

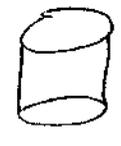
1) $\vec{E} = q \frac{\hat{R}}{R^2}$ $R = \sqrt{x^2 + y^2 + z^2}$ $\hat{R} = \frac{\langle x, y, z \rangle}{R}$

a) $\iint_S \vec{E} \cdot \vec{n} \, ds$ S : Sphere of radius A , centered at origin
 note $\vec{n} = \hat{R}$ since the normal to a sphere is in the radial direction

Flux = $\iint_S \vec{E} \cdot \vec{n} \, ds = \iint_S q \frac{\hat{R}}{R^2} \cdot \hat{R} \, ds$ on the surface of the sphere, $R = A$

$= \frac{q}{A^2} \underbrace{\iint_S ds}_{\text{Surface area of sphere}} = \frac{q}{A^2} (4\pi A^2) \therefore \boxed{\text{Flux} = 4\pi q}$

b) S : cylinder of radius a , & height $2b$
 note break up cylinder into side & top/bottom



Flux_S = $\iint_S \vec{E} \cdot \vec{n} \, ds$ note, on side $\hat{n} = \langle \cos\theta, \sin\theta, 0 \rangle$
 $ds = a \, d\theta \, dz$

Flux_S = $\iint_S \vec{E} \cdot \vec{n} \, ds = \int_0^{2\pi} \int_{-b}^b \frac{q \langle a \cos\theta, a \sin\theta, z \rangle}{(a^2 + z^2)^{3/2}} \cdot \langle \cos\theta, \sin\theta, 0 \rangle a \, dz \, d\theta$

$= \int_0^{2\pi} \int_{-b}^b \frac{q a^2}{(a^2 + z^2)^{3/2}} dz \, d\theta = 2\pi q \int_{-b}^b \frac{a \hat{R} \, dz}{(a^2 + z^2)^{3/2}}$ let $\frac{z}{a} = \tan u$
 $dz = a \sec^2 u \, du$

$= 2\pi q \int \frac{\sec^2 u \, du}{(\sec^2 u)^{3/2}} = 2\pi q \int \cos u \, du = 2\pi q \sin u$ $z = \frac{\sqrt{z^2 + a^2} \sin u}{a}$

$= 2\pi q \left. \frac{z}{\sqrt{z^2 + a^2}} \right|_{-b}^b = \boxed{4\pi q \frac{b}{\sqrt{b^2 + a^2}}}$ Flux through side

Flux_T = $\int_0^{2\pi} \int_0^a q \frac{\langle x, y, z \rangle \cdot \langle 0, 0, 1 \rangle}{(x^2 + y^2 + z^2)^{3/2}} r \, dr \, d\theta$ note, on top $\hat{n} = \langle 0, 0, 1 \rangle$
 $ds = r \, dr \, d\theta$

$= \int_0^{2\pi} \int_0^a q \frac{\langle r \cos\theta, r \sin\theta, b \rangle \cdot \langle 0, 0, 1 \rangle}{(r^2 + b^2)^{3/2}} r \, dr \, d\theta = 2\pi q b \int_0^a \frac{r}{(r^2 + b^2)^{3/2}} dr$

$= 2\pi q b \left[\frac{1}{\sqrt{r^2 + b^2}} \right]_0^a = \boxed{2\pi q b \left[\frac{1}{b} - \frac{1}{\sqrt{a^2 + b^2}} \right]}$ Flux through top, same as Flux through bottom ($b \rightarrow -b$, $\vec{n} \rightarrow -\vec{n}$)

$\therefore \text{Flux} = \text{Flux}_S + 2 \text{Flux}_T = \boxed{4\pi q}$

c) Gauss's Law states that Flux is proportional to charge inside the surface. Since, I have the same charge in a) & b) I expect the same flux, which indeed we calculated.

(iv) $\vec{F} = \sin(y+z)\hat{i} + e^z\hat{j} + xy\hat{k}$

$\text{div } \vec{F} = \vec{\nabla} \cdot \vec{F} = 0$

$\text{curl } \vec{F} = \vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin(y+z) & e^z & xy \end{vmatrix} = \hat{i}(x - e^z) - \hat{j}(y + \cos(y+z)) + \hat{k}(0 - \cos(y+z))$

2) $\oint_S \vec{F} \cdot \hat{n} \, ds$

$\vec{F} = (x^2 + y^2 + z^2)^n (x\hat{i} + y\hat{j} + z\hat{k})$
 $\hat{n} = \frac{\langle x, y, z \rangle}{\sqrt{x^2 + y^2 + z^2}}$

S: surface of sphere with radius a

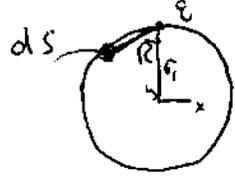
Flux = $\oint_S (a^2)^n \frac{\langle x, y, z \rangle \cdot \langle x, y, z \rangle}{\sqrt{x^2 + y^2 + z^2}} \, ds = a^{2n} \oint_S \underbrace{\sqrt{x^2 + y^2 + z^2}}_{a^2} \, ds$

$= a^{2n+1} \oint_S ds$
 surface area of sphere
 $4\pi a^2$

$\text{Flux} = 4\pi a^{2n+3}$

8) $\vec{F} = \frac{q_1 q_2}{4\pi\epsilon_0} \frac{\vec{R}}{R^3}$ where $\vec{R} = \vec{r}_2 - \vec{r}_1$ ϵ_0 is dielectric constant

note charge density $\rho_c = \frac{nq}{4\pi r_1^2}$ where n = number of electrons, q = charge of each electron, r_1 = radius of sphere at a given time



note we have to "sum" over all possible \vec{R} (at a given time) to find the force due to all the electrons repelling each other
 note need, $\vec{F} = m\vec{a} \Rightarrow \vec{F} = m \frac{d^2\vec{r}}{dt^2}$, you can solve this equation for r_1 , after we find \vec{F} in terms of r_1 .

for a fixed $dF = \frac{nq^2}{4\pi r_1^2 \epsilon_0} \cdot \frac{\vec{R}}{R^3} \, ds$ $ds = r_1^2 \sin\phi \, d\phi \, d\theta$

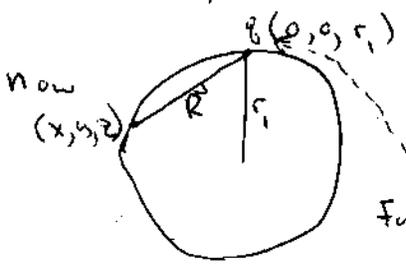
note - the net force will act only in the radial, \hat{R} direction

note - the way we choose to consider this is \uparrow fix one electron at $(0, 0, r_1)$ and how this charge interacts with any other charges, specifically any given $ds \cdot \rho_c$ i.e. the amount of charge we expect to find on any small cross-section of the surface

this is why we have

(all q of same charge)

$$dF = \frac{nq}{4\pi\epsilon_0 r_1^2} \frac{q_2}{4\pi\epsilon_0} \frac{\vec{R}}{R^3} \sin\phi d\phi d\theta$$



$$\vec{R} = \langle 0-x, 0-y, r_1-z \rangle$$

For our charge of interest $\hat{R} = \hat{R}_z$ (because the radial direction for this charge is along the z-axis)

so $\vec{R}_z = r_1 - r_1 \cos\phi$ & $|\vec{R}| = \sqrt{r_1^2 \cos^2\theta \sin^2\phi + r_1^2 \sin^2\theta \sin^2\phi + (r_1 - r_1 \cos\phi)^2}$

$$|\vec{R}| = r_1 \sqrt{\sin^2\phi + (1 - \cos\phi)^2} = r_1 \sqrt{2 - 2\cos\phi}$$

now, we have

$$\int dF = \int_0^{2\pi} \int_0^\pi \frac{nq^2}{16\pi^2\epsilon_0} \frac{r_1(1-\cos\phi)}{r_1^3(1-\cos\phi)^{3/2}} \sin\phi d\phi d\theta$$

$$F = \frac{nq^2}{8\pi\epsilon_0 r_1} \int_0^2 u^{-1/2} du = \frac{nq^2}{4\pi\epsilon_0 r_1^2} u^{1/2} \Big|_0^2 = \frac{\sqrt{2} nq^2}{4\pi\epsilon_0 r_1^2} = F(r_1)$$

now

$$\frac{\sqrt{2} nq^2}{4\pi\epsilon_0 r_1^2} = m \frac{dr_1^2}{dt^2} \Rightarrow \alpha = r_1^2 \frac{d^2 r_1}{dt^2}$$

OK consider $\frac{\alpha}{r_1^2} = \frac{d^2 r_1}{dt^2} \Rightarrow \frac{\alpha}{r_1^2} = \frac{dv}{dt}$ assume $v = v(r)$

make the change of variables $v = v(r)$

now $\frac{d}{dt} v = \frac{dv}{dr_1} \cdot \frac{dr_1}{dt} = v \frac{dv}{dr_1}$

$$\Rightarrow \frac{\alpha}{r_1^2} = v \frac{dv}{dr_1} \Rightarrow \int \frac{\alpha dr_1}{r_1^2} = \int v dv = -\frac{\alpha}{r_1} + C = \frac{1}{2} v^2$$

assume $v(r_0) = 0$ $-\frac{\alpha}{r_0} + C = 0 \Rightarrow C = \frac{\alpha}{r_0}$ $\frac{1}{2} v^2 = \frac{\alpha}{r_0} - \frac{\alpha}{r_1} \Rightarrow \sqrt{\frac{2\alpha(r_1 - r_0)}{r_0 r_1}} = v = \frac{dr_1}{dt}$

$$\frac{dr_1}{dt} = \sqrt{\frac{2\alpha(r_1 - r_0)}{r_0 r_1}} \Rightarrow \int \frac{dr_1}{\sqrt{\frac{2\alpha(r_1 - r_0)}{r_0 r_1}}} = \int dt$$

$$\Rightarrow \int_{r_0}^{2r_0} \sqrt{\frac{r_1 - r_0}{2\alpha}} dr_1 = 2\alpha \int u^{1/2} du = \frac{4}{3} \alpha \left(\frac{r_1 - r_0}{2\alpha} \right)^{3/2} \Big|_{r_0}^{2r_0} = T$$

$$T = \frac{4}{3} \alpha \left(\frac{r_0}{2\alpha} \right)^{3/2}$$

$$T = \frac{\sqrt{2} nq^2}{3 \cdot 4\pi\epsilon_0 m \cdot 24} \left(\frac{r_0 \cdot 4\pi\epsilon_0 m}{nq^2} \right)^{3/2}$$

11) $\vec{F} = 3x^2y^2 \sin z \hat{i} + 2x^3y \sin z \hat{j} + (x^3y^2 \cos z + e^z) \hat{k}$

$\phi_x = 3x^2y^2 \sin z \Rightarrow \phi = x^3y^2 \sin(z) + c(y, z)$

$\phi_y = 2x^3y \sin z \Rightarrow \phi = x^3y^2 \sin(z) + c(x, z)$

$\phi_z = x^3y^2 \cos z + e^z \Rightarrow \phi = x^3y^2 \sin(z) + e^z + c(x, y)$

$\Rightarrow \vec{F} = \nabla \phi$ where $\phi = x^3y^2 \sin(z) + e^z + c$

Since \vec{F} is conservative, the integral around any closed curve will equal zero.

Sec 8.4 / 1, 5, 14, 11

1) $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

i) $\text{div } \vec{r} = \nabla \cdot \vec{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$

ii) $\text{curl } \vec{r} = \nabla \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \hat{i} \left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z} \right) - \hat{j} \left(\frac{\partial z}{\partial x} - \frac{\partial x}{\partial z} \right) + \hat{k} \left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y} \right) = \vec{0}$

iii) $\nabla r = \frac{\partial}{\partial x} (x^2+y^2+z^2)^{1/2} \hat{i} + \frac{\partial}{\partial y} (x^2+y^2+z^2)^{1/2} \hat{j} + \frac{\partial}{\partial z} (x^2+y^2+z^2)^{1/2} \hat{k}$

$\nabla r = \frac{x}{(x^2+y^2+z^2)^{1/2}} \hat{i} + \frac{y}{(x^2+y^2+z^2)^{1/2}} \hat{j} + \frac{z}{(x^2+y^2+z^2)^{1/2}} \hat{k}$

$\nabla r = \frac{\vec{r}}{r}$

recall $r = |\vec{r}|$

iv) $\nabla^2 (\log r) = \nabla^2 \left(\frac{1}{2} \ln(x^2+y^2+z^2) \right)$

$= \frac{\partial}{\partial x} \left(\frac{x}{x^2+y^2+z^2} \right) + \frac{\partial}{\partial y} \left(\frac{y}{x^2+y^2+z^2} \right) + \frac{\partial}{\partial z} \left(\frac{z}{x^2+y^2+z^2} \right)$

$= \frac{x^2+y^2+z^2 - 2x^2}{(x^2+y^2+z^2)^2} + \frac{x^2+y^2+z^2 - 2y^2}{(x^2+y^2+z^2)^2} + \frac{x^2+y^2+z^2 - 2z^2}{(x^2+y^2+z^2)^2}$

$= \frac{1}{(x^2+y^2+z^2)} = \frac{1}{r^2}$

5) Show that $\vec{\nabla} \times (\varphi_1 \vec{\nabla} \varphi_2) = \vec{\nabla} \varphi_1 \times \vec{\nabla} \varphi_2$

$$\vec{\nabla} \times (\varphi_1 \vec{\nabla} \varphi_2) = \varphi_1 \underbrace{\vec{\nabla} \times (\vec{\nabla} \varphi_2)}_{= \vec{0}} + \vec{\nabla} \varphi_1 \times (\vec{\nabla} \varphi_2) = \boxed{\vec{\nabla} \varphi_1 \times \vec{\nabla} \varphi_2} \quad \checkmark$$

Show that $\vec{\nabla} \times (\varphi_1 \vec{\nabla} \varphi_2 + \varphi_2 \vec{\nabla} \varphi_1) = \vec{0}$ $\vec{\nabla} \cdot \vec{c} + \vec{c} \cdot \vec{\nabla}$

$$\begin{aligned} \vec{\nabla} \times (\varphi_1 \vec{\nabla} \varphi_2 + \varphi_2 \vec{\nabla} \varphi_1) &= \vec{\nabla} \times (\varphi_1 \vec{\nabla} \varphi_2) + \vec{\nabla} \times (\varphi_2 \vec{\nabla} \varphi_1) \\ &= \vec{\nabla} \varphi_1 \times \vec{\nabla} \varphi_2 + \vec{\nabla} \varphi_2 \times \vec{\nabla} \varphi_1 \\ &= \vec{\nabla} \varphi_1 \times \vec{\nabla} \varphi_2 - \vec{\nabla} \varphi_1 \times \vec{\nabla} \varphi_2 = \boxed{\vec{0}} \quad \checkmark \end{aligned}$$

Show that $\vec{\nabla} \times (\varphi_1 \vec{\nabla} \varphi_2 - \varphi_2 \vec{\nabla} \varphi_1) = 2 \vec{\nabla} \varphi_1 \times \vec{\nabla} \varphi_2$

$$\begin{aligned} \vec{\nabla} \times (\varphi_1 \vec{\nabla} \varphi_2 - \varphi_2 \vec{\nabla} \varphi_1) &= \vec{\nabla} \times (\varphi_1 \vec{\nabla} \varphi_2) - \vec{\nabla} \times (\varphi_2 \vec{\nabla} \varphi_1) \\ &= \vec{\nabla} \varphi_1 \times \vec{\nabla} \varphi_2 - \underbrace{\vec{\nabla} \varphi_2 \times \vec{\nabla} \varphi_1}_{= -\vec{\nabla} \varphi_1 \times \vec{\nabla} \varphi_2} \\ &= \boxed{2 \vec{\nabla} \varphi_1 \times \vec{\nabla} \varphi_2} \end{aligned}$$

11) $\vec{\nabla} \cdot \vec{p} = 0 \Rightarrow \vec{\nabla} \times \vec{p} = 0$

because with $\vec{p} = \vec{\nabla} \phi$, we have

$$\vec{\nabla} \cdot (\vec{\nabla} \phi) = \boxed{\nabla^2 \phi = 0} \quad \text{Laplace's Equation}$$

$$\vec{\nabla} \times (\vec{\nabla} \phi) = \vec{0}$$

14) i) $\vec{r} \rightarrow$ radial vector, $\vec{a} \rightarrow$ constant vector

$$\vec{\nabla} \cdot (\vec{a} \times \vec{r}) = \vec{r} \cdot (\vec{\nabla} \times \vec{a}) - \vec{a} \cdot (\vec{\nabla} \times \vec{r}) = 0$$

$$\text{ii) } \vec{\nabla} \times (r^2 \vec{a}) = r^2 \underbrace{\vec{\nabla} \times \vec{a}}_{= \vec{0}} + \underbrace{\vec{\nabla} r^2 \times \vec{a}}_{= 2 \vec{r} \times \vec{a}} = \boxed{2 \vec{r} \times \vec{a}} \quad \begin{matrix} \text{From problem 1} \\ \text{again see problem 1} \end{matrix}$$

$$\text{iii) } \vec{a} \times (\vec{\nabla} \times \vec{r}) = \vec{a} \times \vec{0} = \vec{0}$$

$$\begin{aligned} \text{iv) } (\vec{a} \times \vec{\nabla}) \times \vec{r} &= -\vec{r} \times (\vec{a} \times \vec{\nabla}) = -\left[(\vec{\nabla} \cdot \vec{r}) \vec{a} - \vec{\nabla} (\vec{r} \cdot \vec{a}) \right] \\ &= -3 \vec{a} + \vec{\nabla} (a_1 x + a_2 y + a_3 z) \\ &= \boxed{-2 \vec{a}} \end{aligned}$$