ESAM 438
Interdisciplinary Nonlinear Dynamics
Fall 2000
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Take-Home Final

Due December 7, 5 p.m.

The final consists of 4 problems.

1. **One-Dimensional Systems**

   Determine all bifurcations that occur in the systems below when the parameter μ is varied.

   (a)
   $$\frac{dx}{dt} = \mu x + \frac{x^3}{1+x^2}, \tag{1}$$
   (b)
   $$\frac{dx}{dt} = \mu x - \frac{x}{1+x}. \tag{2}$$

2. **Center-Manifold Reduction**

   Consider the dynamical system
   $$\frac{dx}{dt} = y, \tag{3}$$
   $$\frac{dy}{dt} = \mu x - y - x^3 - x^2 y. \tag{4}$$

   (a) Perform a linear stability analysis of the basic solution ($x = 0, u = 0$) of (3,4). Determine also the eigenvectors associated with the two eigenvalues.

   (b) For parameter values close to the bifurcation point use the suspended system to perform a center manifold reduction of (3,4) for small μ. Keep terms up to quartic order in the dependent variable and linear in μ.

   (c) Solve (3,4) numerically. Use suitable initial conditions to obtain good numerical approximations for the center manifold for $\mu = 0.001$, $\mu = 0.01$, $\mu = 0.1$, and $\mu = 0.5$. Plot the center manifolds for these parameters together with the analytical result and discuss the comparison.

3. **Ginzburg-Landau Equation**

   Consider the partial differential equation
   $$\partial_t \psi = \mu\psi - (\partial_x^2 + 1)^2\psi + \psi\partial_x\psi + \alpha\psi^2\partial_x\psi. \tag{5}$$

   For $\alpha = 0$ (5) is the Kuramoto-Sivashinsky equation, which is relevant, for instance, for the description of the dynamics of flame fronts and of fronts of thin films of liquid flowing down an inclined plane.

(a) Perform a linear stability analysis of the basic state $\psi = 0$ of (5).

(b) Derive a Ginzburg-Landau equation that describes the weakly nonlinear behavior near the onset of the instability. Allow the amplitude of the pattern to depend on a slow spatial variable to capture also variations in the local amplitude and wavenumber.

(c) Determine a simple, spatially periodic solution of the resulting Ginzburg-Landau equation. What effect does the term involving α have on the solution? Can you give an argument why it has this effect?

4. **Periodically Forced Hopf Bifurcation**

Consider the following system, which undergoes a Hopf bifurcation to a periodic orbit and which is periodically driven with a frequency close to twice the frequency of that orbit,

$$\frac{du}{dt} = \mu u - v + u^2 + \delta \cos\left(2(1+\Omega)t\right), \tag{6}$$

$$\frac{dv}{dt} = u + bv^3. \tag{7}$$

(a) Use multiple scales to derive an equation for the evolution of the (complex) amplitude A of the oscillation for the case of small μ and weak detuning, i.e. $\Omega = O(\mu)$. The analysis is very similar to that for the amplitude equation for the unforced Hopf bifurcation, i.e. you first determine a fixed-point solution, perform a linear stability analysis of that solution and then do a weakly nonlinear analysis.

(b) The equations derived in 4a will have an explicit time-dependence in them. You can get rid of it by factoring out a periodic function from the complex amplitude A, i.e. set $A = e^{i\tilde{\Omega}t}\mathcal{A}$ with suitably chosen $\tilde{\Omega}$ and obtain an equation for $\mathcal{A}$.

(c) In the amplitude equation derived in part 4a of this problem perform a linear stability analysis of the basic state $\mathcal{A} = 0$ and identify the various instabilities that occur when μ and δ are varied for different values of Ω.

(d) Based on the linear stability analysis, what type of solutions do you expect to arise in the various instabilities? How many frequencies will the solutions exhibit in the various regimes and what are they?[1]

(e) Solve (6,7) numerically for $b = -5$ and investigate the nonlinear behavior arising from the instabilities determined above. Specifically, for $\mu = -0.02$ and $\mu = +0.02$ scan Ω through the range $(-0.05, 0.05)$ for $\delta = 0$, $\delta = 0.05$, $\delta = 0.1$ and $\delta = 0.3$. Describe the various types of behavior you observe (provide suitable plots, e.g. time series of phase space plots) and compare them with your expectation based on the linear stability analysis.

[1] In such forced systems one often can distinguish phase-locked solutions, whose frequency is rationally related to that of the forcing, and unlocked solutions which have an irrational frequency ratio.