Asymptotic and Perturbation Methods in Applied Mathematics
ESAM 420-1

Fall 2002

Hermann Riecke

Take-Home Final

Due Thursday December 12, 2002, Noon.

1. Consider the boundary-value problem
$$\epsilon y'' + a x^\alpha y' + b x^\beta y = 0 \qquad y(0) = A \qquad y(1) = B. \tag{1}$$
If this equation allows a boundary layer at $x = 0$, what will its width be depending on α and β? What will the width be if it is at $x = 1$ or if it is an internal layer at x_0 with $0 < x_0 < 1$?

2. Consider the boundary-value problem
$$\epsilon y'' + (x^2 - \frac{1}{4})y' = 0 \qquad y(0) = 0 \qquad y(2) = 2. \tag{2}$$
   (a) Do you expect any boundary layers, internal, or corner layers? If so, why?
   (b) Determine a solution that is uniformly valid on $0 \leq x \leq 2$ to $\mathcal{O}(1)$, i.e. a one-term expansion.
   (c) Sketch the solution.

3. Consider the boundary-value problem
$$\epsilon y'' - e^{\epsilon y'} - y + x^2 = 0 \qquad y(0) = 0 \qquad y(1) = 0. \tag{3}$$
   (a) Do you expect any boundary layers, internal, or corner layers? If so, why?
   (b) Determine a solution that is uniformly valid on $0 \leq x \leq 1$ to $\mathcal{O}(\epsilon)$. How many terms do you need in the outer and how many in the inner expansion to achieve this accuracy (i.e. neglect terms that are *smaller* than $\mathcal{O}(\epsilon)$).
   (c) Sketch the solution roughly.

4. Consider the periodically forced, weakly damped nonlinear oscillator
$$\frac{d^2y}{dt^2} + y + (\gamma + \alpha y^2)\frac{dy}{dt} + \beta y \frac{dy}{dt} = f\cos(t). \tag{4}$$
   (a) In the absence of forcing, $f = 0$, and without linear damping, $\gamma = 0$, perform a weakly nonlinear analysis of this oscillator, i.e. for small amplitudes use multiple time scales to derive a differential equation ('amplitude equation') for the slow evolution of the amplitude of the oscillator. What would happen in your expansion for y if the amplitude did not satisfy the amplitude equation?
   (b) Now introduce the linear damping term and the forcing term. In order for them to appear simply as additional terms in the amplitude equation they should be weak. How do you have to scale γ and f to achieve this? With this scaling include both terms in the analysis and derive an amplitude equation that includes linear damping and forcing.
   (c) Using the amplitude equation, find a simple stead-state solution for the amplitude. In the absence of linear damping, $\gamma = 0$, how does the amplitude scale with the forcing?
   (d) Without any damping ($\gamma = 0$ and $\alpha = 0$) and without forcing ($f = 0$), find simple periodic solutions for the amplitude. Are these damping terms necessary to keep the amplitude from blowing up when the oscillator is forced?

Note: If you use a symbolic package like Maple or Mathematica include the code with your solution.