Hidden Translation Symmetry in Square–Geomtery Faraday Waves

A report on

"Hidden symmetries of parametrically forced waves."

J.D. Crawford, J.P. Gollub and D. Lane. *Nonlinearity* **6**, 1993.

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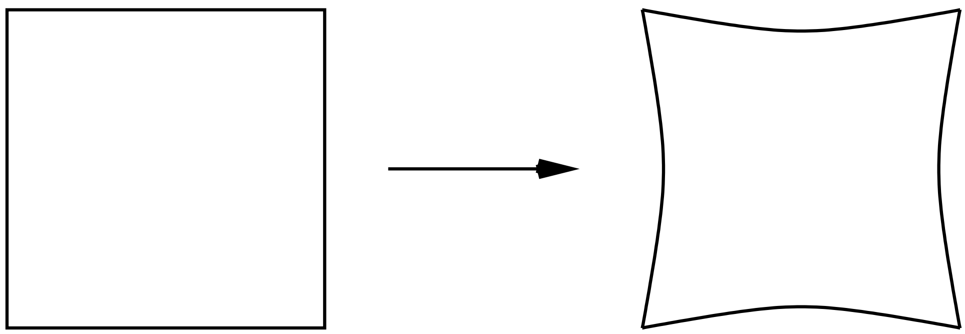
Introduction

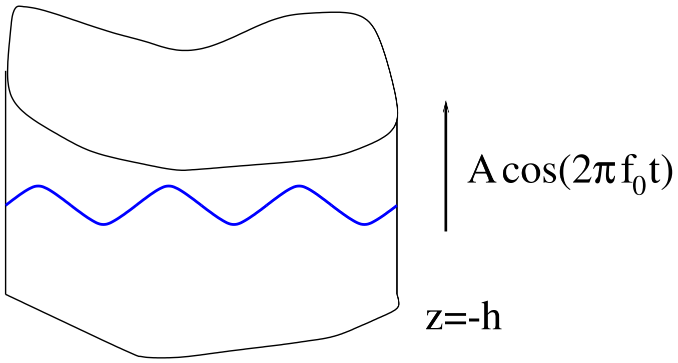
- When surface waves are parametrically excited in a container of square cross–section, there is a "hidden" translation symmetry in addition to the obvious $\mathbf{D_4}$ symmetry.
- This additional symmetry can lead to physcially distinguishable waves being simultaneously excited.
- The hidden symmetry can be broken by deforming the boundary of the cross–section.



- Predictions based on symmetry arguments are verified by performing the Faraday experiment in containers with square and deformed cross–section boundaries.

Mathematical formulation of the fluid problem

(Benjamin and Ursell, 1954)



- Assume fluid is inviscid, irrotational, incompressible.

  $\Rightarrow$ There exists a velocity potential $\nabla\phi = \vec{u}(x, y, z, t)$ which satisfies $\nabla^2\phi = 0$.

- Normal velocity is 0 at sidewalls, bottom.

  $\Rightarrow \hat{n} \cdot \nabla\phi = 0$

$$\phi(x, y, z, t) = \sum_m b_m(t)\psi_m(x, y) \cosh\left[\kappa_m(h + z)\right]$$

$\Rightarrow$ Helmholtz eigenvalue problem satisying Neumann boundary conditions (NBC):

$$\nabla_\perp^2\psi_m + \kappa_m^2\psi_m = 0 \quad \text{in bulk}$$

$$\hat{n} \cdot \nabla_\perp\psi_m = 0 \quad \text{at sidewalls}$$

The Faraday Instability

(Benjamin and Ursell, 1954)

Free surface $\zeta(x, y, t) = \Sigma_m \, a_m(t)\psi(x, y)$

- Navier–Stokes (z–component)
- Kinematic BC at free surface ζ
- linearization

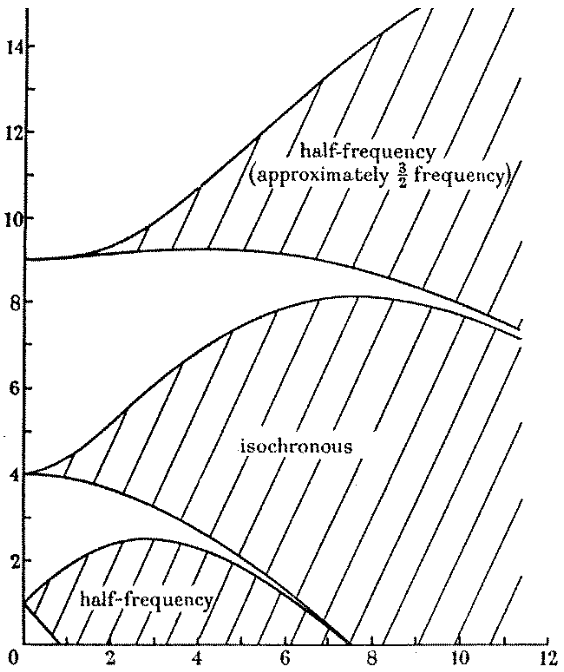
$\Rightarrow$ Time–dependent amplitude a_m obeys Mathieu eqn.:

$$\ddot{a}_m + \left[\omega_m^2 - (2\pi f_0)^2 A \kappa_m \tanh(\kappa_m h) \cos(2\pi f_0 t)\right] a_m = 0$$

$\Rightarrow$ ω_m is natural frequency of unforced wave, from monotonic dispersion relation

Linear stability curves:

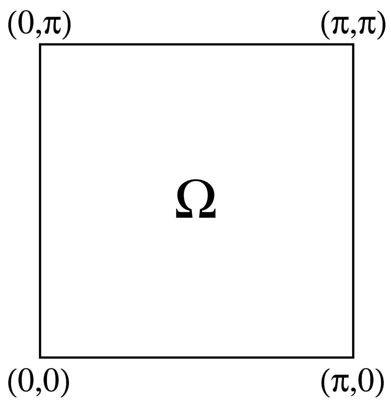
$p(A, f_0, \kappa, h)$



$q(A, f_0, \kappa, h)$

Eigenvalue degeneracy in square geometry

Assume container has square cross–section:



Eigenfunctions/Eigenvalues

$$\begin{aligned} \psi_{\ell,n}(x,y) &= B_{\ell,n}\cos(\ell x)\cos(ny) \\ \kappa^2_{\ell,n} &= \ell^2 + n^2 \end{aligned}$$

Table 1: Representative eigenvalues

κ^2	(ℓ, n)	Dimension of eigenspace
2	$(1,1)$	1
4	$(2,0),(0,2)$	2
5	$(2,1),(1,2)$	2
25	$(5,0),(0,5),(4,3),(3,4)$	4

- $\kappa^2_m = \kappa'^{\,2}_m \longleftrightarrow \omega^2_m = \omega'^{\,2}_m$ due to monotonic dispersion relation.
- Physically distinguishable waves may have same eigenvalue κ^2_m and thus may be excited with same frequency ω^2_m.

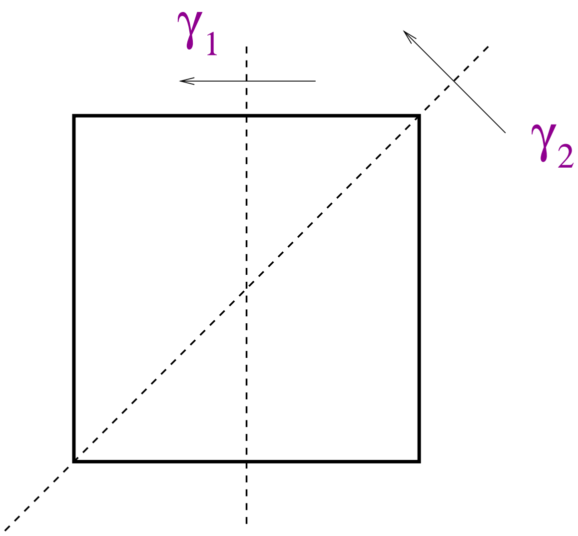
  $\Rightarrow$ This is called eigenvalue degeneracy.
- Degeneracies are due to symmetry, which enters the problem only through the BC since the bulk equation is isotropic.

$\mathbf{D_4}$ symmetry

The algebraic group $\mathbf{D_4}$ is the fourth order dihedral group, corresponding to the symmetries of a square.

$\mathbf{D_4}$ is generated by $\{\gamma_1, \gamma_2\}$, where the reflection γ_i is given by

$$\begin{aligned}\gamma_1 : (x, y) &\longrightarrow (\pi - x, y)\\ \gamma_2 : (x, y) &\longrightarrow (y, x)\end{aligned}$$



- $\mathbf{D_4}$ action commutes with $\nabla_\perp^2$.
- $\mathbf{D_4}$ action preserves the NBC.

$\Rightarrow$ If ψ is a solution with eigenvalue κ^2, then $\gamma \cdot \psi$ is a solution for the same eigenvalue for $\gamma \in \mathbf{D_4}$.

$\Rightarrow$ If ψ and $\gamma \cdot \psi$ are linearly independent, then κ^2 is degenerate.

Mixed mode basis and the $\mathbf{D_4}$ action

Reorganize eigenfunctions into mixed mode solutions:

$$\psi^{\pm}_{\ell,n} \equiv \frac{1}{\sqrt{2}}(\psi_{\ell,n} \pm \psi_{n,\ell})$$

Effect of $\mathbf{D_4}$ action depends on parity of $(\ell + n)$:

$$\begin{aligned} \gamma_1 \cdot \psi^{\pm}_{\ell,n} &= (-1)^{\ell} \begin{cases} \psi^{\pm}_{\ell,n} & (\ell+n) \text{ even} \\ \psi^{\mp}_{\ell,n} & (\ell+n) \text{ odd} \end{cases} \\ \gamma_2 \cdot \psi^{\pm}_{\ell,n} &= \pm\psi^{\pm}_{\ell,n} \end{aligned}$$

- Linearly independent solution generated only by γ_1 and only in case that $(\ell + n)$ is odd.

  $\Rightarrow$ Expect two–fold degeneracy in this case.

Contradictions!! (*cf.* Table 1)

- Two–fold degeneracy can occur for $(\ell + n)$ is even.
- Higher order degeneracies are possible.

$\boxed{\text{Not all degeneracies are due to simple } \mathbf{D_4} \text{ symmetry.}}$

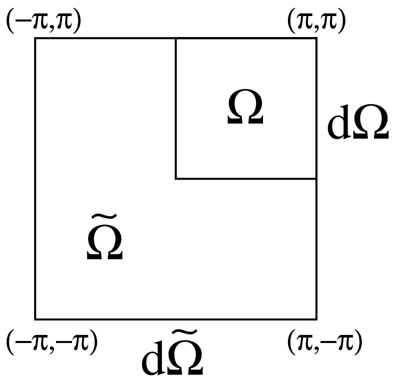
$\boxed{\text{Are there hidden symmetries present?}}$

Symmetries and domain extension

The eigenvalue equation has more symmetry than the NBC. Consider eigenvalue equation on two extended domains:

i) $\tilde{\Omega}: \ -\pi \leq x, y \leq \pi$, ψ even in x and y

$\Rightarrow$ Satisfies periodic BC (PBC) on $d\tilde{\Omega}$



Symmetry group $\widetilde{\mathbf{D}}_4 \dot{+} \mathbf{T}^2$

$T_{(a,b)} \cdot (x, y) = (x + a, y + b) \bmod 2\pi$

ii) $\mathbf{R}^2$, ψ 2π–periodic in x and y

$\Rightarrow$ No BC

Symmetry group $\mathcal{E}(2)$ (Euclidean group)

$\mathcal{E}(2) = \{R(\phi), T_{a,b}, \gamma_2\}$

$R(\phi) \cdot (x, y) = (x \cos\phi - y \sin\phi, x \sin\phi + y \cos\phi)$

$T_{(a,b)} \cdot (x, y) = (x + a, y + b)$

Hidden translation symmetry

There exists a hierarchy of eigenvalue problems:

Domain	BC	Symmetry Group Γ
$\mathbf{R^2}$	none	$\mathcal{E}(2)$
$\tilde{\Omega}$	PBC	$\widetilde{\mathbf{D}_4}\dot{+}\mathbf{T^2}$
Ω	NBC	$\mathbf{D_4}$

Approach: solve problem at a given level by solving problem at higher level and discarding solutions that violate the BCs.

Recall definition of *irreducibility*: the representation of Γ carried by the eigenspace is irreducible if there are no non–trivial proper subspaces of the eigenspace left invariant by Γ.

A mathematical result: the eigenvalue degeneracy is due to the symmetry of Γ only if the eigenspace is irreducible under Γ.

Restrict attention to case $\ell > n = 0$.
There is a $\boxed{\text{hidden symmetry}}$ $\Rightarrow$ translation $T_{\pi/\ell,0} \cdot \psi^{\pm}_{\ell,0} = -\psi^{\mp}_{\ell,0}$

- For ℓ odd, same action as γ_1.
- For ℓ even, generates new solution.
  - Eigenspace $\mathrm{E}_{\ell^2}(\Omega)$ is $\mathbf{D_4}$ reducible.
  - $\mathrm{E}_{\ell^2}(\Omega)$ is irreducible under $\mathbf{D_4}$ enlarged by the hidden symmetry.

Normal form

Ingredients:

- Restrict analysis to simplest nontrivial situation: $\ell > n = 0$.
- Free surface $\zeta(x,y,t) = v_1(t)\psi^+_{\ell,0}(x,y) + v_2(t)\psi^-_{\ell,0}(x,y) + \ldots$
- Instability is period doubling.
- Stroboscopic map $G(v_1,v_2)$ must commute with $\mathbf{D_4}$ and the hidden translation symmetry.

Action of symmetries on amplitudes v_j:

$$\begin{aligned}
\gamma_1 \cdot (v_1,v_2) &= \begin{cases} (v_1,v_2) & (\ell+n) \text{ even} \\ -(v_2,v_1) & (\ell+n) \text{ odd} \end{cases} \\
\gamma_2 \cdot (v_1,v_2) &= (v_1,-v_2) \\
T_{\pi/\ell,0} \cdot (v_1,v_2) &= -(v_2,v_1)
\end{aligned}$$

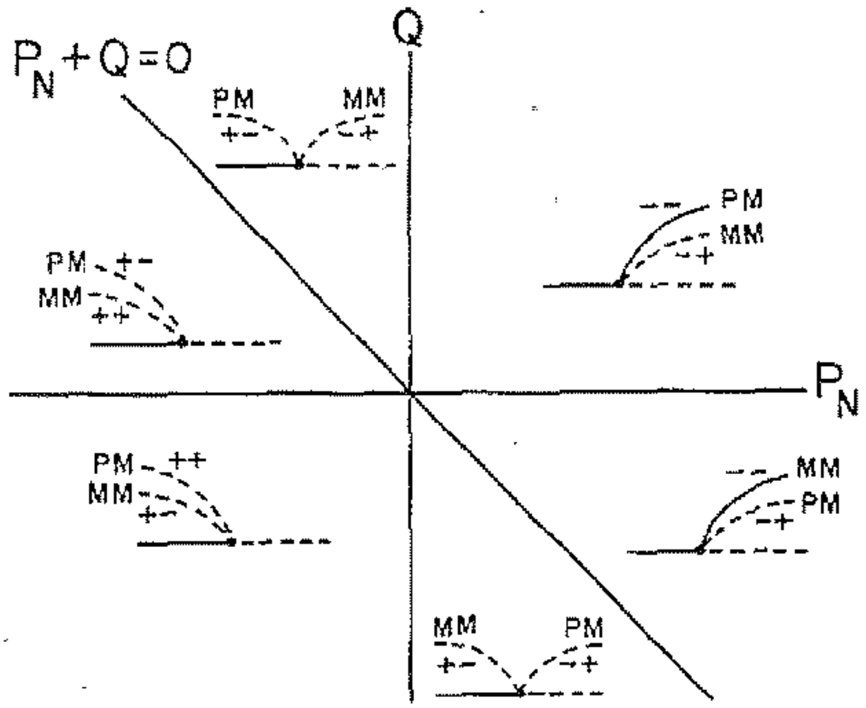
$$G(v_1,v_2) = P(N,\eta^2)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + Q(N,\eta^2)\eta\begin{pmatrix} v_1 \\ -v_2 \end{pmatrix}$$

$$N = v_1^2 + v_2^2$$

$$\eta = v_1^2 - v_2^2$$

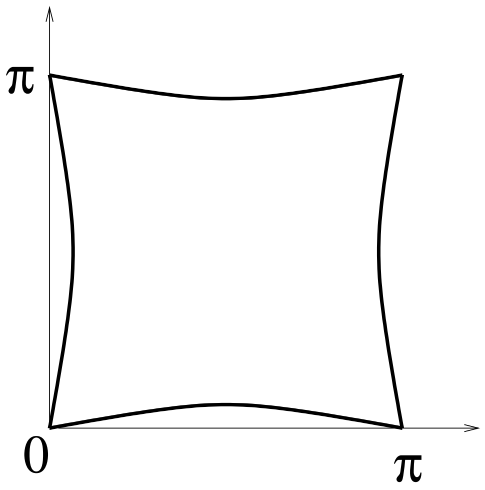
Standing wave solutions

Solution	(v_1, v_2)	Branching	Eigenvalues
mixed mode	$(v,0),(0,v)$	$1+P(v^2,v^4)+v^2Q(v^2,v^4)=0$	$1-4v^2(P_N+Q)$
MM			$1+4v^2Q$
pure mode	$v_1=\pm v_2=\tilde{v}$	$1+P(2\tilde{v}^2,0)=0$	$1-8\tilde{v}^2P_N$
PM			$1-8\tilde{v}^2Q$



Breaking the hidden symmetry

Idea: A sidewall deformation can preserve $\mathbf{D_4}$ symmetry but prohibit an extension to $\tilde{\Omega}$, thus killing the hidden translation symmetry.



Expected effect:

- For $\ell + n$ odd, $\psi^{\pm}_{\ell,n}$ remain degenerate.
- For $\ell + n$ even $\psi^{\pm}_{\ell,n}$ experience eigenvalue/frequency "split" (from perturbation theory).

$\Rightarrow$ Expect that for even parity modes, $\psi^{\pm}_{\ell,n}$ no longer have same stability.

Normal form revisited

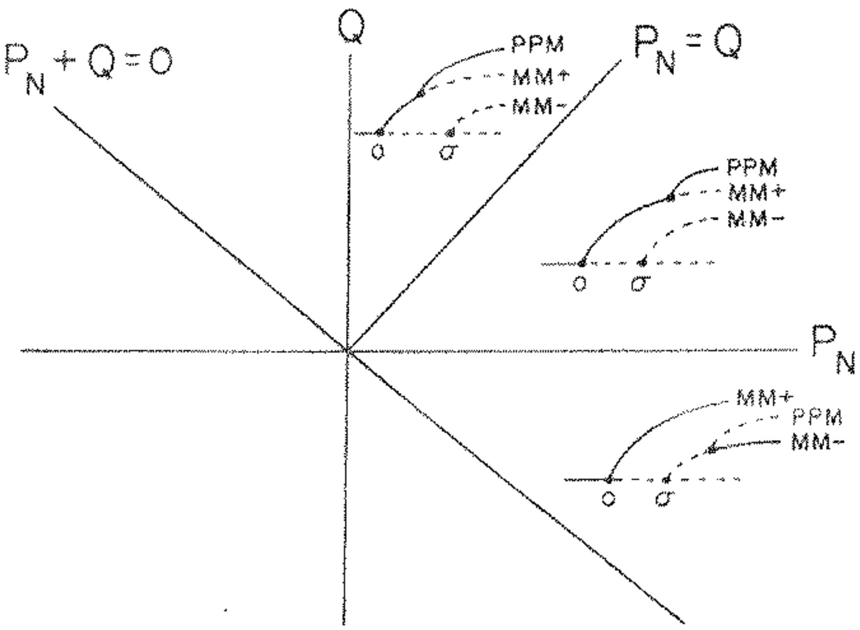
With deformed boundaries:

- odd ℓ: expect same normal form as before
- even ℓ: symmetry is reduced $\Rightarrow$ new normal form

  Extra terms are allowed in new normal form. Reduced symmetry is:

$$\underset{\text{from } \mathbf{D_4}}{\underset{\uparrow}{\mathbf{Z_2}(\gamma_2)}} \times \underset{\text{normal form symmetry}}{\underset{\uparrow}{\mathbf{Z_2}(-1)}}$$

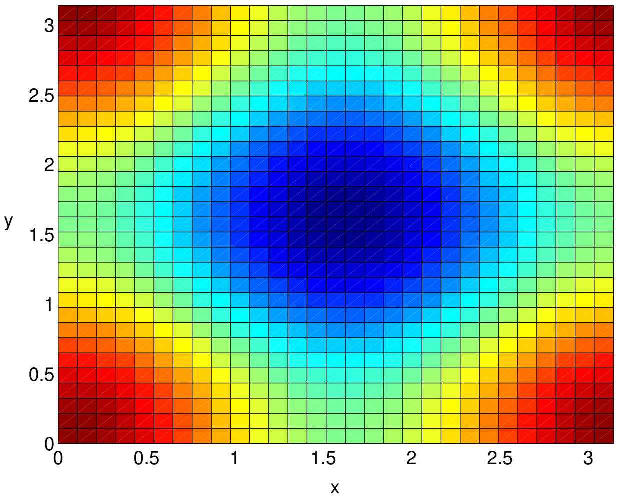
New map: $\hat{G}(v_1, v_2) = G(v_1, v_2) + \sigma \begin{pmatrix} 0 \\ v_2 \end{pmatrix}$



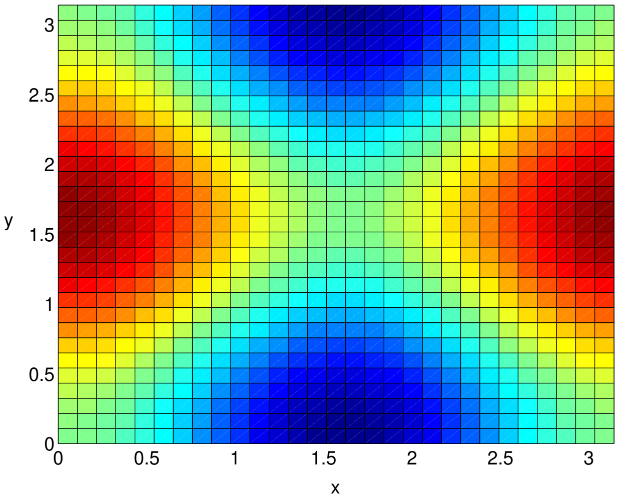
Experimental results

Maple–generated color contour plots of modes:

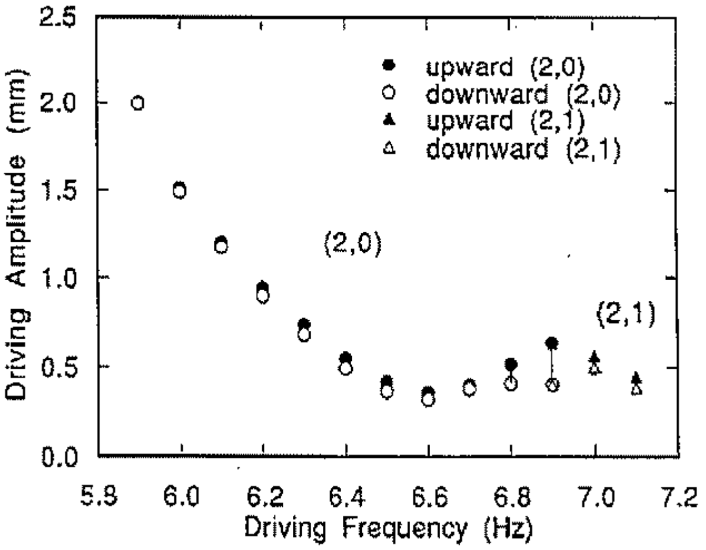
$\psi^{+}_{2,0}$

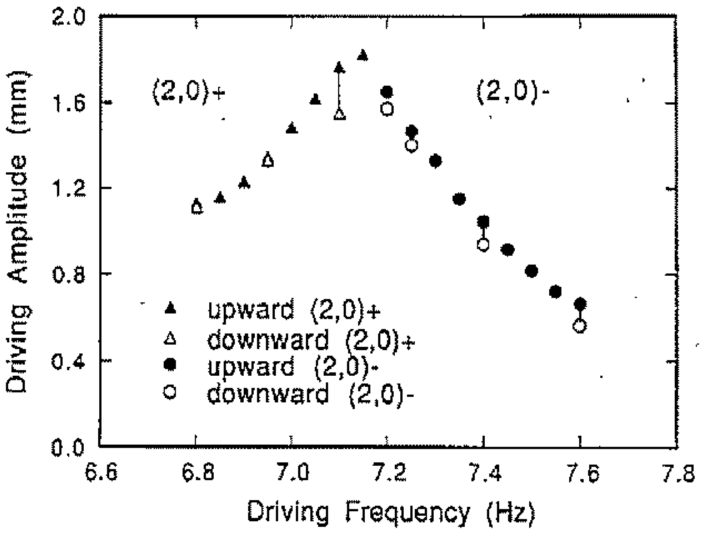


$\psi^{-}_{2,0}$



Linear stability:





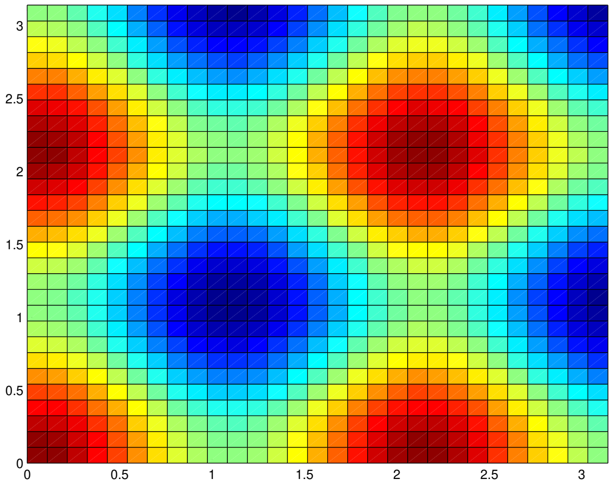
In a square cell (unperturbed boundary), the mixed modes $\psi^{\pm}_{2,0}$ have the same stability.

As predicted by theory, these even parity modes split when the boundary is perturbed to break the hidden translation symmetry.

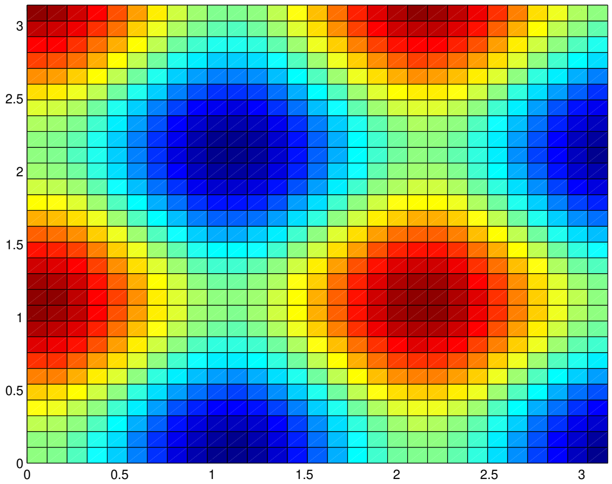
More experimental results

Maple–generated color contour plots of modes:

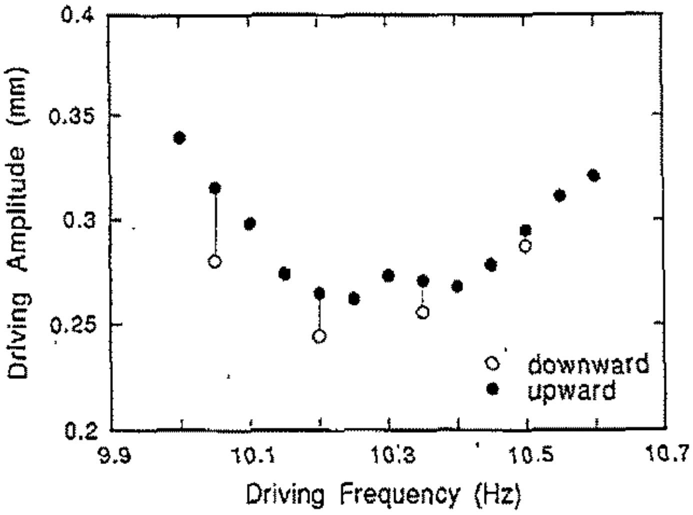
$\psi^+_{3,0}$



$\psi^-_{3,0}$



Linear stability:



Also as predicted, odd parity modes such as $\psi^\pm_{3,0}$ do not split when the boundary is perturbed.