Asymptotic and Perturbation Methods in Applied Mathematics
ESAM 420-1

Fall 2002

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Problem Set 4

Due November 19, 2002.

1. In chemical kinetics, genetics and other fields, the rate at which constituents change may be very different. Two typical systems of two-constituents denoted by $u(t)$ and $v(t)$ are:

   (a)

   $$\dot{u} = -u, \quad \epsilon\dot{v} = u - v$$

   $$u(0) = 1, \quad v(0) = 0.$$

   (b)

   $$\dot{u} = -u + \epsilon v, \quad \epsilon\dot{v} = 2 - u - (1 + 3\epsilon)v$$

   $$u(0) = 1, \quad v(0) = 0.$$

   where $\cdot$ denotes differentiation with respect to time. For $\epsilon \ll 1$, determine the first terms in the inner and outer expansions in each case. Sketch the solutions and write down a uniformly valid first order expansion.

2. For the second order linear equation:

   $$\epsilon y'' + a(x)y' + b(x)y = 0$$

   $$y(\alpha) = A, \quad y(\beta) = B,$$

   we have seen in class, that when $a(x)$ does not change sign in the interval $[\alpha, \beta]$, the location of the boundary layer depends on the sign of $a(x)$. When $a(x) = 0$ somewhere in the interval, many things could happen: two boundary layers, one at each end of the interval, are possible; an interior layer exists at some intermediate point; there are no boundary layers at all; the method of matched asymptotic expansions fails to determine all the constants of integration and some "additional" information is needed to complete the solution. Some examples are:

   (a)

   $$\epsilon y'' + xy' + xy = 0$$

   $$y(-1) = e, \quad y(1) = 2e^{-1}.$$

   (b)

   $$\epsilon y'' - x^2 y' - y = 0$$

   $$y(0) = 1, \quad y(1) = 1.$$

(c)

$$\epsilon y'' - (x - \frac{1}{2})y' = 0$$

$$y(0) = A, \quad y(1) = B$$

In each case, determine the first terms in the inner and outer expansions and sketch the solution.