Asymptotic and Perturbation Methods in Applied Mathematics
ESAM 420-1

Fall 2002

Hermann Riecke

Problem Set 2

Due October 29, 2002.

1. Reconsider the projectile problem, discussed in class, accounting for air resistance [Holmes, p. 31, 3]. The equation of motion in this case is

$$\frac{d^2x}{dt^2} = -\frac{gR^2}{(x+R)^2} - \frac{k}{x+R}\frac{dx}{dt}$$

where k is a nonnegative constant. Assume here that $x(o) = 0$ and $x'(0) = v_0$.

   (a) What is the nondimensional version of this problem if one uses the same scaling as used in class? The problem now depends on two independent parameters: ϵ (as defined in class) and $\alpha = kv_0/gR$.

   (b) Find a two-term asymptotic expansion of the solution for small ϵ.

   (c) Does the addition of air resistance increase or decrease the flight time ?

2. Find a two-term asymptotic expansion, for small ϵ, of the solution of Laplace's equation on the unit disk, namely

$$\nabla^2 u = 0 \qquad \text{for} \quad x^2 + y^2 < 1,$$

where $u = \exp[\epsilon \cos\theta]$ on the boundary $x^2 + y^2 = 1$. [Holmes, p.31, 1(e)]

3. For plane inviscid, incompressible flows, the equations of motion reduce to solving Laplace's equation for the streamfunction ψ; the velocity field is then obtain by direct differentiation. The flow past a body is therefore described (in polar coordinates) by

$$\nabla^2 \psi = 0$$

with boundary conditions

$$\psi(r,\theta) \to Ur\sin\theta \qquad \text{as } r \to \infty \qquad \text{and} \qquad \psi(r,\theta)|_{\text{surface}} = 0.$$

The boundary conditions reflect the fact that the flow approaching the cylinder is unidirectional with speed U and that there is no penetration inside the body. The velocity components are obtained from

$$v_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta}, \qquad v_\theta = -\frac{\partial\psi}{\partial r}.$$

Consider now the flow past the slightly distorted cylinder $r = a(1 - \epsilon\sin^2\theta)$ with $\epsilon \ll 1$.

   (a) Obtain an approximate solution for ψ correct to $O(\epsilon)$.

   (b) Determine the velocity field (v_r, v_θ).

   (c) Show that along the surface of the cylinder the speed is $\sim 2U\sin\theta + \epsilon U\sin 3\theta$.

4. The cooling of a body by convection and radiation is described by the scaled equation

$$\frac{d\theta}{dt} = -(\theta - \theta_s) - \epsilon(\theta^4 - \theta_s^4)$$

where θ is the temperature and ϵ is the ratio of the radiative to convective heat transfer parameters. In the weak radiation regime, $\epsilon \ll 1$, and θ_s can be practically taken as zero.

(a) Obtain an approximate solution for θ, in this limit, valid up to $\mathcal{O}(\epsilon^3)$ (i.e. the error is $\mathcal{O}(\epsilon^3)$).

(b) Compare your answer to the exact solution.

5. Find a two-term asymptotic expansion, valid for small ϵ, of the solution of the initial value problem

$$t\frac{dx}{dt} = x + \epsilon y$$

$$t\frac{dy}{dt} = -(2 + x)y$$

$$y(1) = e^{-1}, \qquad x(1) = 1$$

Note: the form of the solution of this problem may come out complicated.

6. A saturated liquid of semi-infinite extent ($x > 0$) is initially at the freezing temperature T_f. The temperature at its face $x = 0$ is suddenly dropped to a below freezing temperature T_0 and is maintained at that temperature for all time $t > 0$. As heat is extracted from the liquid it begins to freeze; the freezing front being at $x_f(t)$. If the liquid phase is assumed to remain at temperature T_f throughout the process, one needs only to consider the solid phase region $0 < x < x_f(t)$. The temperature in the solid is determined by solving the heat equation

$$\frac{\partial T}{\partial t} - \kappa\frac{\partial^2 T}{\partial x^2} = 0$$

subject to the boundary conditions

$$T(0,t) = T_0, \qquad T\left(x_f(t), t\right) = T_f$$

where $\kappa = \lambda/\rho c$ is the thermal diffusivity in units m^2/s. Here λ is the thermal conductivity [J/m·s·K], ρ is the density [kg/m^3] and c the specific heat [J/kg·K] of the solid. In addition, an energy balance at the freezing front yields

$$\lambda \left.\frac{\partial T}{\partial x}\right|_{x=x_f} = \rho L \frac{dx_f}{dt}$$

where L is the latent heat [J/kg].

(a) By first introducing the dimensionless variables, $\xi = x/\ell$, $\tau = \kappa t/\ell^2$, $s = x_f/\ell$ and the normalized temperature $\theta = (T_f - T)/(T_f - T_0)$, and then making the transformation $(\xi, \tau) \Rightarrow (\xi, s)$, show that the problem reduces to

$$V\frac{\partial \theta}{\partial s} - \frac{\partial^2 \theta}{\partial \xi^2} = 0 \qquad \text{with} \qquad \theta|_{\xi=0} = 1 \qquad \theta|_{\xi=s} = 0$$

$$V = -\epsilon \left.\frac{\partial \theta}{\partial \xi}\right|_{\xi=s}.$$

where $V = ds/d\tau$, and determine ϵ (known as the Stefan number).

(b) Obtain approximate expressions for θ and V correct to $\mathcal{O}(\epsilon^2)$.

(c) Write down expressions for the physical variables $T(x,t)$ and $x_f(t)$.