Asymptotic and Perturbation Methods in Applied Mathematics
ESAM 420-1

Fall 2002

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Problem Set 2

Due December 5, 2002.

1. Use the method of Poincaré-Lindstedt to calculate the periodic solution of

$$\frac{d^2u}{dt^2} + u = \epsilon u (\frac{du}{dt})^2$$

$$u(0) = 1, \qquad \frac{du}{dt}(0) = 0$$

and to determine the correction to the frequency up to $\mathcal{O}(\epsilon^2)$ (inclusively).

2. Consider the motion of a particle of mass m in on a rotating parabola $z = \alpha x^2$. The parabola is rotating at an angular velocity Ω about the z-axis. Friction is neglected. The equation of motion is easily obtained using the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0$$

where $\mathcal{L} = T - V$ is the Lagrangian; T is the kinetic energy and V the potential energy. The Lagrangian is expressed in terms of the generalized coordinates q_i and generalized velocities $\dot{q}_i$. For the present problem

$$T = \frac{1}{2} m (\dot{x}^2 + \Omega^2 x^2 + \dot{z}^2), \qquad V = mgz$$

which, when using the constraint $z = \alpha x^2$, can be expressed in terms of a single coordinate x.

(a) Show that the equation of motion is given by

$$(1 + 4\alpha^2 x^2)\ddot{x} + (2g\alpha - \Omega^2)x + 4\alpha^2 \dot{x}^2 x = 0$$

(b) Show by rescaling the variables that for $\Omega < \sqrt{2g\alpha}$ this equation can be rewritten in the form

$$(1 + 4y^2)\ddot{y} + y + 4\dot{y}^2 y = 0$$

(c) Determine the periodic motion for small amplitudes; that is by assuming that $y = O(\epsilon)$ for some $\epsilon \ll 1$, find a first order uniformly valid solution for $y(\tilde{t}, \epsilon)$ where $\tilde{t}$ is a suitably rescaled time.

(d) Write down the solution in terms of the dimensional (physical) variables for the motion of a particle that starts from rest at $z = z_0$ with $z_0 \ll 1$. What is the period of the motion in this case ?

3. **Brusselator *via* Multiple Scales**

Consider the Brusselator model for the oscillatory reaction

$$\begin{aligned} A &\to U, & (1) \\ U &\to P, & (2) \\ U &\to V, & (3) \\ 2U + V &\to 3U, & (4) \end{aligned}$$

where A represents a substance that is abundant and therefore not a limiting factor in the reaction and P is a product that is removed instantly. This results in the reaction-diffusion system

$$\frac{du}{dt} = A - (B+1)\,u + u^2 v, \qquad (5)$$

$$\frac{dv}{dt} = Bu - u^2 v \qquad (6)$$

for the concentrations u and v. To simplify the calculation take $A = 1$.

(a) Determine the fixed point of (5,6), i.e. a time-independent solution.

(b) Rewrite (5,6) in terms of the deviations from that fixed point.

(c) Perform a linear stability analysis of the fixed point of (5,6). Determine the value B_c of B for which the eigenvalues of the linear problem are purely imaginary.

(d) Perform a weakly nonlinear analysis of (5,6). Assume B is close to B_c and the deviations from the fixed point are small as well. Use multiple time scales to derive an evolution equation for the deviations. This will result in the amplitude equation for a Hopf bifurcation.

(e) What happens in this chemical system as B is varied across B_c?

4. Use multiple scales to obtain an evolution equation for the amplitude of the oscillations described by

$$\begin{aligned} \ddot{u} + u &= \epsilon(1 - v)\dot{u} \\ \alpha\dot{v} + v &= u^2 \end{aligned}$$

where α is a constant and as usually $\epsilon \ll 1$.