Methods of Nonlinear Analysis 412-1

Winter 2002

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Take-Home Final

due Wednesday, March 20, 2002, 5 p.m.

1. **Amplitude Equation *via* Multiple Scales**

   Consider the partial differential equation,

$$\partial_t \psi = R\psi + v\partial_x\psi - (\partial_x^2 + 1)^2\psi + \alpha\psi\partial_x\psi + \beta\psi^2\partial_x\psi. \tag{1}$$

   (a) Perform a linear stability analysis of the basic state $\psi = 0$. What kind of a bifurcation do you expect?

   (b) Consider R as the control parameter and perform a weakly nonlinear analysis to obtain an evolution equation for the amplitude of the pattern. (Take the amplitude space-independent, i.e. do not introduce slow spatial variables.)

   (c) Find a set of simple, spatially periodic solutions to the resulting evolution equation. Are they qualitatively similar to the simple periodic solutions we found for the Swift-Hohenberg model? Is there a qualitative difference between eq.(1) and the Swift-Hohenberg model?

2. **Center-Manifold Reduction**

   Consider the coupled equations

$$\frac{dx}{dt} = (\alpha - 1)x + y + \beta xy + (x+y)^3, \tag{2}$$

$$\frac{dy}{dt} = x + (\alpha - 1)y. \tag{3}$$

   (a) Perform a linear stability analysis of the trivial state $(x, y) = (0, 0)$ and determine the eigenvectors associated with the two eigenvalues.

   (b) Considering the symmetries of eqs.(2,3) and the result of your linear stability analysis, what kind of bifurcation do you expect?

   (c) Introduce new coordinates in which the linear operator is diagonal.

   (d) In these new coordinates, determine a center manifold of $(0, 0)$ and the evolution equation on that manifold. Based on your symmetry argument, how should you scale the amplitude of the bifurcating mode relative to the control parameter α? In your expansion keep terms up to third order in the amplitude.
   Note: you may want to consider small β, i.e. β being of the same order as the amplitude.

   (e) Sketch a typical bifurcation diagram showing the dependence of the solution to the resulting amplitude equation on α.

3. **Convection in Anisotropic System**

   Nematic liquid crystals have a preferred axis of orientation indicated by their director. Quite a few convection experiments have been performed in this system with the director aligned horizontally, e.g. in the x-direction [1]. These systems are not isotropic. In certain parameter regimes the convection modes that destabilize the motionless state have the form

$$\psi(x,t) = Ae^{iqx+ipy} + Be^{iqx-ipy} + c.c. + h.o.t. \tag{4}$$

With periodic boundary conditions the translation ($x \to x + \Delta x$,$y \to y + \Delta y$) and reflection symmetries ($x \to -x$, $y \to -y$) of the system induce certain symmetry operations on the amplitudes A and B. By suitable combinations of the translations in the x- and y-direction and of the two reflections the symmetry operations can be written as

$$\begin{aligned} (\phi,\psi)(A,B) &= (e^{i\phi}A, e^{i\psi}B), \qquad \phi,\psi \in \mathcal{R}, & (5)\\ \kappa_1(A,B) &= (B,A), & (6)\\ \kappa_2(A,B) &= (A^*,B^*). & (7) \end{aligned}$$

The operations $\{(\phi,\psi),\kappa_1,\kappa_2\}$ form the symmetry group Γ of the system. In his problem we identify the symmetries of all possible states of this system.

(a) Show that an arbitrary element $\gamma \in \Gamma$, which can be thought of as an arbitrary combination of the basic elements $\kappa_{1,2}$ and (ϕ,ψ), can be expressed as

$$\gamma = \kappa_1^l \kappa_2^m (\phi,\psi), \qquad l,m \text{ integer}. \tag{8}$$

(b) Show that all elements $(A,B) \in \mathcal{C}^2$ are on some group orbit of $(a,b) \in \mathcal{R}^2$ with $b \geq a \geq 0$.

(c) By considering the action of the general group element $\kappa_1^l \kappa_2^m(\phi,\psi)$ on the general element $(a,b) \in \mathcal{R}^2$ identify all isotropy subgroups $\Sigma_{(A,B)}$ of Γ.

(d) For each isotropy subgoup $\Sigma_{(A,B)}$, identify the corresponding fixed-point subspace.

(e) Considering the linear mode given in eq.(4), what kind of patterns do the elements of the various fixed-point subspaces correspond to?

References

[1] M. Dennin, D.S. Cannell, and G. Ahlers. Patterns of electroconvection in a nematic liquid crystal. *Phys. Rev. E*, 57:638, 1998.