ESAM 412-1 Methods in Nonlinear Analysis

Spring Quarter 2008

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Final Due 4pm Friday June 13, 2008

NOTE:

In contrast to the usual homeworks this final is to be done without consulting any of your colleagues!

1. **Normal-Form Transformations**
   Consider the system

$$\frac{du}{dt} = \mu u + v - u^2 - v^2 \qquad (1)$$

$$\frac{dv}{dt} = -u + \mu v \qquad (2)$$

   (a) Perform a linear stability analysis of the fixed point $(u, v) = (0, 0)$. What kind of bifurcation does the instability suggest?

   (b) Rewrite (1,2) in terms of a complex amplitude A that is made up from the amplitudes of two vectors that span the center eigenspace.

   (c) Assuming $\mu = \mathcal{O}(A^2)$, perform a normal-form transformation of the evolution equation for A that removes all quadratic terms in A and A^*. Give the equation for A to third order in A and A^* , i.e. retain all cubic terms.

   (d) Again assuming $\mu = \mathcal{O}(A^2)$, building on part 1c, extent the normal-form transformation to remove as many cubic terms in A or A^* as possible and give the equation for A to cubic order.

   (e) What kind of bifurcation does the resulting equation describe? Do you notice something unexpected about the bifurcation to this order?

2. **Hexagon and Stripe Patterns**
   Consider a two-dimensional system with translation symmetry in all directions that undergoes a steady bifurcation involving a mode with wavenumber q.

   (a) Focusing on the modes A_{0°, A_{120°, and A_{240° corresponding to the wavevectors $(q, 0)$, $q(-\frac{1}{2}, \frac{\sqrt{3}}{2})$, and $q(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$, respectively, exploit the translation symmetry and the rotation symmetry by 120° to show that the amplitude equations have the form

$$\frac{d}{dt}A_{0^\circ} = \mu A_{0^\circ} + \alpha A^*_{120^\circ} A^*_{240^\circ} + \beta A_{0^\circ}\left(|A_{0^\circ}|^2 + \gamma\left(|A_{120^\circ}|^2 + |A_{240^\circ}|^2\right)\right) \qquad (3)$$

   to cubic order, with the equations for A_{120° and A_{240° obtained by cyclic permutations of the modes.

   (b) Writing $A_{0^\circ} = R_{0^\circ} e^{i\phi_{0^\circ}}$etc., derive coupled equations for the magnitudes R_{0°, R_{120°, R_{240° and a suitable combination of the three phases.

   (c) Using the equations derived in part 2b, compute the solutions for stripe patterns, $R_{0^\circ} \neq 0$, $R_{120^\circ} = R_{240^\circ} = 0$, and for hexagon patterns, $R_{0^\circ} = R_{120^\circ} = R_{240^\circ} \neq 0$, assuming $\alpha = \mathcal{O}(A)$, $\mu = \mathcal{O}(A^2)$ and $\beta < 0$. Sketch the bifurcation diagram for both solutions.

(d) Assuming again $\alpha = \mathcal{O}(A)$, $\mu = \mathcal{O}(A^2)$ and $\beta < 0$, compute the linear stability of stripe patterns within (3) and the associated two other equations for A_{120° and A_{240°. How does the linear stability depend on γ and μ? [1]

3. **Patterns Fronts and Droplets**
Consider the Ginzburg-Landau equation for a pitch-fork bifurcation at which a spatially *homogeneous* mode destabilizes the basic state $A = 0$. In addition, assume that this pitch-fork bifurcation is weakly perturbed by a quadratic term. Thus, consider the equation

$$\partial_t A = \frac{1}{2}\partial_x^2 A + A - A^3 + \epsilon\alpha A^2, \qquad \epsilon \ll 1, \qquad -\infty < x < \infty \tag{4}$$

for A real. For $\alpha = 0$ this system exhibits a bistability between the two states $A = +1$ and $A = -1$.

(a) Confirm that for $\epsilon = 0$ eq.(4) has a continuous family of stationary front-solutions of the form

$$A = A_0(x) \equiv \tanh(x - x_0),$$

each of which connects the two states $A = +1$ and $A = -1$.

(b) Write now (4) more generally as

$$\partial_t A = \frac{1}{2}\partial_x^2 A - \partial_A V(A) - \epsilon\partial_A \hat{V}(A), \qquad \epsilon \ll 1. \tag{5}$$

Assume that for $\epsilon = 0$ eq.(5) has a stationary front solution $A = A_0(x - x_0)$. Determine the impact of $\hat{V}(A)$ on the front solution by making the ansatz

$$A(x, T) = A_0(x - x_0(T)) + \epsilon A_1(x - x_0(T), T) + \mathcal{O}(\epsilon^2)$$

where $T = \epsilon t$ is a slow time. Obtain a solvability condition that results in an evolution equation for x_0,

$$\frac{d}{dT}x_o(T) = v$$

and express v in terms of $\hat{V}(A)$. This condition will make make use of the translation symmetry of (5), which generates a translation mode.

(c) If one wanted to describe a localized 'droplet' of the state $A = +1$ surrounded by a background of $A = -1$ (cf. figure below) one could make the ansatz

$$A(x, t) = A_L(x - x_L(T)) + A_R(x - x_R(T)) - A_\infty + \epsilon A_1 + \ldots$$

where for (4) with $\alpha = 0$ one would have

$$A_L(x - x_L) = \tanh(x - x_L), \qquad A_R(x - x_R) = -\tanh(x - x_R), \qquad A_\infty = 1.$$

The resulting perturbation calculation would be quite involved. Instead consider the following, simplified problem, based on (4) with $\alpha = 0$. For widely separated fronts, $x_R - x_L \gg 1$, the left front A_L, located at x_L with $-x_L \gg 1$, experiences the right front A_R as a weak perturbation,

$$A_R(x - x_R) \sim -e^{2(x - x_R)} \qquad \text{for } x < 0 \text{ and } x_R \gg 1. \tag{6}$$

The full problem involves some subtleties. We therefore 'cheat' and consider the equation

$$\partial_t A = \frac{1}{2}\partial_x^2 A + A - A^3 - \epsilon e^{(x - x_R)} \tag{7}$$

[1] Just for completeness: with some more effort one can show that the hexagons are linearly stable close to their saddle-node bifurcation.

with x_R kept constant. Note that the exponential in (7) is growing more slowly than in (6). Use part 3b to obtain an evolution equation for the position $x_L(T)$ of the left front[2]. Without discussing the justification, ignore in this problem the fact that for $2(x - x_R) > \ln \epsilon^{-1}$ the perturbation becomes $\mathcal{O}(1)$ [3].
What kind of motion of the front A_L does the perturbation by the front A_R induce? What overall dynamics would you expect if the initial condition consisted of a large number of domains with $A \sim +1$ and $A \sim -1$.

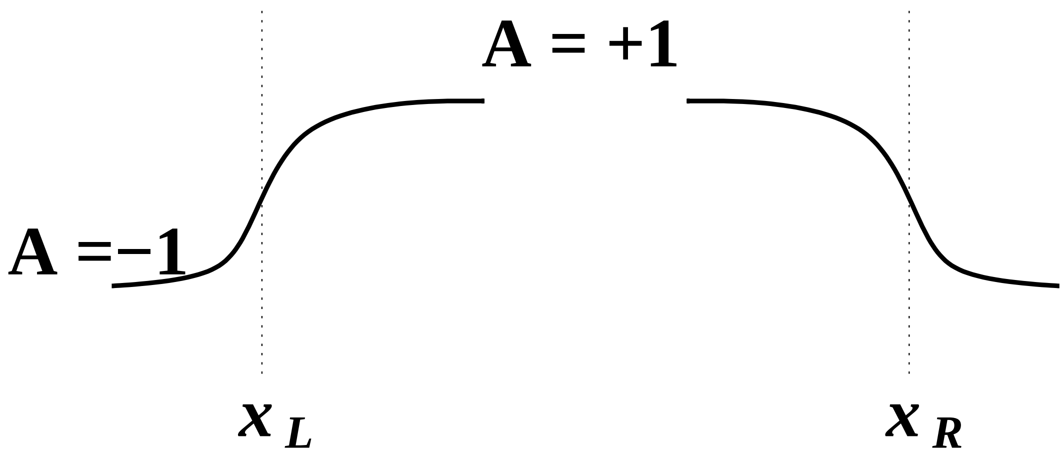

Figure 1: Sketch of the two interacting fronts.

[2]You may want to use Maple or mathematica to solve the integral.
[3]A detailed analysis shows that this is effectively justified.