Methods of Nonlinear Analysis 412-1

Winter 2002

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Homework Assignment 1

due Thursday, January 31, 2002

1. **Tricritical Point**
   Consider the dynamical system
   $$\partial_t u = f(u, \lambda) \tag{1}$$
   with $f(-u, \lambda) = -f(u, \lambda)$. For $u = 0$ and $\lambda = 0$ assume that $\partial_u f = 0$ and $\partial^3_{uuu} f$ is small. For small λ, identify the leading-order terms in u and λ and obtain an equation for the fixed points of this system in the vicinity of $u = 0$. Sketch the various bifurcation diagrams that arise. For $\partial^3_{uuu} f = 0$ the system is said to be at a tricritical point. Why is this point of particular interest if one wants to discuss hysteretic (jump) transitions?

2. **Trajectories in a Simple Dynamical System**
   Consider the dynamical system
   $$\partial_t x = -\alpha x - x^2, \tag{2}$$
   $$\partial_t y = -\beta y \tag{3}$$
   with β positive.
   (a) Determine the solution $(x(t), y(t))$ of $(2, 3)$ and the trajectory $y = y(x)$ for $\alpha > 0$.
   (b) Determine the solution $(x(t), y(t))$ of $(2, 3)$ and the trajectory $y = y(x)$ for $\alpha = 0$.
   (c) Compare the trajectories in the two cases. In particular, comment on the effect of the initial conditions on the trajectories as they approach the fixed point $(0, 0)$.
   (d) The case $\alpha = 0$ demonstrates the non-uniqueness of the center manifold. Comment on the distance between the different center manifolds and whether they can be distinguished in a power series expansion around the fixed point $(0, 0)$.

3. **Center Manifold Reduction for a Steady Bifurcation**
   Consider the dynamical system
   $$\partial_t x = y - x^3, \tag{4}$$
   $$\partial_t y = \alpha x - x^3 - \beta \frac{y}{1 + x^2}. \tag{5}$$
   (a) Perform a linear stability analysis of the fixed point $(x, y) = (0, 0)$. Depending on α and β what kind of instabilities can you identify? Determine the stable, unstable, and center linear eigenspaces in each of the cases.
   (b) Identify the symmetries of the system $(4,5)$. Based on this information together with the linear stability analysis, what types of bifurcations do you expect in this system?
   (c) For $\beta = 1$ and α small, perform a center manifold reduction to obtain an equation for a center manifold and an equation for the slow evolution on that manifold. Keep only the leading-order terms in this expansion.
   (d) For $\beta = 1$ and α small, obtain the equivalent result by performing a multiple-scale calculation. Identify clearly the right and left eigenvectors of the linear operator (matrix) obtained in the linear stability analysis in (a).

(e) Solve (4,5) numerically (using `Matlab/pplane5` or `dstool`; see web site) in the regime for which your center manifold reduction is supposed to be valid. How well does the analytical approach do near the bifurcation point? (For a quantitative comparison with the center manifold you would have to write your own code (using ode45, for instance, in matlab) to be able to print the trajectories.)

4. **Center Manifold Equations for a Hopf Bifurcation**

Consider the dynamical system

$$\partial_t x = -x - y + z^2, \tag{6}$$

$$\partial_t y = 2x + (1+\epsilon)y - z^2, \tag{7}$$

$$\partial_t z = x + 2y - z. \tag{8}$$

It exhibits a Hopf bifurcation at $\epsilon = 0$ leading to a limit cycle (periodic orbit).

(a) Determine a center manifold that describes the dynamics of this system for small amplitudes x, y, and z and for small values of the bifurcation parameter ϵ (which is not necessarily 0).

(b) Determine the evolution equations describing the dynamics on the center manifold. Express them also in terms of the complex amplitude $A = u + iv$ and compare the resulting equation with the normal form for a Hopf bifurcation

$$\partial_t A = \alpha A + \gamma |A|^2 A. \tag{9}$$

Is the bifurcation forward or backward (i.e. super- or subcritical)?

(c) Solve the equations numerically (e.g. using `dstool` or write your own matlab etc. code). Is the result qualitatively consistent with your analytic result? Explore what happens to the limit cycle (periodic orbit) when ϵ is increased.