Methods of Nonlinear Analysis 412-1

Winter 2002

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Homework Assignment 3

due Friday, March 8, 2002

1. $\mathcal{D}_n$ **Equivariant Vector Field**

   Consider the group $\Gamma = \mathcal{D}_n$ with the representation on $\mathcal{C}$ given by

   $$\kappa z = z^*, \qquad \zeta z = e^{i\frac{2\pi}{n}} z. \tag{1}$$

   Show that the most general Γ-invariant function $f(z, z^*) = \sum_{m,n} f_{m,n}\, z^m (z^*)^n$ can be written as a function of the two basic invariants

   $$u = |z|^2, \qquad v = z^n + (z^*)^n. \tag{2}$$

   Show that the most general Γ-equivariant vector field $g(z, z^*)$ can be written as

   $$g(z, z^*) = zp(u, v) + (z^*)^{n-1} q(u, v). \tag{3}$$

2. **Isomorphism of Representations**

   (a) Consider two representations of a group Γ, one on $\mathcal{R}^m$, the other on $\mathcal{R}^n$. Can these representations be isomorphic to each other for any m and n?

   (b) Are the following two representations of $\mathcal{Z}_2$ on $\mathcal{C}$ isomorphic to each other,

   $$\text{i)}\quad \kappa z = z^*,\ z \in \mathcal{C} \qquad \text{ii)}\quad \kappa w = -w, w \in \mathcal{C}\,? \tag{4}$$

   (c) For which values of the integers m and n are the following two representations of $SO(2)$ on $\mathcal{C}$ isomorphic to each other,

   $$\text{i)}\quad Rz = e^{i\,m\theta} z, z \in \mathcal{C} \qquad \text{ii)}\quad Rw = e^{i\,n\theta} w, w \in \mathcal{C}\,? \tag{5}$$

3. $O(2)$**-Hopf Bifurcation**

   Consider a system that is reflection and translation symmetric, which undergoes a Hopf bifurcation to traveling wave modes

   $$\psi = Ae^{i(\omega t - qx} + Be^{i(\omega t + qx)} + c.c. + h.o.t., \tag{6}$$

   where $A \in \mathcal{C}$ and $B \in \mathcal{C}$ correspond to the amplitudes of left- and right-traveling waves, respectively. Convince yourself that translations θ and reflections κ act as

   $$\theta(A, B) = (e^{-i\theta} A, e^{i\theta} B), \qquad \kappa(A, B) = (B, A). \tag{7}$$

   In normal form the equation describing the waves,

   $$\dot{A} = g_A(A, A^*, B, B^*) \tag{8}$$
   $$\dot{B} = g_B(A, A^*, B, B^*), \tag{9}$$

have the additional symmetry

$$\phi(A, B) = (e^{i\phi}A, e^{i\phi}B), \tag{10}$$

which can be seen to be related to time-translation symmetry. The overall symmetry of the system is therefore given by $O(2) \times S^1$, with a general element of the group given by $\kappa^l(\theta, \phi)$, $l = 0, 1$. Please note that κ does not commute with θ but it does commute with ϕ.

The goal is to determine the isotropy subgroup lattice for this system, i.e. to determine all isotropy subgroups of $O(2) \times S^1$ for this representation and determine the associated fixed-point subspaces.

(a) Show that it is sufficient to consider the action of the group elements on the elements $(a, b) \in \mathcal{C}^2$ with a and b real and $a \geq b \geq 0$. All other $(A, B) \in \mathcal{C}^2$ are then on a suitable group orbit of (a, b).

(b) By considering all possible elements (a, b) determine all isotropy subgroups. What are the associated fixed-point subspaces? What is their dimension? Based on this information are any solutions exhibiting these symmetries guaranteed?

(c) Consider eq.(6), what kind of waves do the elements in the various isotropy subgroups correspond to?

Note: Due to the normal-form symmetry the equations for the amplitudes A and B can be separated into equations for the magnitudes a and b and for the phases of A and B. The equations for a and b decouple from those for the phases and one obtains a new set of equations with $\mathcal{D}_4$ symmetry. The fixed-point subspaces you have determined become one-dimensional in this setting. What can you conclude then?