

$$\begin{aligned} \vec{r}_1 &= t \left( \frac{\vec{A} + \vec{B}}{2} \right) \\ \vec{r}_2 &= \vec{B} + t \left( \frac{1}{2} \vec{A} - \vec{B} \right) \quad \text{Assume} \\ \vec{r}_3 &= \vec{A} + t \left( \frac{1}{2} \vec{B} - \vec{A} \right) \quad \vec{r}_1 = \vec{r}_2 = \vec{r}_3 \end{aligned}$$

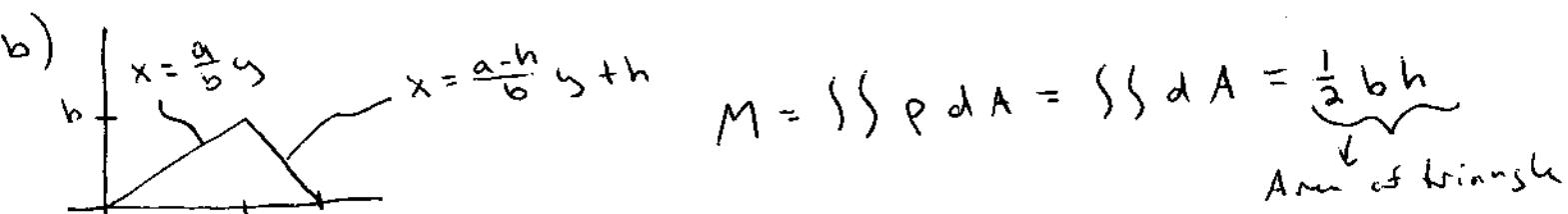
$$u + \vec{r}_1 = \vec{r}_2 \quad t \left( \frac{\vec{A} + \vec{B}}{2} \right) = \vec{B} + t \left( \frac{1}{2} \vec{A} - \vec{B} \right)$$

$$t \left( \frac{\vec{B}}{2} + \vec{B} \right) = \vec{B} \Rightarrow \vec{B} \left[ \left( \frac{3}{2} \right) t - 1 \right] = 0 \Rightarrow t = \frac{2}{3}$$

If we plug  $t$  into  $\vec{r}_1$ ,  $\vec{r}_2$ , &  $\vec{r}_3$  we see that

$$\vec{r}_1 = \frac{1}{3} (\vec{A} + \vec{B}) \quad \vec{r}_2 = \frac{1}{3} (\vec{A} + \vec{B}) \quad \vec{r}_3 = \frac{1}{3} (\vec{A} + \vec{B})$$

This is where the medians intersect, i.e. the centroid.



$$\bar{x} = \frac{2}{bh} \iint_{\text{triangle}} x \, dy \, dx = \frac{2}{bh} \iint_0^b \left[ \frac{1}{2} \left( \frac{a-h}{b}y + h \right)^2 - \frac{1}{2} \left( \frac{a}{b}y \right)^2 \right] dy \, dx$$

$$= \frac{1}{bh} \iint_0^b \left( \frac{h^2}{b^2} - \frac{2ah}{b^2} \right) y^2 + \frac{2ah}{b} y - \frac{2h^2}{b} y + h^2 dy \, dx$$

$$= \frac{1}{bh} \left[ -\frac{2ahb^3}{3b^3} + \frac{h^2}{3b^2} b^3 + \frac{2ahb^2}{2b} - \frac{2h^2b^2}{b} + h^2b \right] = \frac{a}{3} + \frac{h}{3}$$

$$\bar{y} = \frac{2}{bh} \iint_0^b y \, dy \, dx = \frac{2}{bh} \iint_0^b \left[ \frac{a-h}{b} y^2 + hy - \frac{a}{b} y^2 \right] dy \, dx$$

$$= \frac{2}{bh} \left[ -\frac{h}{b} y^3 + \frac{hy^2}{2} \right]_0^b = \frac{2}{bh} \left[ \frac{h^4}{6} \right] = \frac{b}{3}$$

$$\langle \bar{x}, \bar{y} \rangle = \left\langle \frac{a}{3} + \frac{h}{3}, \frac{b}{3} \right\rangle = \frac{\langle a, b \rangle}{3} + \frac{\langle h, 0 \rangle}{3}$$

Therefore the center of mass of this triangle is the same as the centroid.

$$② \frac{\partial (R, \theta, \phi)}{\partial (x, y, z)} = \begin{vmatrix} \frac{\partial R}{\partial x} & \frac{\partial R}{\partial y} & \frac{\partial R}{\partial z} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} & \frac{\partial \theta}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix} \quad \begin{aligned} x &= R \cos \theta \sin \phi & PS 3-2 \\ y &= R \sin \theta \sin \phi \\ z &= R \cos \phi \end{aligned}$$

$$R = \sqrt{x^2 + y^2 + z^2} \Rightarrow \frac{\partial R}{\partial x} = \frac{x}{\sqrt{x^2 + y^2 + z^2}}, \quad \frac{\partial R}{\partial y} = \frac{y}{\sqrt{x^2 + y^2 + z^2}}, \quad \frac{\partial R}{\partial z} = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

$$\frac{y}{x} = \tan \theta \Rightarrow \frac{\partial \theta}{\partial x} = -\frac{y}{x^2 + y^2}, \quad \frac{\partial \theta}{\partial y} = \frac{x}{x^2 + y^2}, \quad \frac{\partial \theta}{\partial z} = 0$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$x^2 + y^2 = R^2 \sin^2 \theta \Rightarrow \frac{\partial \phi}{\partial x} = \frac{x z}{\sqrt{x^2 + y^2} (x^2 + y^2 + z^2)}$$

$$\tan^2 \phi = \frac{x^2 + y^2}{z^2} \Rightarrow \frac{\partial \phi}{\partial y} = -\frac{\sqrt{x^2 + y^2}}{(x^2 + y^2 + z^2)}$$

$$\frac{\partial \phi}{\partial z} = \frac{y^2}{\sqrt{x^2 + y^2} (x^2 + y^2 + z^2)}$$

where we have  $x^2 + y^2 + z^2$ , replace with  $R$

$$\frac{\partial (R, \phi, \theta)}{\partial (x, y, z)} = \begin{vmatrix} \frac{x}{R} & \frac{y}{R} & \frac{z}{R} \\ \frac{x z}{\sqrt{x^2 + y^2} R^2} & \frac{y z}{\sqrt{x^2 + y^2} R^2} & -\frac{\sqrt{x^2 + y^2}}{R^2} \\ -\frac{y}{x^2 + y^2} & \frac{x}{x^2 + y^2} & 0 \end{vmatrix} = \frac{\frac{x}{R} \left(0 + \frac{x}{R^2 \sqrt{x^2 + y^2}}\right) - \frac{y}{R} \left(0 - \frac{y}{R^2 \sqrt{x^2 + y^2}}\right)}{+ \frac{z}{R} \left(\frac{x^2 z}{(x^2 + y^2)^{3/2} R^2} + \frac{y^2 z}{(x^2 + y^2)^{3/2} R^2}\right)}$$

$$= \frac{x^2 + y^2}{R^3 \sqrt{x^2 + y^2}} + \frac{z^2 (x^2 + y^2)}{R^3 (x^2 + y^2)^{3/2}} = \frac{x^2 + y^2 + z^2}{R^3 (x^2 + y^2)^{1/2}} = \frac{1}{R \sqrt{x^2 + y^2}}$$

Note  $x^2 + y^2 = R^2 \sin^2 \theta \Rightarrow \sqrt{x^2 + y^2} = R \sin \theta$

$$= \boxed{\frac{\partial (R, \phi, \theta)}{\partial (x, y, z)} = \frac{1}{R^2 \sin \theta} = \frac{1}{\frac{\partial (x, y, z)}{\partial (R, \phi, \theta)}}}$$

$$③ \vec{F} = GM \iiint \frac{\rho(x,y,z) \hat{R}}{R^2} dx dy dz \quad R^2 = x^2 + y^2 + z^2 \quad \hat{R} = \frac{\vec{R}}{R}$$
 note - inverted cone has form  $z^2 = \frac{b}{a} (x^2 + y^2)$

a)  $\vec{F} = GM \rho_0 \iiint \frac{\langle x, y, z \rangle}{R^2} dx dy dz = \langle F_x, F_y, F_z \rangle$ 
 $\vec{F} = GM \rho_0 \iiint \frac{\langle R \cos \theta \sin \phi, R \sin \theta \sin \phi, R \cos \phi \rangle}{R^2} \frac{1}{R^2} R^2 \sin \theta d\theta d\phi$

note  $\int_0^{2\pi} \cos \theta d\theta = \int_0^{2\pi} \sin \theta d\theta = 0 \therefore$  if we integrate in  $\theta$  first

$\vec{F} = GM \rho_0 \iiint \langle 0, 0, \cos(\phi) 2\pi \rangle \sin \phi d\phi dR$

$\therefore \vec{F}$  only has a  $\hat{z}$  component

$\vec{F} = GM \rho_0 \hat{z} \left\{ \int_0^{\tan^{-1}(a/b)} \sin \phi \cos \phi d\phi \right\} 2\pi dR$

$\vec{F} = GM \rho_0 2\pi \hat{z} \left\{ \int_0^{\tan^{-1}(a/b)} b \sin \phi d\phi = GM \rho_0 2\pi \hat{z} b \cos \phi \right\}_0$

$\vec{F} = 2\pi \rho_0 GM \hat{z} \left( 1 - \frac{b}{\sqrt{a^2+b^2}} \right)$

b)  $\vec{F} = GM \rho_0 \left\{ \int_0^a dr \left\{ \int_0^{b/r} dz \right\} \frac{\langle r \cos \theta, r \sin \theta, z \rangle}{\sqrt{r^2+z^2}} \cdot \frac{r d\theta}{z^2+r^2}$

again, integrate in  $\theta$  first

$\vec{F} = GM \rho_0 2\pi \hat{z} \left\{ \int_0^a dr \left\{ \int_0^{b/r} dz \frac{zr}{(z^2+r^2)^{3/2}} \right\} \right\}$

$= 2\pi GM \rho_0 \hat{z} \left\{ \left[ \frac{-r}{\sqrt{r^2+z^2}} \right]_{b/r}^a \right\} = 2\pi GM \rho_0 \hat{z} \left\{ \left[ \frac{a}{\sqrt{1-(\frac{b}{a})^2}} - \frac{r}{\sqrt{r^2+b^2}} \right] dr \right\}$

$= 2\pi GM \rho_0 \hat{z} \left[ \frac{a}{\sqrt{1+(\frac{b}{a})^2}} - \frac{1}{2} \int \frac{du}{u^{1/2}} \right] = 2\pi GM \rho_0 \hat{z} \left[ \frac{a^2}{\sqrt{a^2+b^2}} - \sqrt{a^2+b^2} + b \right]$

$= 2\pi GM \rho_0 \hat{z} \left[ \frac{a^2}{\sqrt{a^2+b^2}} - \frac{a^2+b^2}{\sqrt{a^2+b^2}} + b \right] = 2\pi GM \rho_0 b \left[ 1 - \frac{b}{\sqrt{a^2+b^2}} \right] \hat{z}$

$$\textcircled{4} \quad (r, \theta, z) \quad (r-a)^2 + z^2 = b^2$$

$$\text{a)} \int_0^{2\pi} d\theta \int_{-b}^b dz \left( r dr \right) = \int_0^{2\pi} d\theta \int_{-b}^b dz \left[ (\sqrt{b^2-z^2}+a)^2 - (-\sqrt{b^2-z^2}+a)^2 \right]$$

$$= \int_0^{2\pi} d\theta \int_{-b}^b dz \left[ 4ab \sqrt{1-\left(\frac{z}{b}\right)^2} \right]$$

$$= \int_0^{2\pi} d\theta \int_{-\pi/2}^{\pi/2} dz \left[ 4ab^2 \cos^2(\theta) \right] = \int_0^{2\pi} d\theta 4ab^2 \left[ \frac{1}{2}\theta + \frac{1}{2} \sin(2\theta) \right]_{-\pi/2}^{\pi/2}$$

$$= \boxed{ab^2 2\pi^2}$$

$$\text{b)} \quad r = a + \rho \cos \phi \quad \theta = \phi \quad z = \rho \sin \phi \quad (r, \theta, z) \rightarrow (\rho, \phi, \phi)$$

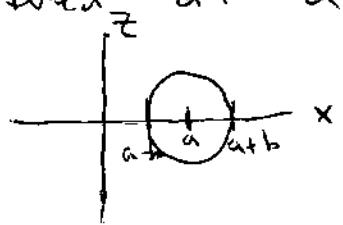
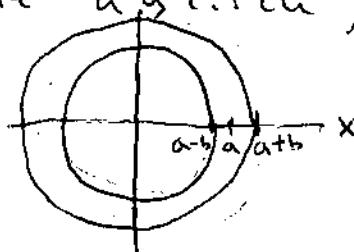
$$\frac{\partial(r, \theta, z)}{\partial(\rho, \phi, \phi)} = \begin{vmatrix} \cos \phi & 0 & -\rho \sin \phi \\ 0 & 1 & 0 \\ \sin \phi & 0 & \rho \cos \phi \end{vmatrix} = \cos \phi (\rho \cos \phi) + -\rho \sin \phi (-\sin \phi) = \rho$$

$$\int_0^{2\pi} \int_0^{2\pi} d\theta \int_0^b (a + \rho \cos \phi) \rho d\rho = \int_0^{2\pi} \int_0^{2\pi} d\theta \int_0^b \left[ \frac{a}{2} \rho^2 + \frac{\rho^3}{3} \cos \phi \right]_0^b$$

$$= \int_0^{2\pi} d\theta \int_0^{2\pi} \left[ \frac{a}{2} b^2 \phi + \frac{b^3}{3} \sin \phi \right]_0^b$$

$$= \boxed{ab^2 2\pi^2}$$

A torus is like a donut, so in the x-y plane it looks like a washer with radius  $b$ , in the z-x plane it looks like a circle centered at  $a$



(S) See 7.3 / 24, 25 PS 3 - 5  
 24)  $\int_0^a \int_0^{\pi/2} \int_0^{\pi/2} dz dy dx$   $x^2$

$\frac{1}{8}$  sphere  $x > 0, y > 0, z > 0$ , sphere  
 centered at origin with radius  $a$

$$\begin{aligned} & \int_0^{\pi/2} \int_0^{\pi/2} \int_0^a r^4 \cos^2 \theta \sin^3 \phi dr = \int_0^{\pi/2} \int_0^{\pi/2} d\theta \cos^2 \theta \sin^3 \theta \frac{1}{5} a^5 \\ & = \frac{1}{5} a^5 \int_0^{\pi/2} \cos^2 \theta d\theta \int_0^{\pi/2} (\sin \theta - \sin \theta \cos^2 \phi) d\phi = \frac{1}{5} a^5 \int_0^{\pi/2} \cos^2 \theta d\theta [\cos \phi + \frac{1}{3} \cos^3 \phi]_0^{\pi/2} \\ & = \frac{1}{5} a^5 \left[ \frac{1}{2} + \frac{1}{2} \cos(2\theta) \right] d\theta \Big|_0^{\pi/2} = \frac{2}{15} a^5 \left[ \frac{1}{2} + \frac{1}{4} \sin(2\theta) \right]_0^{\pi/2} = \boxed{\frac{a^5 \pi}{30}} \end{aligned}$$

25)  $\int_0^2 \int_0^{\pi/2} \int_0^{\pi/2} dz dy dx$   $\frac{(2x-z^2)^{1/2} (2z-z^2-y^2)^{1/2}}{(x^2+y^2+z^2)^{1/2}}$   $\frac{1}{4}$  "slice" of a sphere (like an orange wedge) with  $x > 0, y > 0, 0 > z > 1$  & the original sphere is centered at  $(0, 0, 1)$  with radius 1

$$\begin{aligned} & \int_0^{\pi/2} \int_0^{\pi/2} \int_0^{2\cos \phi} \frac{1}{r} r^2 dr \\ & = \int_0^{\pi/2} \int_0^{\pi/2} \sin \phi \left[ \frac{1}{2} r^2 \right]_0^{2\cos \phi} = \int_0^{\pi/2} \int_0^{\pi/2} 2 \sin \phi \cos^2 \phi d\phi \\ & = \int_0^{\pi/2} d\phi \left[ -\frac{2}{3} \cos^3 \phi \right]_0^{\pi/2} = \int_0^{\pi/2} d\phi \cdot \frac{2}{3} = \boxed{\frac{\pi}{3}} \end{aligned}$$

note  $x^2 + y^2 + z^2 - 2z \leq 0 \Rightarrow$   
 $r^2 - 2r \cos \phi \leq 0$

$$\textcircled{6} \quad \int_S z \, dS \quad \underbrace{\begin{array}{l} z = xy \\ 0 < x < 1, 0 < y < 1 \end{array}}_S$$

$$dS = \sqrt{1+z_x^2+z_y^2} \, dx dy \quad z_x = y \quad z_y = x \quad ds = \sqrt{1+x^2+y^2} \, dx dy$$

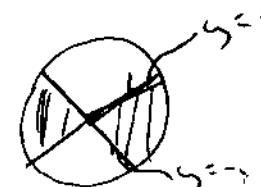
$$\begin{aligned} I &= \iint_S xy \sqrt{1+x^2+y^2} \, dx dy \quad u = 1+x^2+y^2 \quad du = 2x \, dx \\ &= \frac{1}{2} \int_0^1 y \, dy \int u^{1/2} \, du = \frac{1}{2} \int_0^1 y \left[ \frac{2}{3} u^{3/2} \right] \, dy = \frac{1}{3} \int_0^1 y \, dy \left[ (1+x^2+y^2)^{3/2} \right] \\ &= \frac{1}{3} \int_0^1 y \left[ (2+y^2)^{3/2} - (1+y^2)^{3/2} \right] \, dy \quad \text{let } u_1 = y^2 + a_1; \quad du = 2y \, dy \\ &= \frac{1}{6} \left[ \int u_1^{3/2} \, du_1 + \int u_2^{3/2} \, du_2 \right] = \frac{1}{6} \frac{2}{5} \left[ (2+y^2)^{5/2} - (1+y^2)^{5/2} \right] \\ &= \frac{1}{15} \left[ 3^{5/2} - 2^{5/2} - 2^{5/2} + 1 \right] = \boxed{\frac{1}{15} [3^{5/2} - 2^{7/2} + 1]} \end{aligned}$$

$$\textcircled{7} \quad z = x^2 - y^2 \quad \text{above } x^2 + y^2 \leq 1$$

$$ds = \sqrt{1+z_x^2+z_y^2} \, dx dy \quad z_x = 2x \quad z_y = -2y \quad ds = \sqrt{1+4x^2+4y^2} \, dx dy$$

Let's use polar coordinates

$$ds = \sqrt{1+4r^2} r dr d\theta \quad R:$$



$$\begin{aligned} I &= \int_0^{2\pi} d\theta \int_0^1 \sqrt{1+4r^2} r \, dr \, d\theta \quad u = 1+4r^2 \quad du = 8r \, dr \\ &= \int_0^{2\pi} d\theta \int \frac{1}{8} u^{1/2} \, du = \frac{1}{8} \int_0^{2\pi} d\theta \left[ \frac{1}{8} \frac{2}{3} u^{3/2} \right] = \frac{1}{32} \int_0^{2\pi} d\theta \left[ \frac{1}{8} \frac{2}{3} (1+4r^2)^{3/2} \right] \end{aligned}$$

$$I = \frac{2\pi}{2} \left[ \frac{1}{12} (5^{3/2} - 1) \right]$$

$$\boxed{I = \frac{\pi}{12} (5^{3/2} - 1)}$$