Methods of Nonlinear Analysis 412-1

Winter 2002

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Homework Assignment 2

due Friday, February 22, 2002

1. **Hopf Bifurcation *via* Multiple Scales**

   Consider the Brusselator model for the reactions

$$\begin{aligned} A &\rightarrow U, & (1)\\ U &\rightarrow P, & (2)\\ U &\rightarrow V, & (3)\\ 2U + V &\rightarrow 3U, & (4) \end{aligned}$$

   where A represents a substance that is abundant and therefore not a limiting factor in the reaction and P is a product that is removed instantly. This results in the reaction-diffusion system

$$\begin{aligned} \partial_t u &= \nabla^2 u + A - (B+1)\,u + u^2 v, & (5)\\ \partial_t v &= D\nabla^2 v + Bu - u^2 v & (6) \end{aligned}$$

   for the concentrations u and v [1, 2] below). To simplify the calculation take $A = 1$. In this problem we ignore any spatial dependence.

   (a) Perform a linear stability analysis of the Brusselator. Under which conditions is the destabilizing mode of the homogeneous steady state oscillatory?

   (b) Use multiple time scales to derive the amplitude equation for the Hopf bifurcation.

2. **Normal Form for Takens-Bogdanov Bifurcation**

   Consider a dynamical system of the form

$$\begin{aligned} \partial_t x &= y + F(x,y), & (7)\\ \partial_t y &= G(x,y), & (8) \end{aligned}$$

   where the functions $F(x,y)$ and $G(x,y)$ are strictly nonlinear, i.e. at least quadratic in x and y, but otherwise arbitrary. Thus, the linear part $\mathbf{L}$ consists of a Jordan block $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ (cf. Guckenheimer & Holmes Ch. 7.2, Wiggins Example 2.2.2).

   (a) Perform a linear stability analysis of (7,8) and determine all eigenvectors of the linear operator.

   (b) Envision a near-identify transformation that simplifies the nonlinear terms as much as possible. The nonlinear terms that cannot be removed by such transformations form a subspace and are invariant under the transformation $\exp(\mathbf{L}^t s)$, i.e. each of the vectors $\mathbf{V} \equiv (u,v)^t$ in this subspace satisfies

$$e^{-\mathbf{L}^t s}\,\mathbf{V}(e^{\mathbf{L}^t s}(x,y)^t) = \mathbf{V}((x,y)^t). \qquad (9)$$

   Use this normal-form symmetry to identify to all orders in x and y which nonlinearities cannot be removed. First determine $e^{-\mathbf{L}^t s}$ explicitly. To make use of the invariance condition (9) it is then useful to take the derivative of the invariance condition with respect to s.

(c) Determine explicitly the range of $\mathcal{L}$ in

$$\mathcal{H}^{(2)} \equiv \{(u(x,y),v(x,y))^t|(u(ax,ay),v(ax,ay))^t = a^2(u(x,y),v(x,y))^t\}. \tag{10}$$

Note that the results of different near-identify transformations differ by multiples of vectors from the range of $\mathcal{L}$. Determine the range of $\mathcal{L}$ and use this to show that at the bifurcation point (codimension-2 point) the normal form for the Takens-Bogdanov bifurcation in the absence of any symmetries of the original equations (7,8) can be written to second order as

$$\begin{aligned} \partial_t x &= y, & (11)\\ \partial_t y &= c_1 xy + c_2 x^2. & (12) \end{aligned}$$

3. **Squares and Stripes in a SH-Model**

Consider the modified Swift-Hohenberg model

$$\partial_t \psi = R\psi - (\nabla^2+1)^2\psi - \frac{a}{3}\psi^3 + b\nabla\cdot\left(\nabla\psi\,(\nabla\psi)^2\right). \tag{13}$$

It is a modification of a model derived by Gertsberg and Sivashinsky [3] for the description of convection with poorly conducting top and bottom boundaries. With such boundary conditions the critical wavenumber for the onset of convection becomes very small and a systematic expansion in the small wavenumber can be performed. A detailed analysis of the possible planforms in an equation of this type has been performed in [4].

(a) Use the method of your choice to derive coupled amplitude equations for the amplitudes of rolls in the x- and in the y-direction that are valid close to threshold, i.e. for $R - R_c = \mathcal{O}(\epsilon^2)$, $\epsilon \ll 1$. Allow the wavevector (q,p) of the modes to differ slightly from the critical wavevector, $(q,p) = (1+\epsilon Q, 0)$ and $(q,p) = (0, 1+\epsilon P)$, respectively.

(b) Calculate all fixed points of the resulting amplitude equations. Determine their stability. What kind of patterns do the fixed points represent?

(c) As a function of a and b, determine all possible phase diagrams, i.e. delineate the regions of existence and of stability of all fixed points in the R-Q-plane with $P = 1$ fixed.

(d) Sketch the flow in the phase plane in each of the regions identified in 3c).

(e) Can any persistent dynamics arise in eq.(13)?

References

[1] G. Nicolis and I. Prigogine. *Self-organization in nonequilibrium systems: from dissipative structures to order through fluctuations.* Wiley, New York, 1977.

[2] A. Dewit, D. Lima, G. Dewel, and P. Borckmans. Spatiotemporal dynamics near a codimension-2 point. *Phys. Rev. E*, 54:261, 1996.

[3] V.L. Gertsberg and G.I. Sivashinsky. Large cells in nonlinear Rayleigh-Bénard convection. *Prog. Theor. Phys.*, 66:1219, 1981.

[4] E. Knobloch. Pattern selection in long-wavelength convection. *Physica D*, 41:450, 1990.