ESAM 412-1 Methods in Applied Mathematics

Spring Quarter 2008

Hermann Riecke

Problem Set 2 Due Thursday May 8, 2008

1. **Eigenspaces of Fixed Points**

   (a) Consider the general dynamical system

   $$\dot{\underline{x}} = \underline{\underline{\mathcal{L}}}\,\underline{x} + \underline{\mathcal{N}}(\underline{x})$$

   where $\underline{x}$, $\underline{\underline{\mathcal{L}}}$, and $\underline{\mathcal{N}}$ are real. If $\underline{\underline{\mathcal{L}}}$ has an unrepeated pair of complex eigenvalues λ and λ^*, give a basis for the real eigenspace associated with these eigenvalues in terms of their complex eigenvectors. Show that the vectors $\underline{v}$ in this eigenspace satisfy the equation

   $$\left(\underline{\underline{\mathcal{L}}} - \lambda \underline{\underline{I}}\right)\left(\underline{\underline{\mathcal{L}}} - \lambda^* \underline{\underline{I}}\right)\underline{v} = 0$$

   where $\underline{\underline{I}}$ is the identity matrix.

   (b) Consider now the three-dimensional dynamical system

   $$\begin{aligned}
   \frac{d}{dt}x &= \frac{1}{2}x + \left(\frac{1}{2} - \alpha\right)y + z - x^3, \\
   \frac{d}{dt}y &= \left(\frac{1}{2} - \alpha\right)x + \frac{1}{2}y + z + x^2 - y^3, \\
   \frac{d}{dt}z &= -x - y + (1-\alpha)\,z - z^3.
   \end{aligned}$$

   Determine the stable, unstable, and center eigenspaces of the fixed opint $(0,0,0)$ as a function of α.
   (Technical hint: one eigenvalue σ of $\underline{\underline{\mathcal{L}}}$ is given by $\sigma = \alpha$).

2. **Center Manifold Equation for Pitch-Fork Bifurcation**

   Consider the dynamical system

   $$\begin{aligned}
   \frac{d}{dt}x &= y, \\
   \frac{d}{dt}y &= \mu x - y - x^3 - x^2 y.
   \end{aligned}$$

   (a) Determine the center manifold of the fixed point $(x,y) = (0,0)$ for $|\mu| \ll 1$ to lowest non-trivial order in the center-manifold variables.

   (b) Determine the evolution equation describing the dynamics on $W^{(c)}$ to lowest non-trivial order.

3. **Evolution on the Center Manifold via Multiple Scales**

In class we considered the dynamical system

$$\begin{aligned}\frac{d}{dt}x &= \mu x + xy - \gamma x^3,\\ \frac{d}{dt}y &= -y + x^2 - y^2\end{aligned}$$

and derived the center manifold and the evolution on it up to $\mathcal{O}(x^4)$ and $\mathcal{O}(x^5)$, respectively.

(a) Use multiple scales with an expansion $x(t) = \epsilon x_1(T) + \epsilon^2 x_2(T) + \ldots$ to derive evolution equations for $x_1(T)$ and $x_2(T)$ and the corresponding expressions for y as a function of x.

(b) Compare your results with those of the center-manifold reduction.
(Hint: combine the two equations for x_1 and x_2 to a single equation for x and compare it with the equation obtained in the center-manifold reduction.)

(c) As discussed in class, the quintic term in the equation for x is only relevant if the cubic term is sufficiently small. How do you have to choose γ to keep the cubic and the quintic term simultaneously? Discuss the resulting bifurcation diagram.

4. **Center Manifold for a Hopf Bifurcation**

Consider the dynamical system

$$\begin{aligned}\frac{d}{dt}x &= -x - y + z^{2,} - 2x^2 - \frac{3}{4}y^2,\\ \frac{d}{dt}y &= 2x + (1+\epsilon)y - z^2 + 2x^2 + \frac{3}{4}y^2,\\ \frac{d}{dt}z &= x + 2y - z.\end{aligned}$$

It exhibits a Hopf bifurcation at $\epsilon = 0$ leading to a periodic orbit (limit cycle).

(a) Determine a center manifold that describes the dynamics of this system for $\epsilon = 0$ in the neighborhood of the fixed point $(0, 0, 0)$.

(b) Give the evolution equation for the complex amplitude of the oscillations that describes the dynamics on that center manifold.