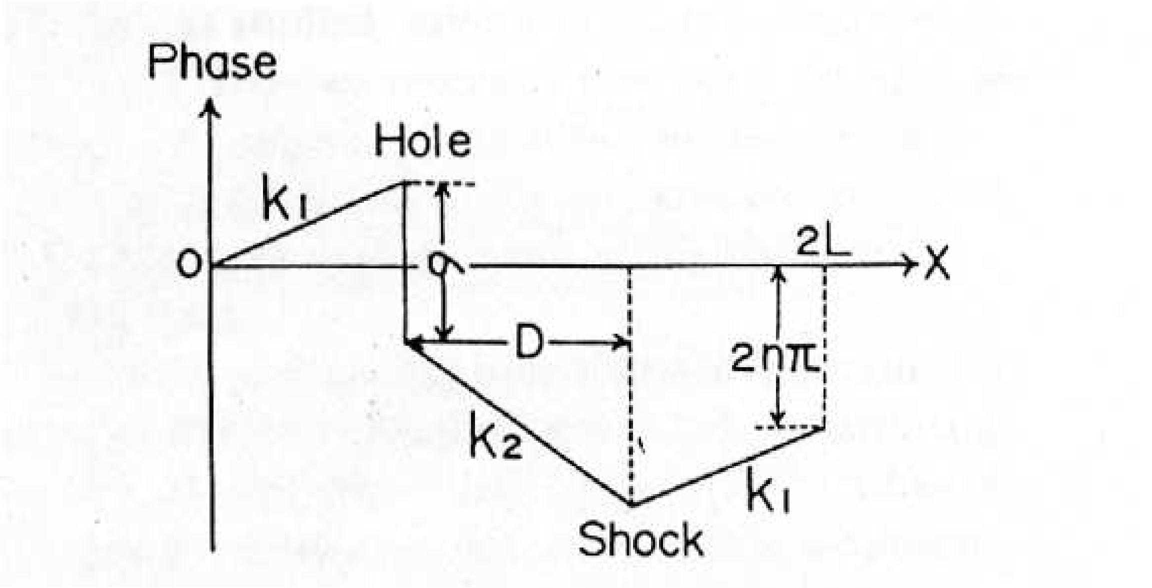

Figure 8: Schematic diagram of phase conservation in hole-shock pair. Nozaki, Bekki, Jour. Phys. Soc. Japan Vol. 53 No.5, (1984)

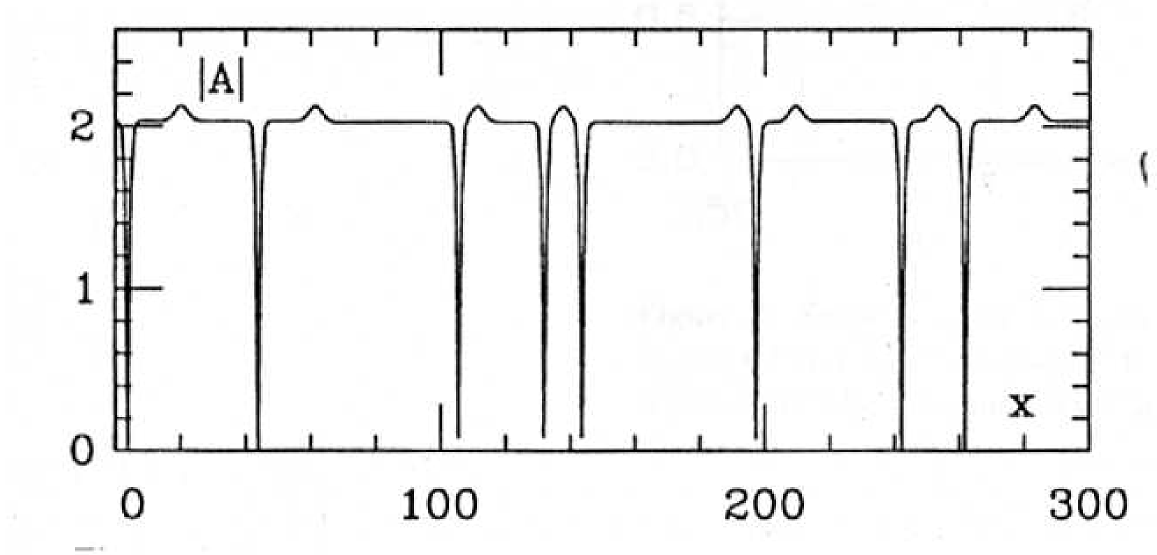

Figure 9: Stable, steady, multiple hole-shock solution for α = -0.9, β = -5.5. Note that there are eight holes and eight shocks: phase is conserved. Chaté, Nonlinearity 7 (1994) 185-204.

Are BN-Holes important in Intermittency?

It is not clear exactly how large a role BN-holes play in intermittency. By this we mean, “When I look at intermittent dynamics, do I see BN-holes or not?”, and ”Does the appearance and stability of BN-holes determine *when* I will observe intermittent dynamics?”

- The first question is not too difficult to answer, although little quantitative work has been done to determine the regions of existence of BN-holes in (α,β) space. One *does* see BN-holes, both in experiment and in numerical simulations of the CGL, and they appear both within and outside of the intermittent region. In fact, the intermittent regime has been defined as that region of parameter space for which there is a finite density of localized objects at infinite time (very long times in simulation).[8]

[8] Chaté. “Spatiotemporal intermittency regimes of the one-dimensional complex Ginzburg-Landau equation

- The density of localized objects must somehow be correlated to the stability of these objects (if they are all unstable to plane-waves, the density will be zero!). However, it is not enough to study the stability of a single Bekki-Nozaki hole. In fact, there is no reason to exclude nonlinear interaction between holes and between holes and other localized objects (shocks). We could just integrate (1) for long times and see if everything settles down to plane-waves, thereby ignoring the "machinery" behind stability. From this we could extract some information on hole stability in a statistical sense. A more ambitious approach would be to determine the weakly nonlinear stability of such a hole as well as hole-hole hole-shock interaction and *build up* a picture of intermittency. This has not yet been done.

Homoclons

- Recently, it has been observed in numerical simulations that there often exist holes connecting two waves of the *same* wavenumber in the intermittent regime.[9] Bekki-Nozaki holes diverge in width as the wavenumbers of the asymptotic waves approach one another. Therefore, the new holes are not BN-holes; they are something different.

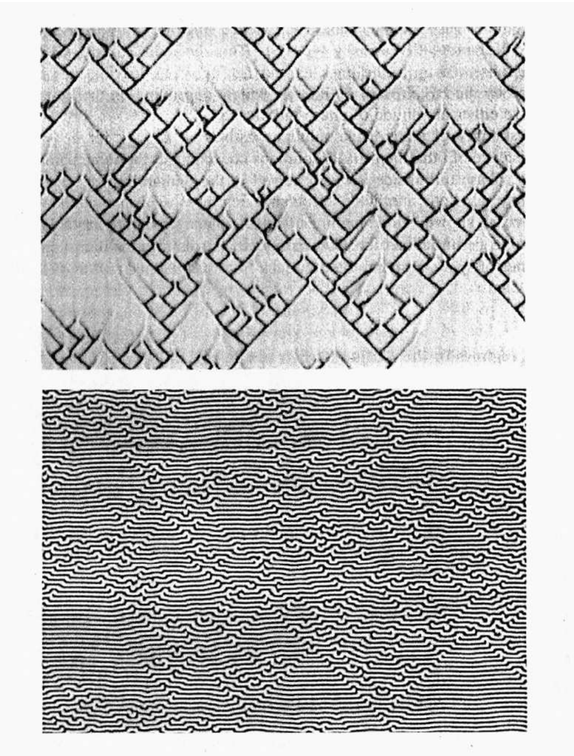

Figure 10: Space-time plot of holes connecting plane-waves of equal wavenumber ≈ 0. Upper: Amplitude, where lighter means greater amplitude. The black lines are holes (Homoclons). Note that the amplitude to either side of the hole is the same (same shade). Lower: Space-time plot of the phase. Note that the phase to either side of the holes is approximately constant, i.e. 0 wavenumber. Chaté, Nonlinearity

[9] van Hecke. "The building Blocks of Spatiotemporal Intermittency" chao-dyn/9707010 (1999)

• If we once again consider objects travelling at a constant speed and decompose the CGL into a system of 3 ODE's, we can imagine a trajectory in phase space which connects a fixed point (plane wave) with itself (figure 11). This homoclinic orbit is precisely what our new holes are. They carry the name "Homoclons".

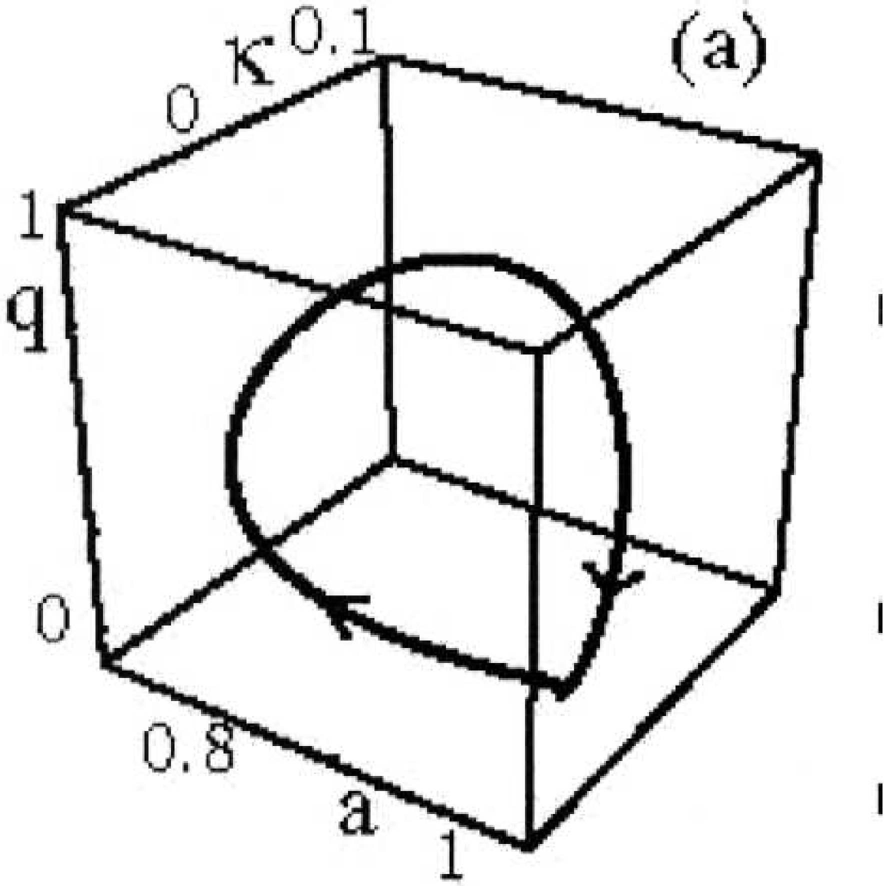

Figure 11: Homoclinic orbit in constant velocity subspace of CGL equation (see central panel). The fixed point corresponds to a plane-"wave" with wavenumber 0 at $(a,q,\kappa) = (1,0,0)$. The orbit exists for a unique value of the velocity v. van Hecke, Howard, nlin.CD/0002031 (2000)

Stability of Homoclons

- Homoclons are *not* exact solutions of the CGL equation and so their stability is a subtler issue than that of the Bekki-Nozaki holes. The 3 ODE's of the "uniformly propagating" CGL-subspace are parameterized by ω, the frequency, and v, the speed of the objects. However, for q $= q_r = q_l = 0$, which has been observed in simulations, ω is fixed and only v remains. The fixed point for q = 0 "waves" has a 1-D outgoing manifold, and a 2-D inspiraling manifold. The two connect to form a homoclinic loop for a unique value of v which fixes the speed of the homoclon. This calculated speed agrees well with the speed of the holes in simulation, so it seems Homoclons do exist.

- However, like the BN-holes, Homoclons have a core instability given by a single "core" mode.[10] If they are unstable, why are they observed? Like the BN-holes, the Homoclons are a local attractor in the CGL phase space. Plane waves are attracted on or near to the multi-dimensional stable manifold of the Homoclon and then escape slowly along its unstable manifold, either evolving into an amplitude defect and phase slip, or slowly decaying as the phase-gradient relaxes (figure 12).

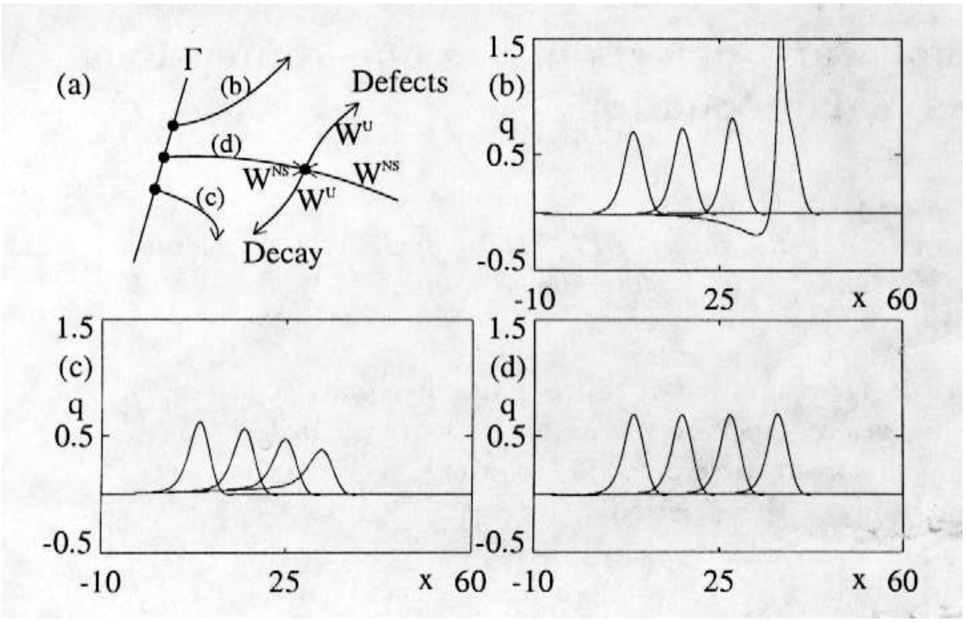

Figure 12: Evolution of holes near Homoclon fixed point. The underlying wavenumber is 0 with $\alpha = 0.6$, β = -1.4. Plots (a),(b),(c) show local wavenumber. (a) Schematic of CGL phase space. Holes are attracted to the fixed point and then evolve slowly along the unstable manifold W^U. (b) Evolution towards a defect and phase-slip. (c) Decay of hole. (d) Evolution very near a coherent Homoclon.

[10] van Hecke, Phys. Rev. Lett. 80, 1896 (1998)

More Stability of Homoclons

- Homoclons can lead to true amplitude defects where the amplitude dips to zero and the phase is undefined. A phase-slip occurs and the *winding number* $\frac{1}{2\pi}\int dxq$ decreases by one. Just before the phase slip, the wavenumber of the trailing edge of the Homoclon goes negative to conserve phase. This new dip, which corresponds to a hole propagating in the opposite direction from the original, then generates another phase-slip and so on. We see that holes will be generated in a zig-zag pattern (figure 10 again).

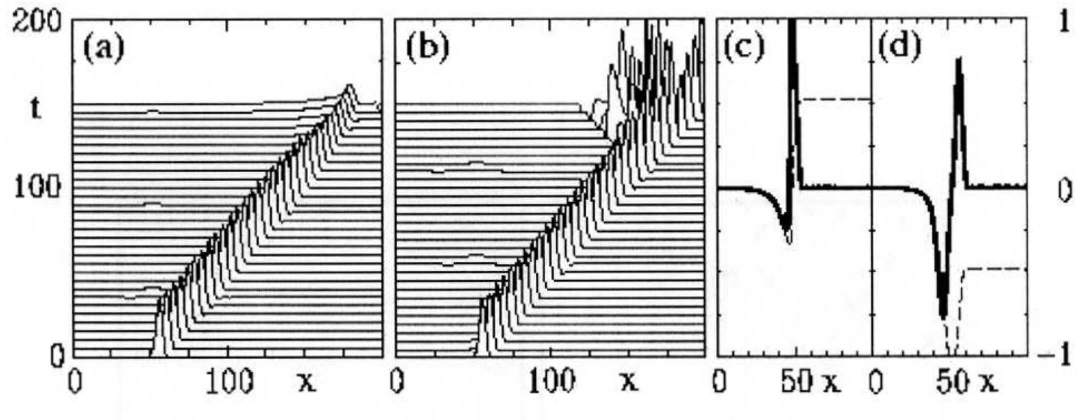

Figure 13: Evolution of near-Homoclon. (a) Space-time plot of wavenumber for a decaying hole. (b) Space-time plot of wavenumber evolving to a phase-slip. (c) Profile of wavenumber before phase-slip. Note the dip in wavenumber at the trailing edge of the hole to conserve phase. (d) Hole after phase-slip. dashed line indicates phase- winding number; note that it has decreases by one from (c). van Hecke chao-dyn/9707010 (1997) ?

- We see from figure 8 that not only do the holes produce a zig-zag pattern, but that the generation of new holes seems to occur periodically. Numerical simulations show that this period τ is dependent only on α, β and the wavenumber of the underlying plane-wave state q_{pw}

- Furthermore, the period diverges when the wavenumber is equal to the wavenumber of a coherent homoclon ($q_{pw} = q_{homoclon}$). Turning to the phase-space diagram of figure 12 we can explain this behavior qualitatively by noting that holes evolving near the critical wavenumber will evolve quickly towards the saddle point and then diverge locally exponentially along the unstable manifold. As a first approximation we say,
  $q_{pw} - q_{homoclon} = e^{\frac{-(\tau-\tau_O)}{s}}$[11]
  where s is a characteristic time. In fact, s should be the inverse of the unstable eigenvalue for the coherent Homoclon near the saddle point which only depends on α and β. Numerical calculations of the eigenvalue show this is so.

[11] van Hecke and Howard, "Ordered and self-disordered dynamics of holes and defects in the one-dimensional complex Ginzburg-Landau equation." nlin.CD/0002031 (2000)

So then what is intermittency?

We now understand how to classify Bekki-Nozaki holes and Homoclons, both of which appear to be building-blocks of intermittency. We know some things about their dynamics. This does not mean we understand intermittency. An important unanswered question is:

– Can we determine when chaotic dynamics will persist *a priori*, that is, from the stability of the constituent objects of intermittency and their interaction?

Another look at the CGL phase diagram, and the various dynamics we have seen at each point visited indicate that this question is not easily answered. Consider the figure on the next page which shows both Bekki-Nozaki holes and Homoclons coexisting. The interaction between the various holes is evident and nontrivial.

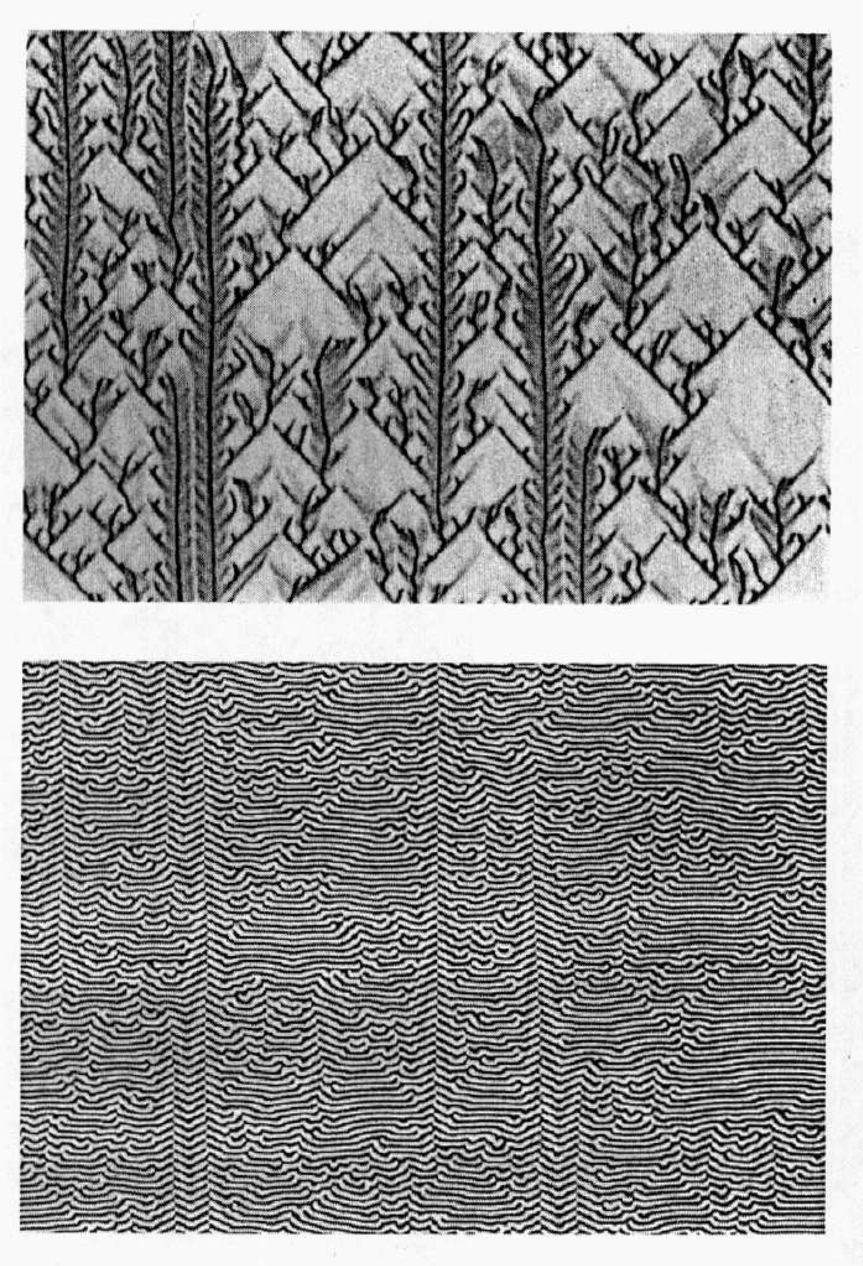

Figure 14: Top: Space-time plot of amplitude; dark lines are holes. Bottom: Space-time plot of phase. See if you can distinguish the Bekki-Nozaki holes and Homoclons given what we know about the waves they connect and their dynamics.

Central Panel: Bekki-Nozaki Hole Solution

– Finding the exact hole solution to the complex Ginzburg-Landau equation requires a procedure similar to that used in integrating the Nonlinear Schrödinger equation, called Hirota's Method. Let's forgo that here and instead examine the result itself.[12]

We substitute an ansatz of the form $A(x,t) = \frac{G(x)e^{i(Kx-\Omega t)}}{F(x)^{1+i\gamma}}$ into (1) and after solving, obtain,

$$A(x,t) = ge^{i(\gamma\kappa x-\Omega t)}\frac{1-e^{2\kappa x}}{(1+e^{2\kappa x})^{1+i\gamma}} \qquad (2)$$

where g, Ω, κ and γ all depend on the coefficients of the CGL equation in a complicated way.

– We see that for x near zero there is an exponential dip in the order parameter and so it is, in fact, a hole. Furthermore, we should be able to see how the hole asymptotically connects two plane-waves of different wavenumbers. Indeed, taking the appropriate limits yields the following.

$$\lim_{x\to-\infty} ge^{i(\gamma\kappa x-\Omega t)}\frac{1-e^{2\kappa x}}{(1+e^{2\kappa x})^{1+i\gamma}} = ge^{i(\gamma\kappa x-\Omega t)} \qquad (3)$$

$$\lim_{x\to\infty} ge^{i(\gamma\kappa x-\Omega t)}\frac{1-e^{2\kappa x}}{(1+e^{2\kappa x})^{1+i\gamma}} = ge^{i(-\gamma\kappa x-\Omega t+\pi)} \qquad (4)$$

[12] We examine the case of a non-propagating hole (v = 0).

So to the left the "wing" of the hole approaches a plane-wave of wavenumber $\alpha\kappa$, while to the right it approaches a plane-wave with wavenumber -$\alpha\kappa$. Furthermore, we see there is a phase jump of π. Refering to the figure below which shows the wavenumbers of the asymptotic waves as a function of the speed of the hole, we see that for a non-propagating hole everything works out.[13]

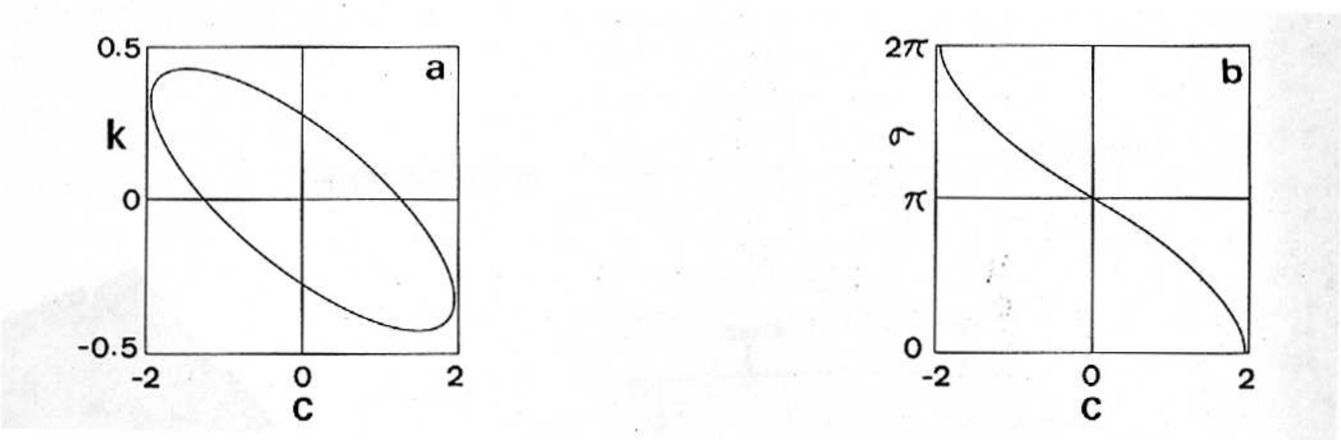

Figure 15: Left: Wavenumbers of the asymptotic plane-waves as a function of hole velocity (here velocity is c). Right: Phase jump as a function of hole velocity. Note that for a stationary hole the two wavenumbers are equal in magnitude and opposite in sign and the phase jump is π as we have found.

[13]Starting with the appropriate ansatz, the case of a propagating wave can be worked out, but it is considerably more complicated

Central Panel: Constant Velocity Subspace of the Complex Ginzburg Landau Equation

Here we consider solutions of the CGL equation which travel at a constant velocity v. Introducing the variable ξ = x - vt and considering an ansatz of the form A = $\hat{A}(\xi)e^{-i\omega t}$, we obtain from (1),

$$-v\partial_\xi\hat{A} - i\omega\hat{A} = \hat{A} + (1+i\alpha)\partial_\xi^2\hat{A} - (1+i\beta)\hat{A}^3 \quad (5)$$

which we rewrite as,

$$\frac{\partial_\xi^2\hat{A}}{\hat{A}} = \frac{1}{(1+i\alpha)}(-1 - i\omega + (1+i\beta)|\hat{A}|^2 - \frac{v\partial_\xi\hat{A}}{\hat{A}}) \quad (6)$$

Letting the complex quantity z = $\frac{\partial_\xi\hat{A}}{\hat{A}}$, we see that $\partial_\xi z = \frac{\partial_\xi^2\hat{A}}{\hat{A}} - (\frac{\partial_\xi\hat{A}}{\hat{A}})^2$ or,
$\frac{\partial_\xi^2\hat{A}}{\hat{A}} = \partial_\xi z + z^2$, which allows us to rewrite (6) as,

$$\partial_\xi a = \kappa a \quad (7)$$

$$\partial_\xi z = -z^2 + \frac{1}{(1+i\alpha)}(-1 - i\omega + (1+i\beta)a^2 - vz) \quad (8)$$

where $z = \kappa + iq, a = |\hat{A}|, q = \partial_\xi arg(\hat{A})$

Finally, since we have 3 real quantities, a, κ and q, we can write,

$$\begin{aligned} \partial_\xi a &= \kappa a & (9) \\ \partial_\xi \kappa &= -(\kappa^2 - q^2) + \frac{1}{(1+\alpha^2)}((1+\alpha\beta)a^2 - \alpha(vq+\omega) - 1) & (10) \\ \partial_\xi q &= -2\kappa q + \frac{1}{(1+\alpha^2)}((\beta-\alpha)a^2 - vq + \alpha v\kappa + (\alpha-\omega)) & (11) \end{aligned}$$

Fixed points of this system (v=0) exist for $a^2 = 1 - q^2$ and $\omega = \alpha q^2 + \beta a^2$. i.e. they are the plane-wave solutions of (1).