ESAM 412-1 Methods in Applied Mathematics

Spring Quarter 2008

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Problem Set 1 Due Wednesday, April 16, 2008

1. **The van der Waals Equation and the Cusp Catastrophe** (see Drazin p.65)
   Consider the van der Waals equation

$$\left(P + \frac{a}{V^2}\right)(V - b) = RT \tag{1}$$

   for a non-ideal gas, where V is the volume of the gas, T its temperature, and P its pressure.

   (a) Show that (1) has a triple root for V at the critical point given by $P_c = \frac{a}{27b^2}$ and $T_c = \frac{8a}{27bR}$.

   (b) Sketch the function $P(V)$ for $V > 0$ and $P > 0$ for low and high temperatures, showing that for $T < T_c$ the inverse function $V(P)$ is multi-valued, whereas for $T > T_c$ it is single-valued for $P > 0$.

   (c) Using

$$p = \frac{P}{P_c} - 1, \qquad \rho = \frac{V_c}{V} - 1, \qquad t = \frac{T}{T_c} - 1$$

   show that (1) can be written as

$$\rho^3 + \frac{1}{3}\rho\,(8t + p) + \frac{2}{3}(4t - p) = 0. \tag{2}$$

   Show that (2) has a double zero for ρ along the curve

$$81\,(4t - p)^2 = -\,(8t + p)^3\,.$$

   Sketch that curve. Does this curve have a cusp? For each region defined by this curve mark the number of solutions of (2) .

2. **Perturbed Pitch-Fork Bifurcation**

   In class we considered the equation

$$\partial_t x = \mu x - x^3 + h \tag{3}$$

   to describe a preturbed pitch-fork bifurcation. Often, instead the equation

$$\partial_t y = \nu y - y^3 + \alpha + \beta y^2 \tag{4}$$

   is considered. It makes the fact more apparent that a special feature of the pitch-fork bifurcation is the missing of all even terms in x.

(a) Introduce a suitable variable transformation relating y and x as well as ν and μ that brings (4) into the form (3). What is the connection between the coefficients ν, α, β, and μ and h?

(b) Use (4) to show that among the perturbations of the pitch-fork bifurcation are also diagrams of the form shown below.

Can such a bifurcation diagram also be obtained in (3)? If so, how do the parameters μ and h have to be varied to get it?

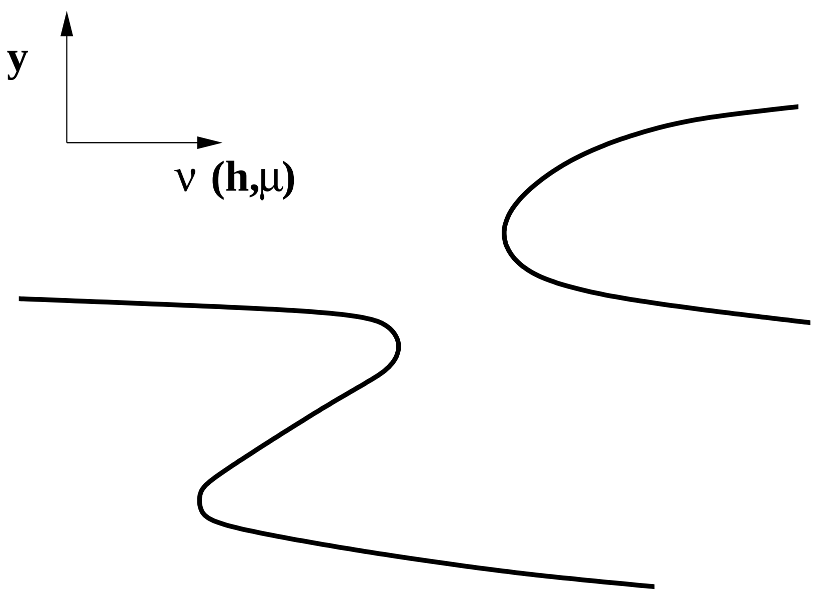

Figure 1: One possible perturbation of the pitch-fork bifurcation

An example of an experimentally investigated perturbed pitch-fork bifurcation can be found in the paper by Aitta, Ahlers, and Cannell, Phys. Rev. Lett. 54 (1985) 673. There the bifurcation becomes subcritical as the parameters are changed.

3. **Non-generic Scaling Laws for SNIC**

Consider a situation in which the saddle-node bifurcation that generates the oscillations is not generic and is described by

$$\dot{x} = \mu + x^{2n} \qquad \text{with } n > 1. \tag{5}$$

Rewrite (5) in terms of the rescaled variables τ and ξ given by $t = \mu^{\alpha}\tau$ and $x = \mu^{\beta}\xi$. Assuming that all three terms in the equation are of the same order, determine α and β. For small μ the period of the oscillation will again be dominated by the time the system spends in the vicinity of $x = 0$ where x satisfies (5). How does the period scale with μ?